Symmetrische Rollungen mit sphärischen Punktbahnen
Symmetric Rollings with Spherical Trajectories

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39. Fortbildungstagung für Geometrie, Strobl, 8.–10. November 2018

Austrian Science Fund FWF

Overview

1. **Symmetric Rolling Motion**
2. **Basics on Hexapods and Self-Motions**
3. **Known Symmetric Rollings with Spherical Paths**
4. **Problem Formulation using Study Parameters**
5. **Duporcq Hexapod**

Presented results are published in (and references are given with respect to):

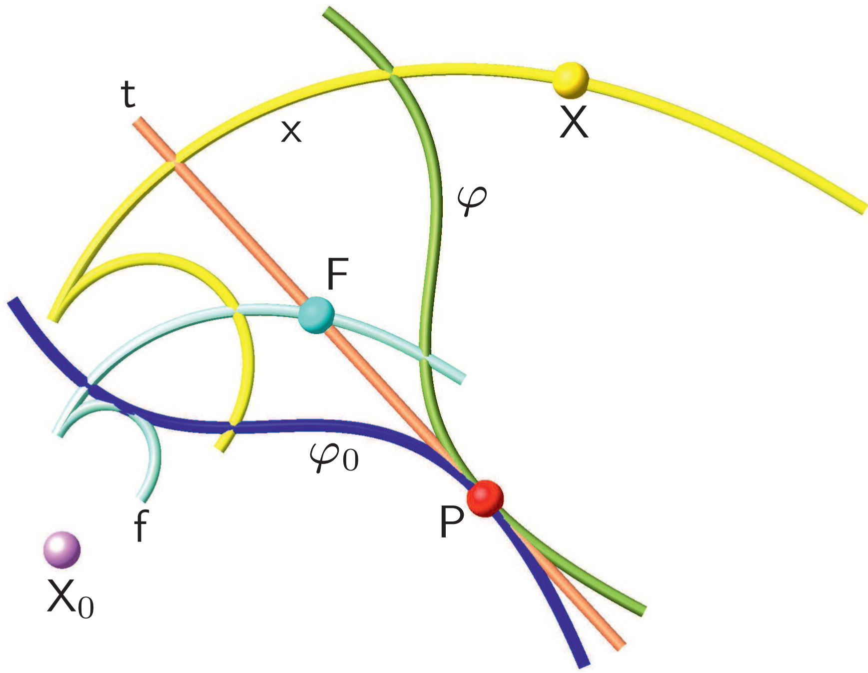
G. Nawratil: Hexapods with plane-symmetric self-motions. In special issue Kinematics and Robot Design (R. Di Gregorio ed.), Robotics 7(2) #27 (2018)

1. Planar Symmetric Rolling

During the constrained motion of a planar mechanism the instantaneous pole P traces the so-called fixed/moving polode in the fixed/moving system.

It is well known that this motion can also be generated by the rolling of the moving polode φ along the fixed polode φ_0 without sliding.

If the polodes are symmetric with respect to the pole tangent t, then the motion is called *planar symmetric rolling*.



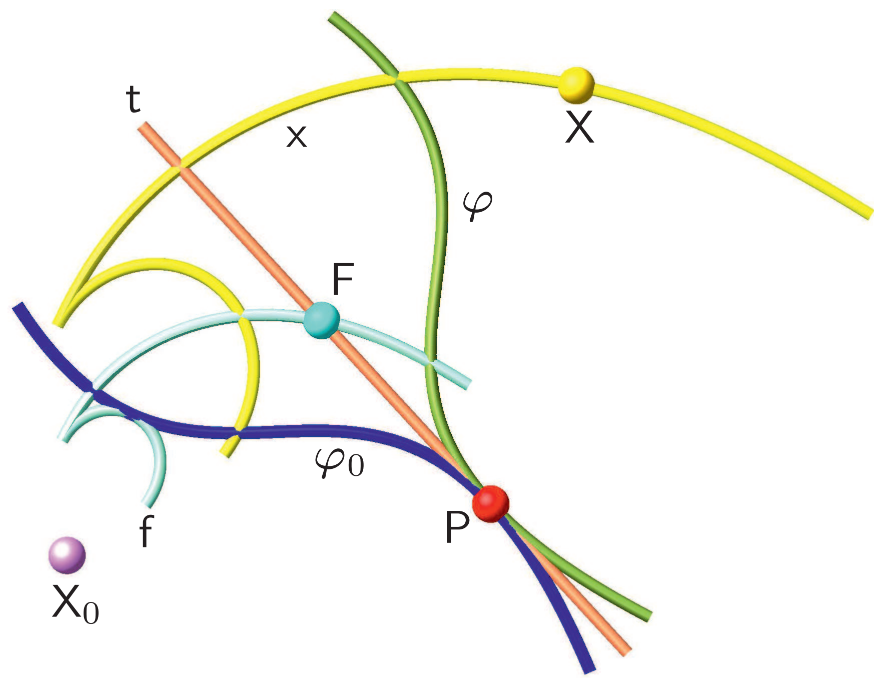
1. Planar Symmetric Rolling

In 1826, this motion was first studied by Quetelet [1], who pointed out that:

The path x of a point X under this special planar motion can be generated by the reflexion of a point X_0 of the fixed system on each tangent of φ_0.

This can also be reformulated as follows:

x can be obtained by a central dilation with center X_0 and scale factor 2 (i.e., central doubling) of X_0's pedal-curve f with respect to φ_0.

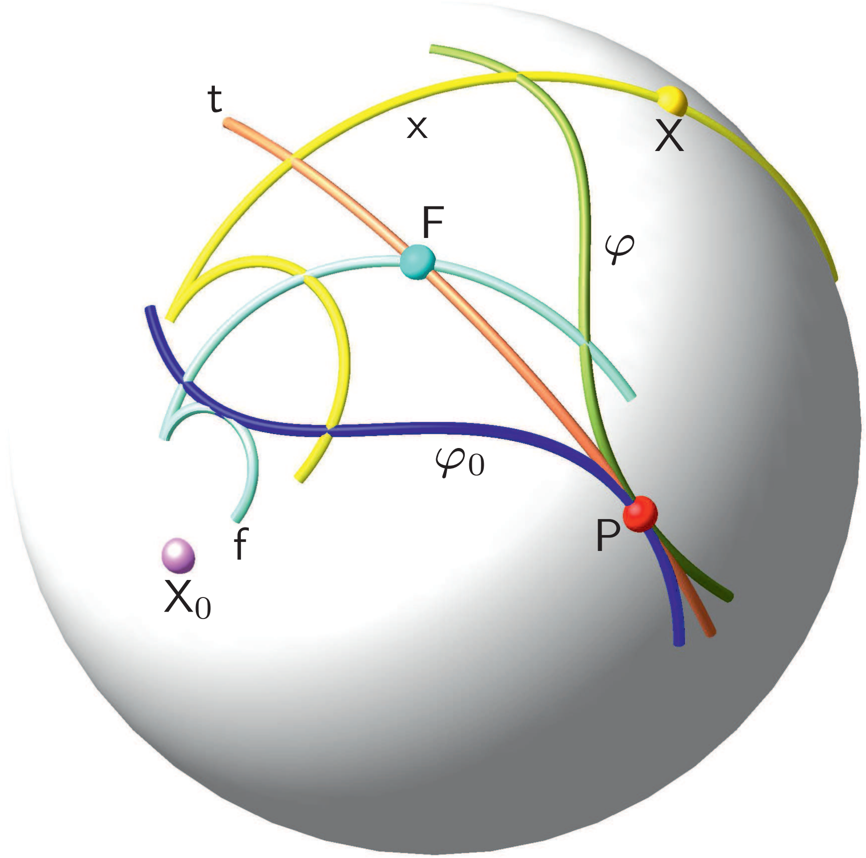


1. Spherical Symmetric Rolling

The spherical counterpart of this motion is called *spherical symmetric rolling* and was extensively studied by Tölke [2].

The spherical version of the above given characterization also holds true for the spherical symmetric rolling.

From another perspective, a planar/spherical symmetric rolling is generated by reflecting the fixed system in a 1-parametric continuous set of lines/great circles.



1. Spatial Generalization of Symmetric Rollings

The latter point of view is of importance for the spatial generalization of symmetric rollings, which can be done in multiple ways:

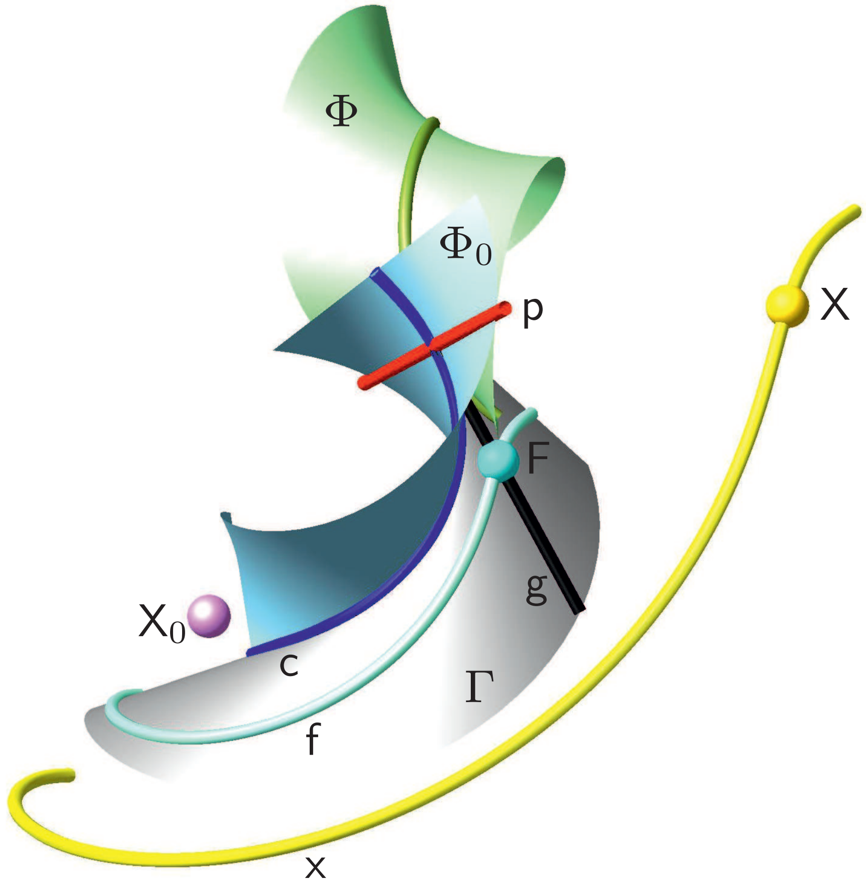
1. Darboux [5] noted a 2-parametric spatial motion, which is generated by the rolling of a moving surface Φ on an indirect congruent fixed surface Φ_0.

   It also holds that the path-surface of a point X can be generated by the reflexion of a point X_0 of the fixed system on each tangent-plane of Φ_0; i.e. the path-surface can be obtained by a central doubling of X_0's pedal-surface with respect to Φ_0's tangent-planes.

1. Spatial Generalization of Symmetric Rollings

2. Krames [6] considered line-symmetric motions as the 1-parametric spatial analogue. These motions are obtained by reflecting the moving system in a 1-parametric set of lines forming the so-called *basic surface* Γ.

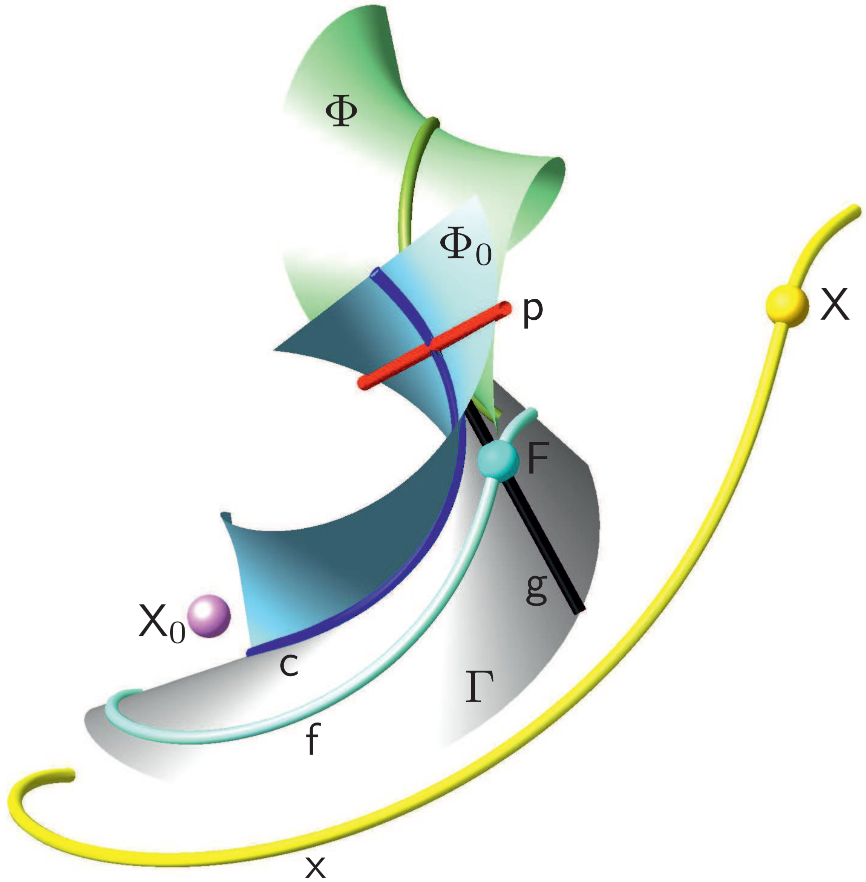
   The path x of a point X under a line-symmetric motion can be generated by the reflexion of a point X_0 on each generator g of Γ; i.e. x can be obtained by a central doubling of X_0's pedal-curve f with respect to Γ's rulings.



1. Spatial Generalization of Symmetric Rollings

However, it should be pointed out that Γ differs from the fixed axode Φ_0, which is generated by Γ's central tangents (i.e. common normals of infinitesimal neighboring lines of Γ).

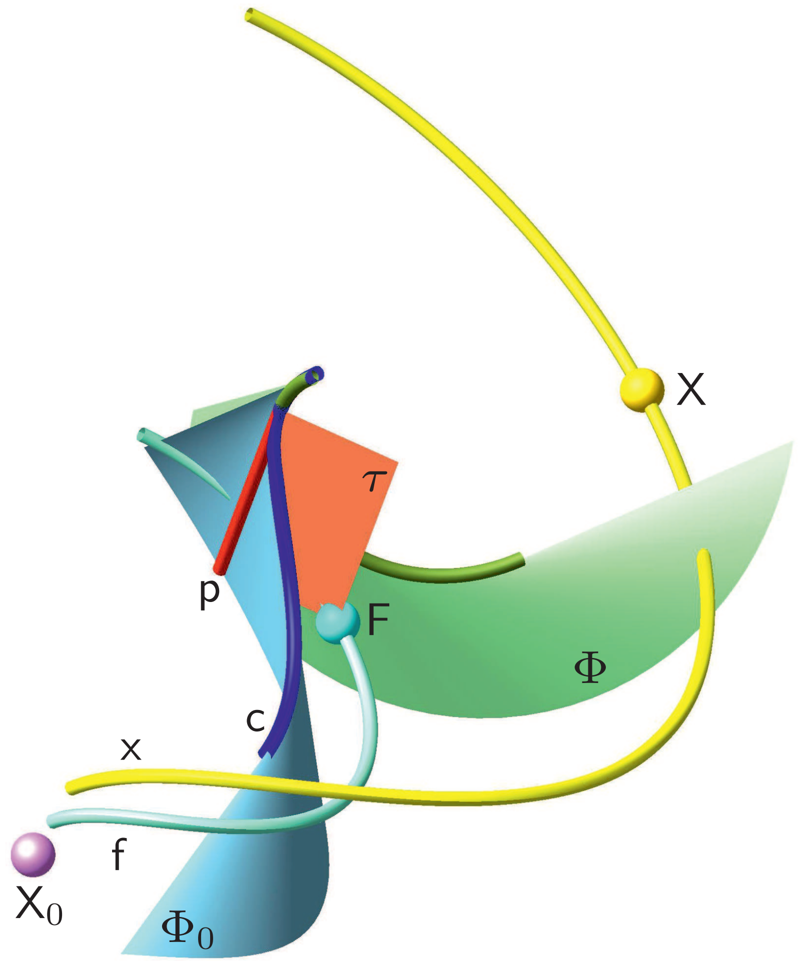
Φ_0 and the moving axode Φ are at each time instant symmetric with respect to the axis p of the instantaneous screw, which is in general not an instantaneous rotation.



1. Spatial Generalization of Symmetric Rollings

3. It is astonishing that neither Tölke [2] nor Krames [6] mentioned the more apparent generalization by reflecting the fixed system in a 1-parametric continuous set of planes.

   Less attention was paid to these so-called *plane-symmetric motions* in the literature until now. The basic properties of this motion type were reported by Bottema and Roth [8].

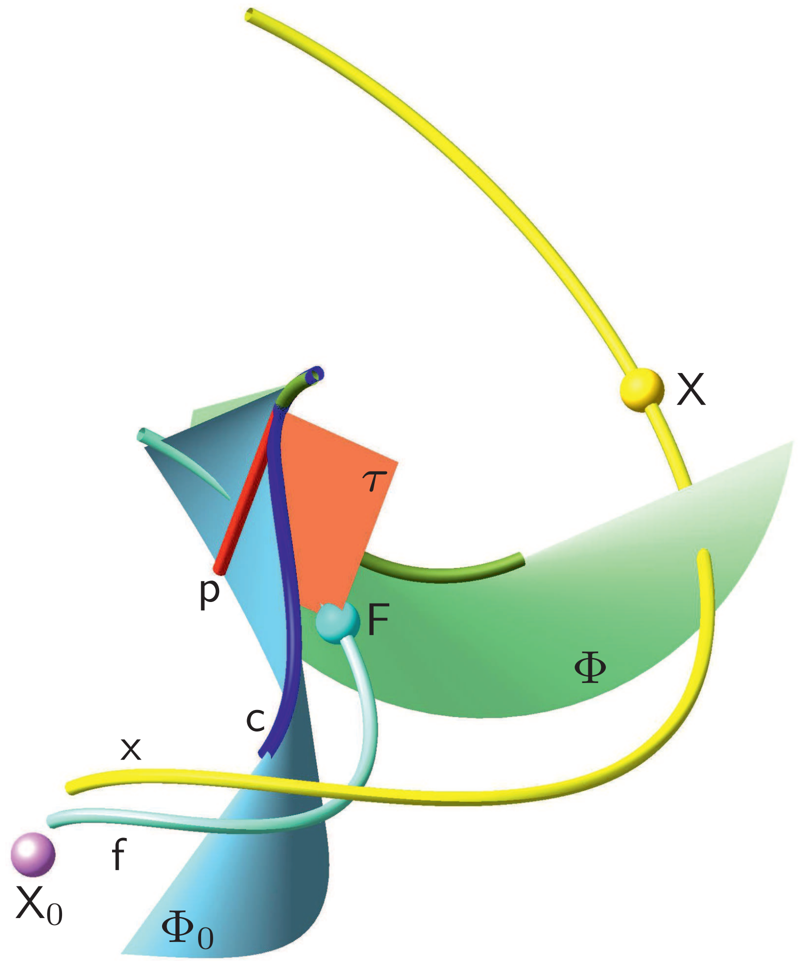


1. Plane-Symmetric Motion

Given is a 1-parametric continuous set of planes $\tau(t)$ with time t. By reflecting the fixed frame $\mathfrak{F}_0$ on the plane $\tau(t)$, we obtain the pose $\mathfrak{F}_0^t$.

Two infinitesimal neighboring poses $\mathfrak{F}_0^t$ and $\mathfrak{F}_0^{t+\Delta t}$ can be transformed into each other by a reflexion on $\tau(t)$ followed by a further reflexion on $\tau(t+\Delta t)$.

This is a pure rotation about the line p of intersection of $\tau(t)$ and $\tau(t+\Delta t)$.

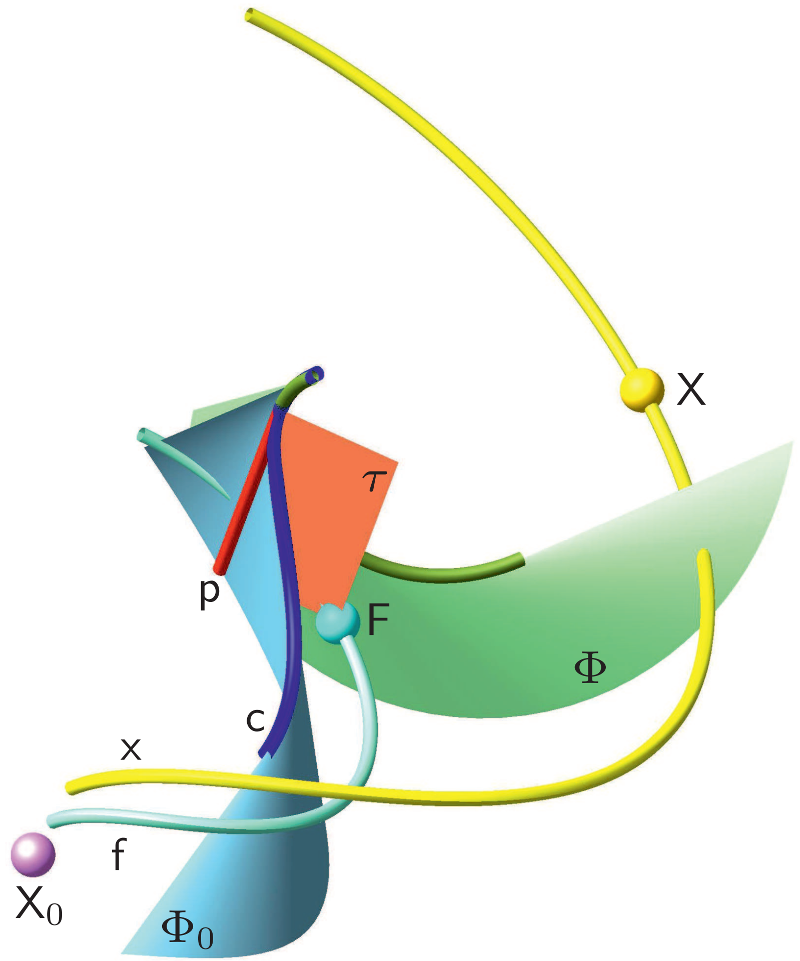


1. Plane-Symmetric Motion

The axis p of this instantaneous rotation is a torsal ruling of the surface enveloped by the given 1-parametric set of planes.

Therefore the fixed axode Φ_0 is a developable surface and the corresponding moving axode Φ is obtained by reflecting Φ_0 in Φ_0's tangent-plane τ along p.

The path x of a point X under a plane-symmetric motion can be generated by the reflexion of a point X_0 on each tangent-plane τ of Φ_0



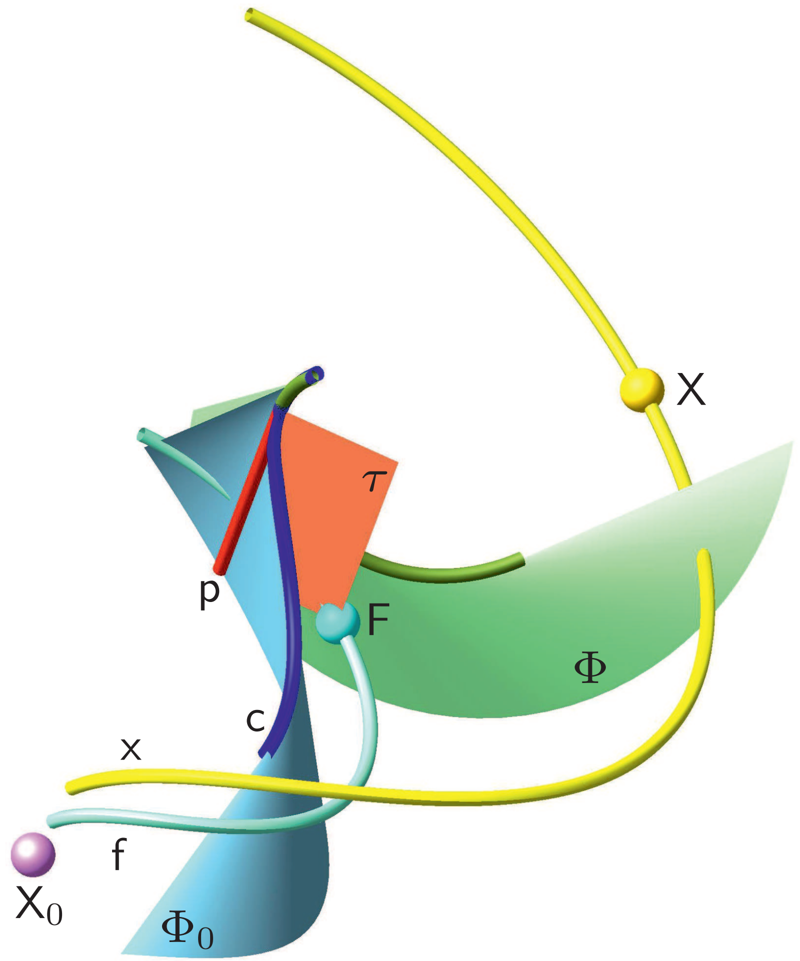
1. Plane-Symmetric Motion

This can also be reformulated as follows:

x can be obtained by a central doubling of X_0's pedal-curve f with respect to Φ's tangent-planes.

Due to these properties, the plane-symmetric motion is the straightforward spatial counterpart of the planar/spherical symmetric rolling.

Therefore the plane-symmetric motion is also called *spatial symmetric rolling*.

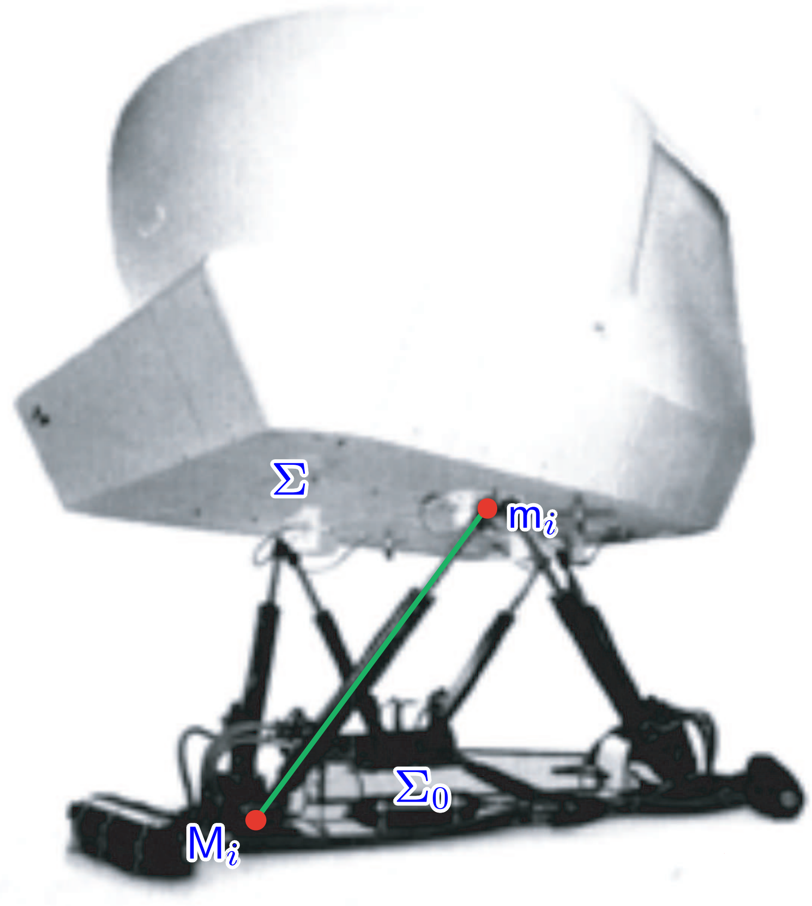


2. Basics on Hexapods and Self-Motions

A hexapod consists of a moving platform Σ and the fixed base Σ_0, which are linked with six legs anchored via spherical joints.

The geometry of a hexapod is given by the six base anchor points $\mathrm{M}_i \in \Sigma_0$ and by the six platform points $\mathrm{m}_i \in \Sigma$.

For fixed leg lengths a hexapod is in general rigid. But, in some particular cases the hexapod can perform an n-parametric motion $(n > 0)$, which is called *self-motion*.

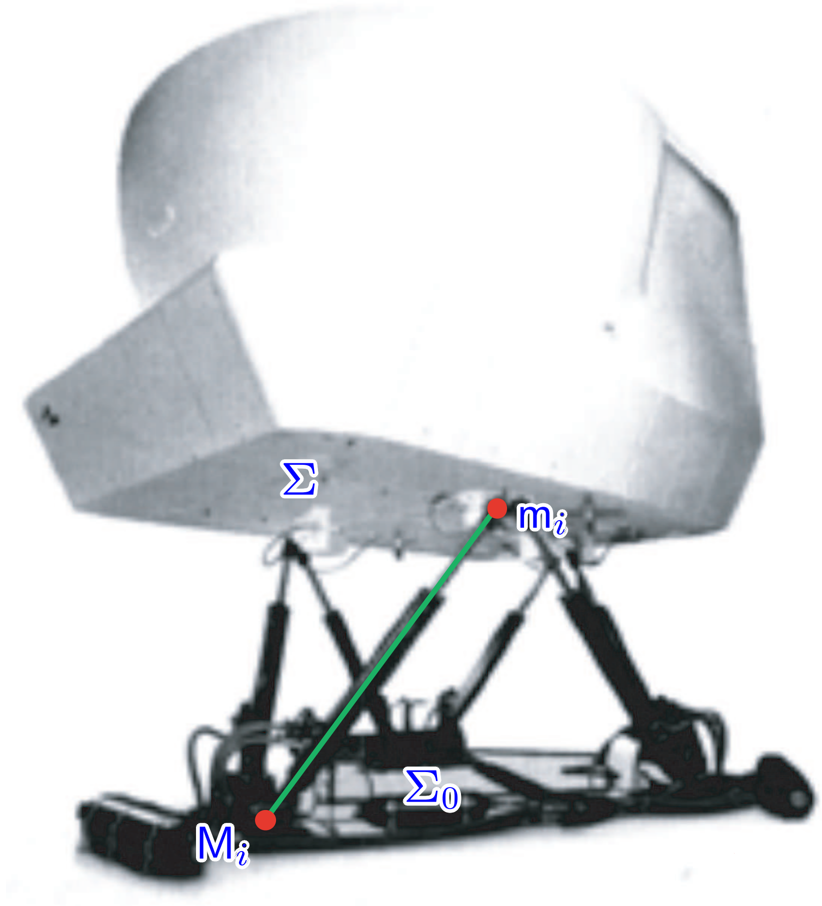


2. Basics on Hexapods and Self-Motions

All self-motions are solutions to the problem posed by the French Academy of Science for the *Prix Vaillant* (1904), also known as

Borel–Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

Clearly, in each pose of a self-motion the hexapod has to be *infinitesimal flexible*.

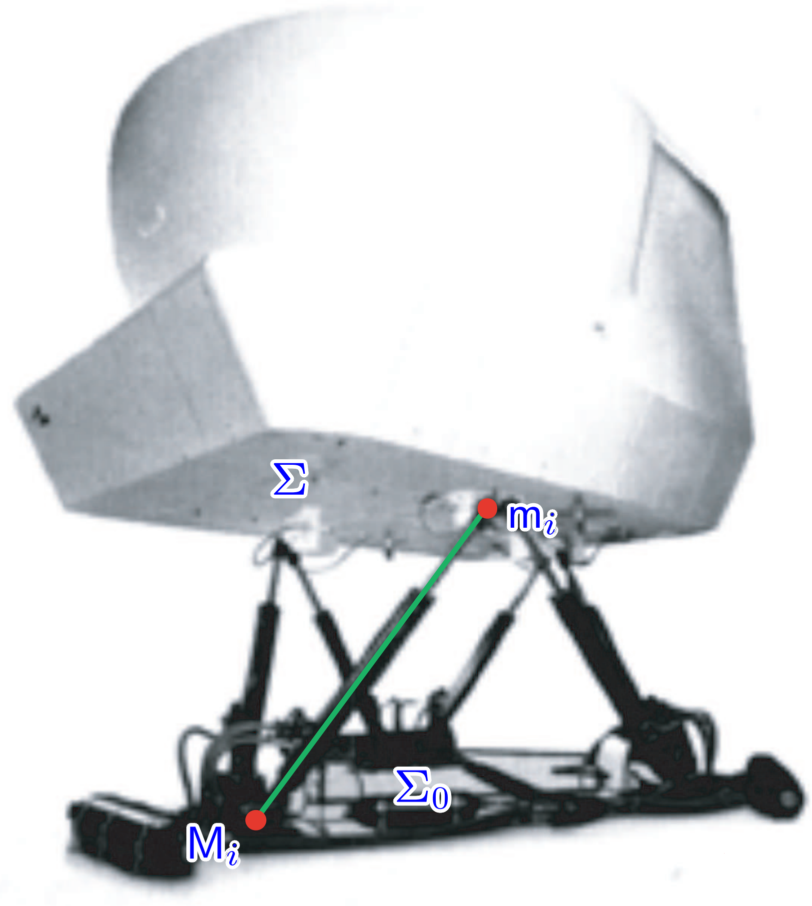


2. Basics on Hexapods and Self-Motions

A hexapod is *singular* (*infinitesimal flexible*, *shaky*), if and only if, the carrier lines of the six legs belong to a linear line complex.

A hexapod is called *architecturally singular* if the six legs belong in each relative pose of the platform with respect to the base to a linear line complex.

These special solutions to the Borel–Bricard problem are already well studied (review on this topic is given in [15]).



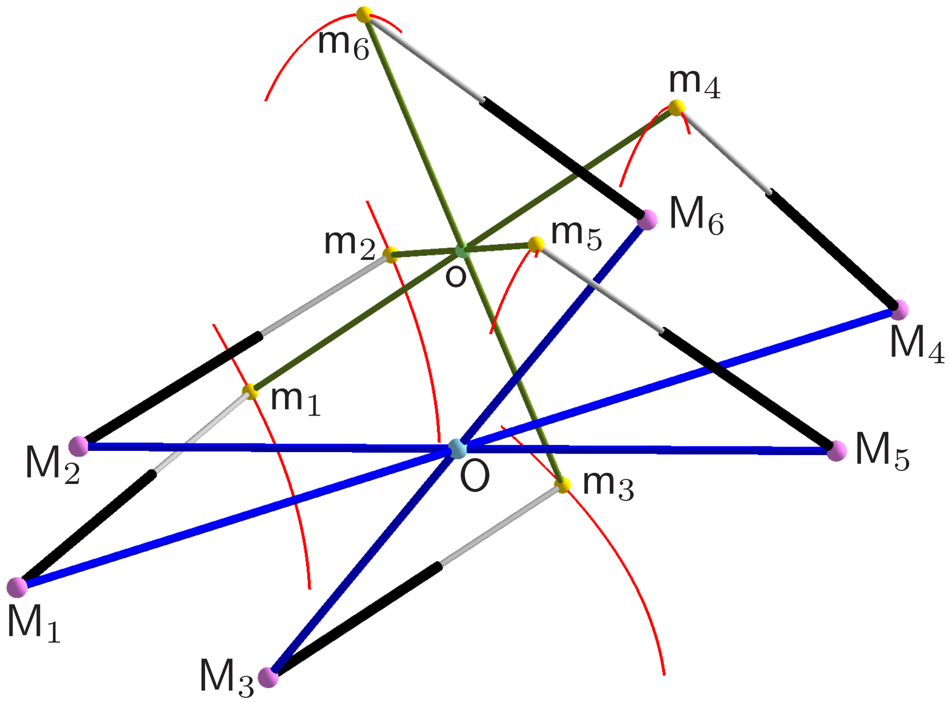
2. Basics on Hexapods and Self-Motions

The approaches for the determination of non-architecturally singular hexapods recorded in the literature, can roughly be divided into the following two groups:

1. Assumptions on the geometry of the platform and base; e.g.,

   ⋆ linear mapping between platform and base [16–22],
   ⋆ symmetry properties of platform and base [20–24] (cf. Figure),
   ⋆ special topology (e.g. octahedral structure [25]),

   or a combination of these assumptions (e.g. [20–22]).

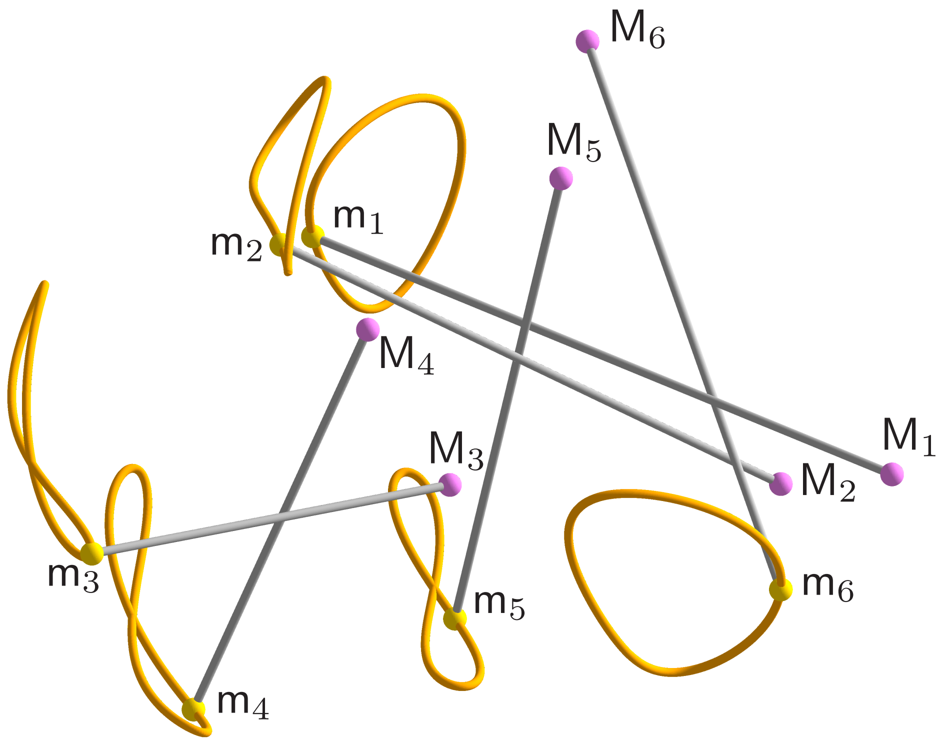


2. Basics on Hexapods and Self-Motions

2. Assumptions on the self-motion; e.g.,

   ⋆ line-symmetric self-motion [9],
   ⋆ type II Darboux–Mannheim self-motion [26],
   ⋆ Schoenflies self-motion [27],
   ⋆ translational self-motion [28],
   ⋆ self-motion of maximal degree [29] (cf. Figure),

   or more generally characterizations like *linear relations between direction cosines* [30–33].



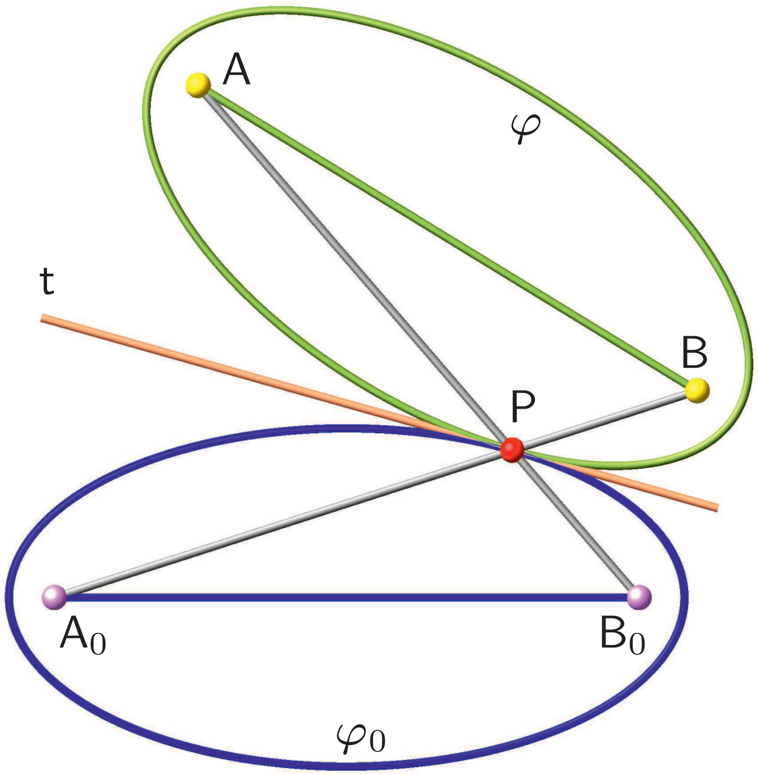
These assumptions are done to reduce the complexity of the problem, as one has to deal with 29 design parameters and 6 degrees of freedom.

3. Planar Symmetric Rollings with Circular Paths

All planar symmetric rollings with points running on circular paths were determined by Bereis [3].

a) The polodes are ellipses and the focals of the moving ellipse are running on circles.

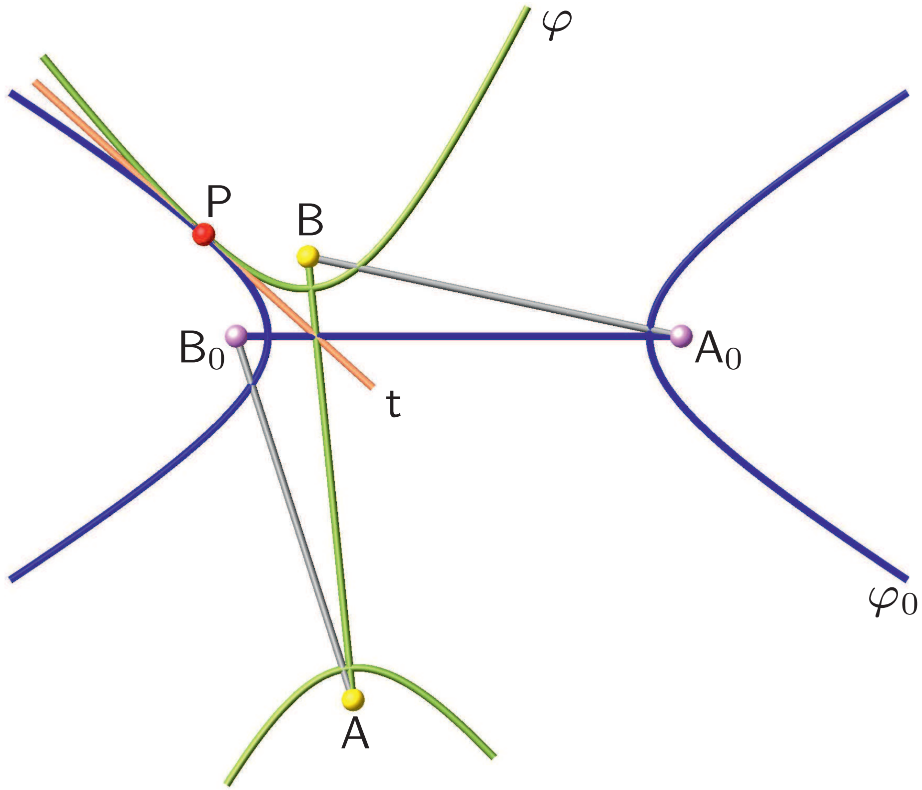
   This motion can be realized by a twin-crank mechanisms with non-counter-rotating cranks.



3. Planar Symmetric Rollings with Circular Paths

b) The polodes are hyperbolas and the focals of the moving hyperbola are running on circles.

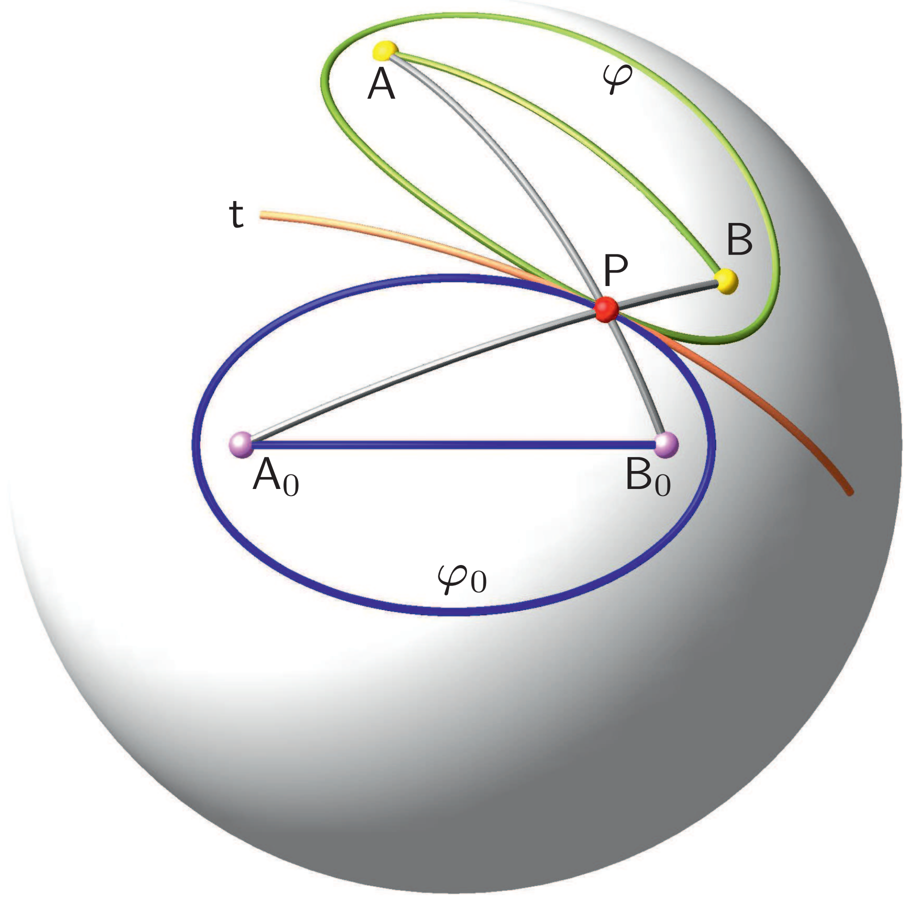
This motion can be realized by a twin-crank mechanisms with counter-rotating cranks.



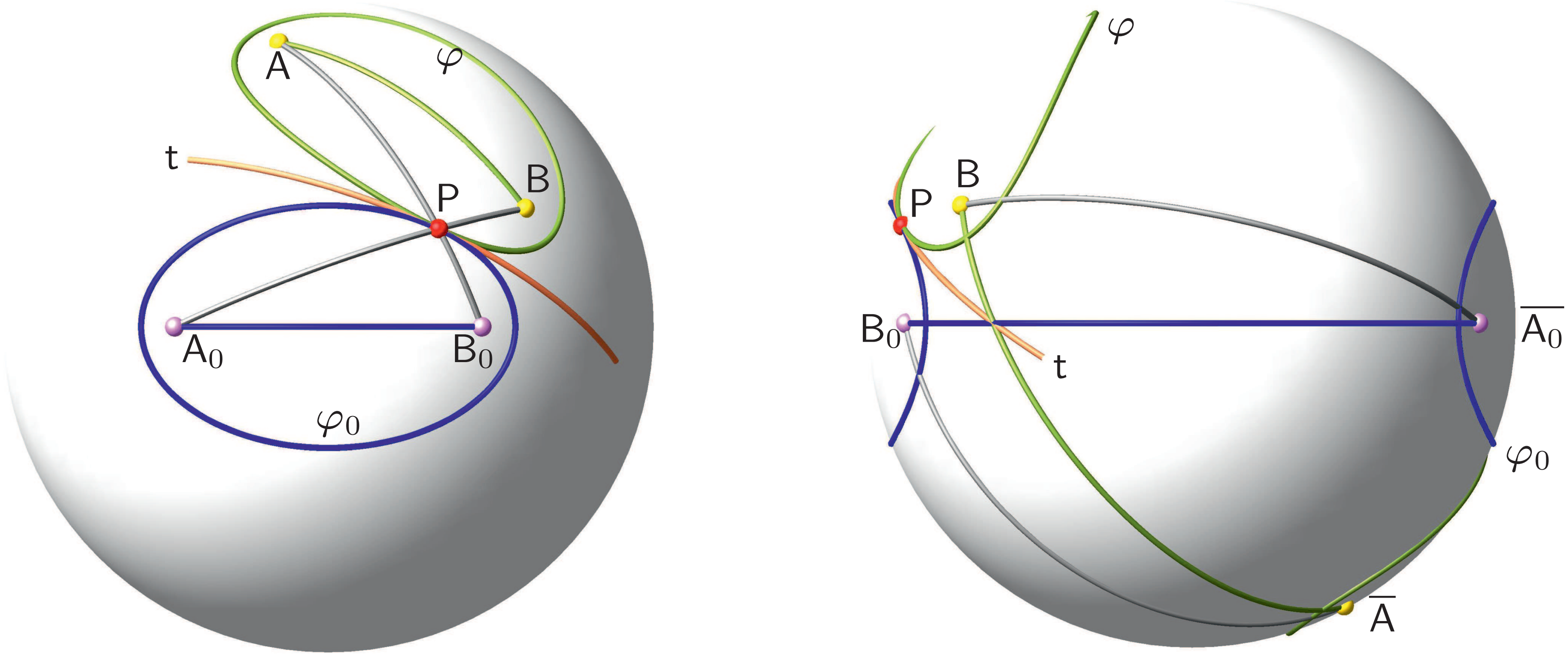
3. Spherical Symmetric Rollings with Circular Paths

Spherical symmetric rollings with circular paths are generated by spherical isograms, which are studied in more detail in [34].

The moving and fixed polode are spherical conics. As on the sphere points can be replaced by their antipodes, it can be shown [35] that every spherical conic can be interpreted as a spherical ellipse.

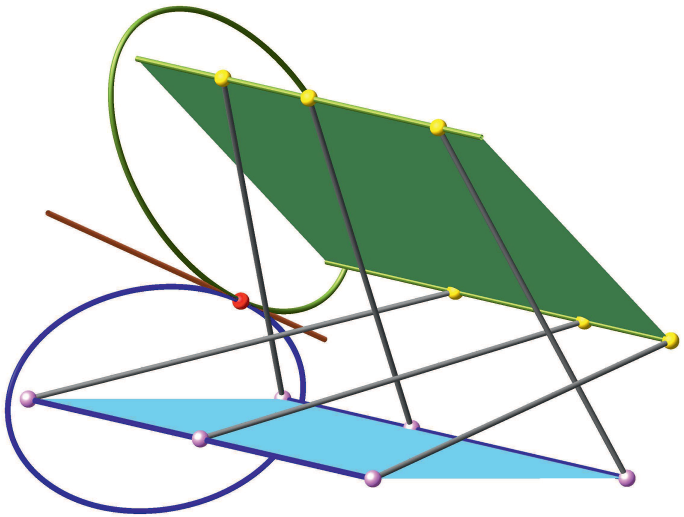


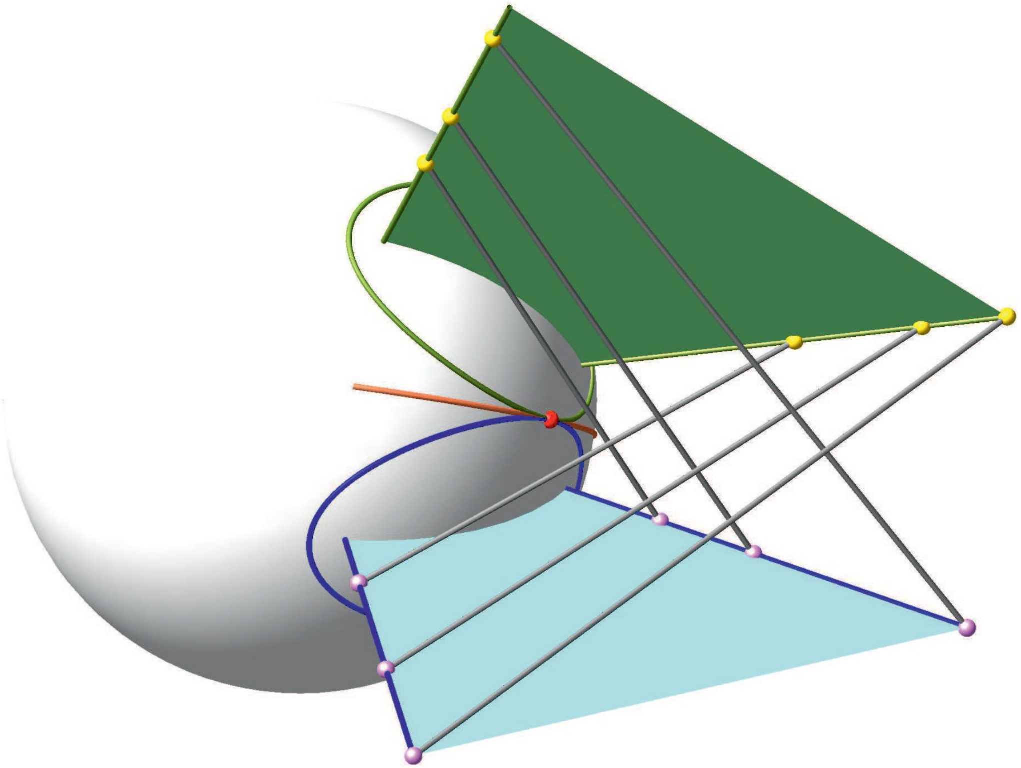
3. Spherical Symmetric Rollings with Circular Paths



If we replace A and A_0 by their antipodal points $\overline{A}$ and $\overline{A}_0$, respectively, and look on the sphere from the right side, then we get the figure illustrated on the right.

3. Resulting Hexapods with Symmetric Self-Motions





The axodes of the self-motions are cylinders (left) and cones (right), respectively, but only the planar/spherical directrices of these singular quadrics are illustrated.

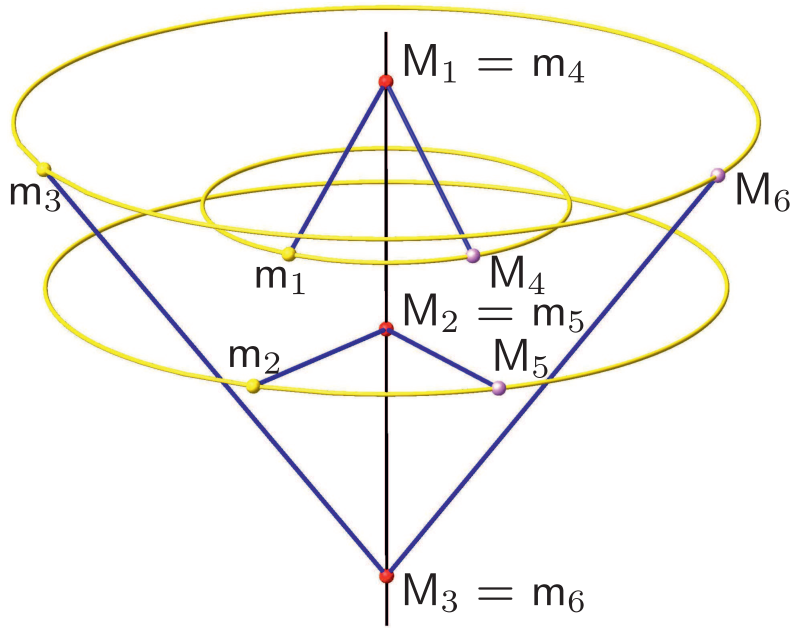
3. Hexapods with Trivial Symmetric Self-Motions

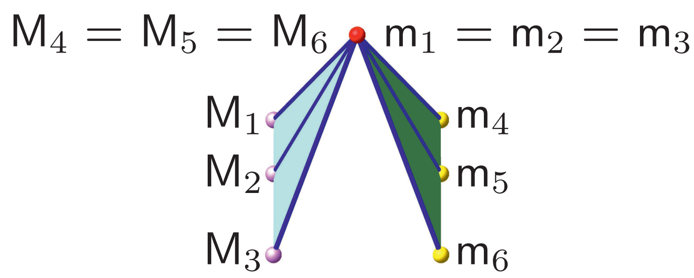
Clearly, a pure rotation is a symmetric rolling where every point of the moving system traces a circle. This trivial case (butterfly self-motion) arises for:

- ⋆ $M_1, \ldots M_i$ are collinear and
- ⋆ $m_{i+1}, \ldots, m_6$ are collinear

with $1 \leq i \leq 5$.

If the collinearity is replaced by coincidence then there exists even a 2-dimensional spherical self-motion.





3. Known Self-Motions are Line-Symmetric

If we embed the planar/spherical symmetric rollings into the group SE(3) of Euclidean displacements, then they can also be seen as line-symmetric motions.

Therefore, all the self-motions of the hexapods given so far are symmetric rollings and line-symmetric motions at the same time.

This raises also the question of whether self-motions exist, which are symmetric rollings but not line-symmetric.

An answer can be obtained by formulating the problem in terms of algebraic geometry by means of Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$.

4. Problem Formulation using Study Parameters

There is a bijection between SE(3) and real points of the 7-dimensional Study parameter space P^7, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$ sliced along the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$.

For $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$ the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and the rotation matrix $\mathbf{R} := (r_{ij})$ of the corresponding Euclidean displacement $\mathbf{x} \mapsto \mathbf{Rx} + \mathbf{t}$ are given by:

$$t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2),$$

$$t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),$$

$$t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),$$

$$\mathbf{R} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

4. Problem Formulation using Study Parameters

The reflection on a plane is an orientation-reversing congruence transformation, which cannot be described directly by the Study parameters. Therefore, we follow the approach of Selig and Husty [7], which is as follows:

- We start with a reflexion on a fixed plane; say the xy-plane of the fixed frame $\mathfrak{F}_0$. By this plane-reflection of $\mathfrak{F}_0$, we obtain $\overline{\mathfrak{F}}_0$.
- Then we apply the reflexion on the plane $\tau(t)$, which finally yields the pose $\overline{\mathfrak{F}}_0^t$.

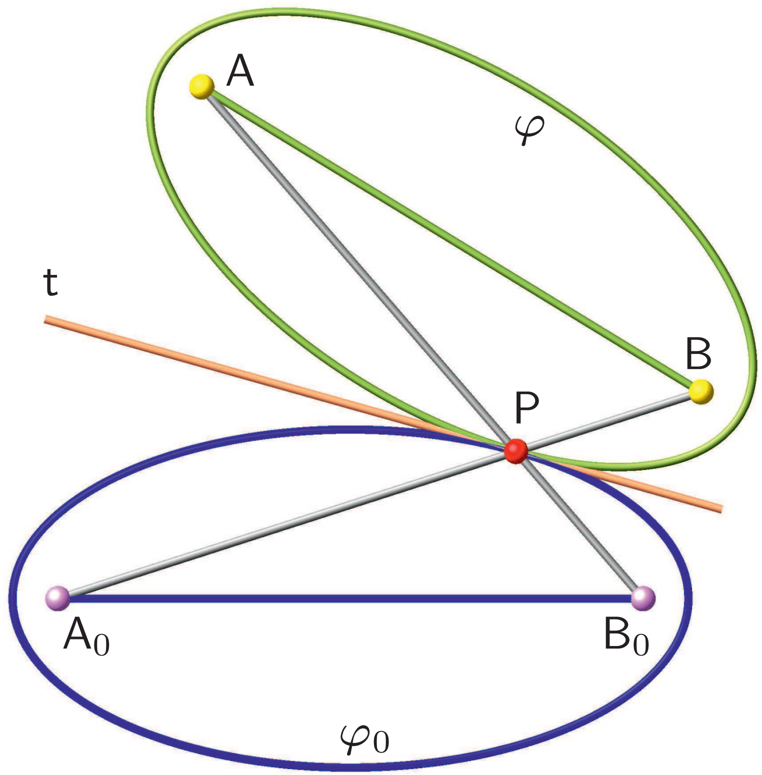
As the composition of two plane-reflexions is again a direct congruence transformation, we can describe the plane-symmetric motions in this way.

As a consequence plane-symmetric motions are given by $e_3 = f_0 = f_1 = f_2 = 0$; i.e. this corresponds to a 3-dimensional generator space P^3 of Ψ.

4. Symmetric Rollings with Spherical Paths

Due to the symmetry of the motion the following theorem holds:

Theorem 2. If a point A of the moving system traces a spherical path with center B_0 during a symmetric rolling, then also the point B of the moving system has a spherical trajectory about the point A_0, where A and A_0 as well as B and B_0 are the symmetric points of the moving and fixed system.



Due to this symmetric-leg replacement the set of base points and platform points are indirectly congruent.

4. Spatial Symmetric Rollings with Spherical Paths

The condition that the point m_i is located on a sphere centered in M_i is a quadratic homogeneous equation in the Study parameters according to Husty [36].

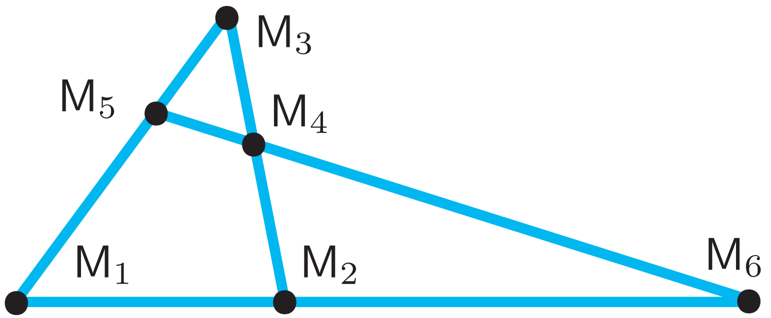
Therefore it corresponds to a quadric Λ_i in the 3-dimensional projective space P^3 with homogenous coordinates $(e_0 : e_1 : e_2 : f_3)$.

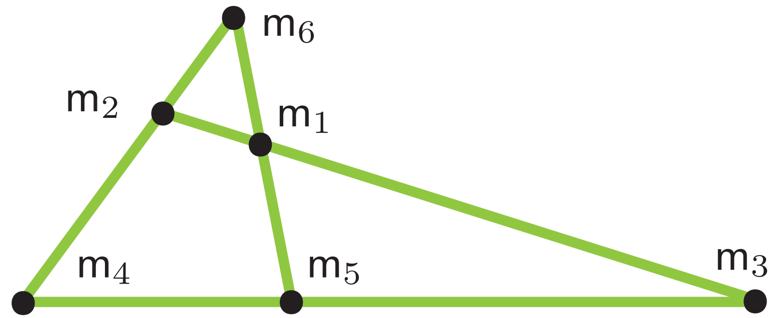
Due to the symmetric-leg replacement, we only have to find spatial rolling motions where three points run on spheres. This means that the corresponding three quardrics Λ_1, Λ_2 and Λ_3 of P^3 have to have a curve in common, which can be a:

- ⋆ straight line
- ⋆ conic section
- ⋆ cubic curve
- ⋆ quartic curve

5. Duporcq Hexapod

A discussion of cases shows that there exists only one further hexapod with plane-symmetric self-motions, which are neither planar nor spherical. This manipulator is the so-called *Duporcq hexapod*.



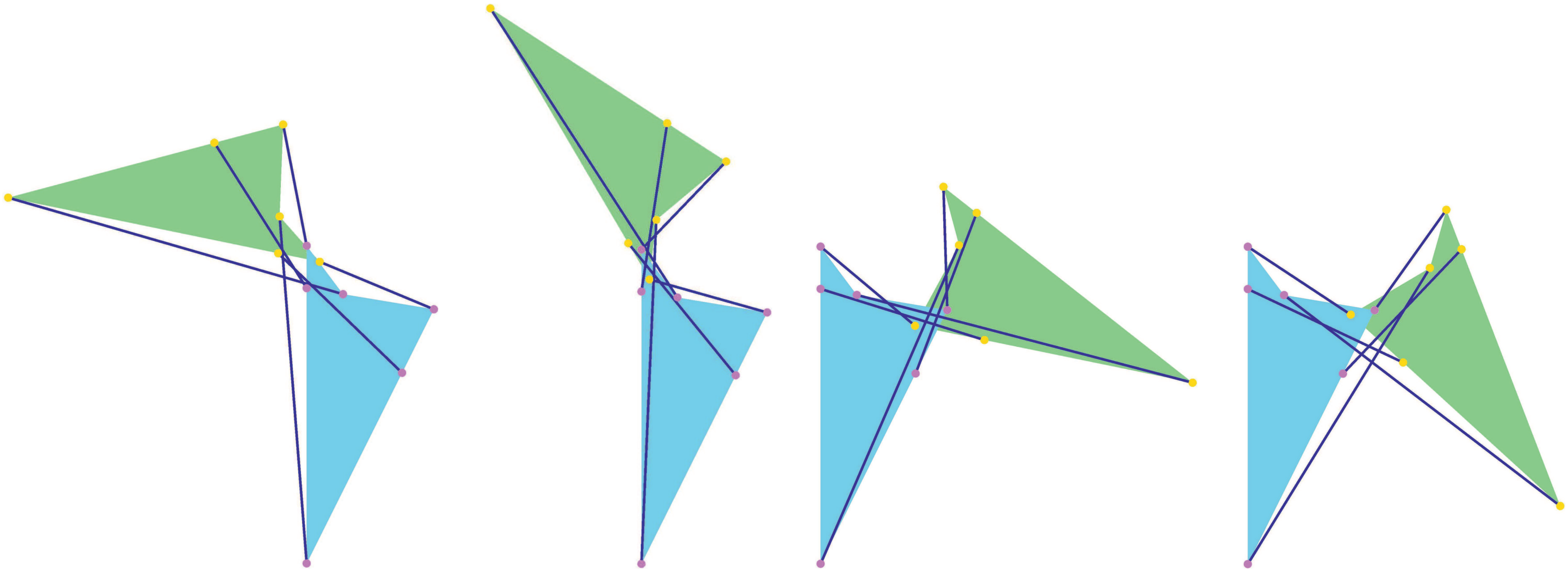


$M_1, \ldots, M_6$ and $m_1, \ldots, m_6$ be the vertices of two congruent complete quadrilaterals, which are labeled in a way that m_i is the opposite vertex of M_i.

It is well known [39] that the Duporcq hexapod is architecturally singular.

5. Duporcq Hexapod

Due to Duporcq [38] this hexapod has a 2-parametric line-symmetric self-motion. There are four transition poses between this line-symmetric self-motion and the 1-parametric plane-symmetric self-motion. All four branching singularities are totally flat configurations.



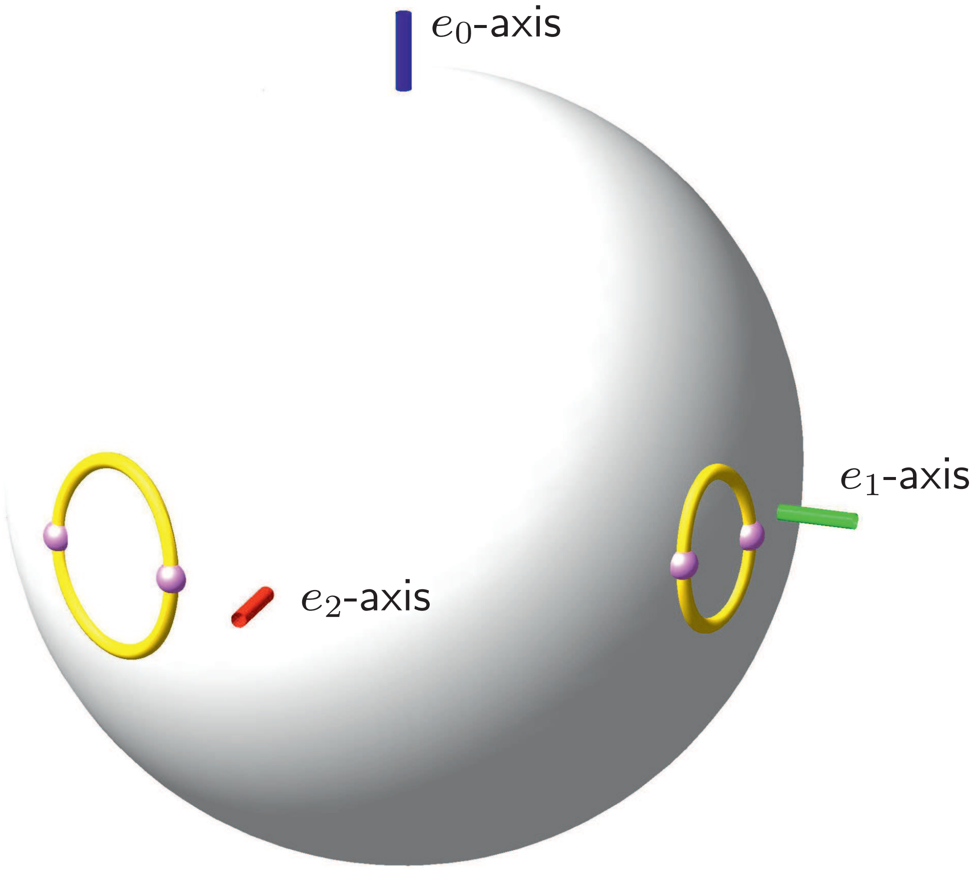
5. Duporcq Hexapod

Recall that the sphere condition Λ_i is a homogenous quadratic equation in e_0, e_1, e_2, f_3.

We express f_3 from the condition $\Lambda_2 - \Lambda_1$, which is linear in f_3.

Plugging the resulting expression into Λ_1 implies a homogenous quartic equation Υ in e_0, e_1, e_2, which represents the plane-symmetric self-motion.

The quartic Υ is displayed under consideration of the normalization condition $e_0^2 + e_1^2 + e_2^2 = 1$.

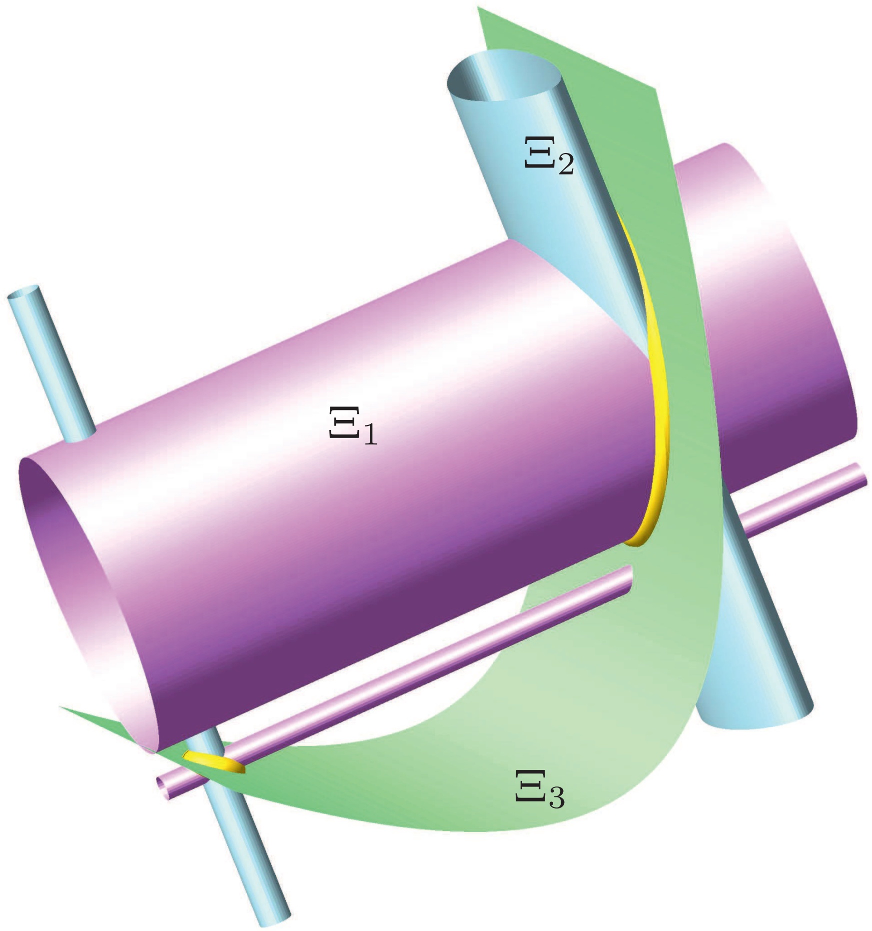


5. Duporcq Hexapod

If $ax + by + cz + d = 0$ is the equation of the plane of symmetry, then its dual representation is given by the homogenous quadruplet $(u_0 : u_1 : u_2 : u_3) = (d : a : b : c)$ according to [11].

The fixed axode corresponds to an algebraic curve of degree 4 in the dual representation, which results from the intersection of two cylinders of degree 4 and one quadratic cylinder.

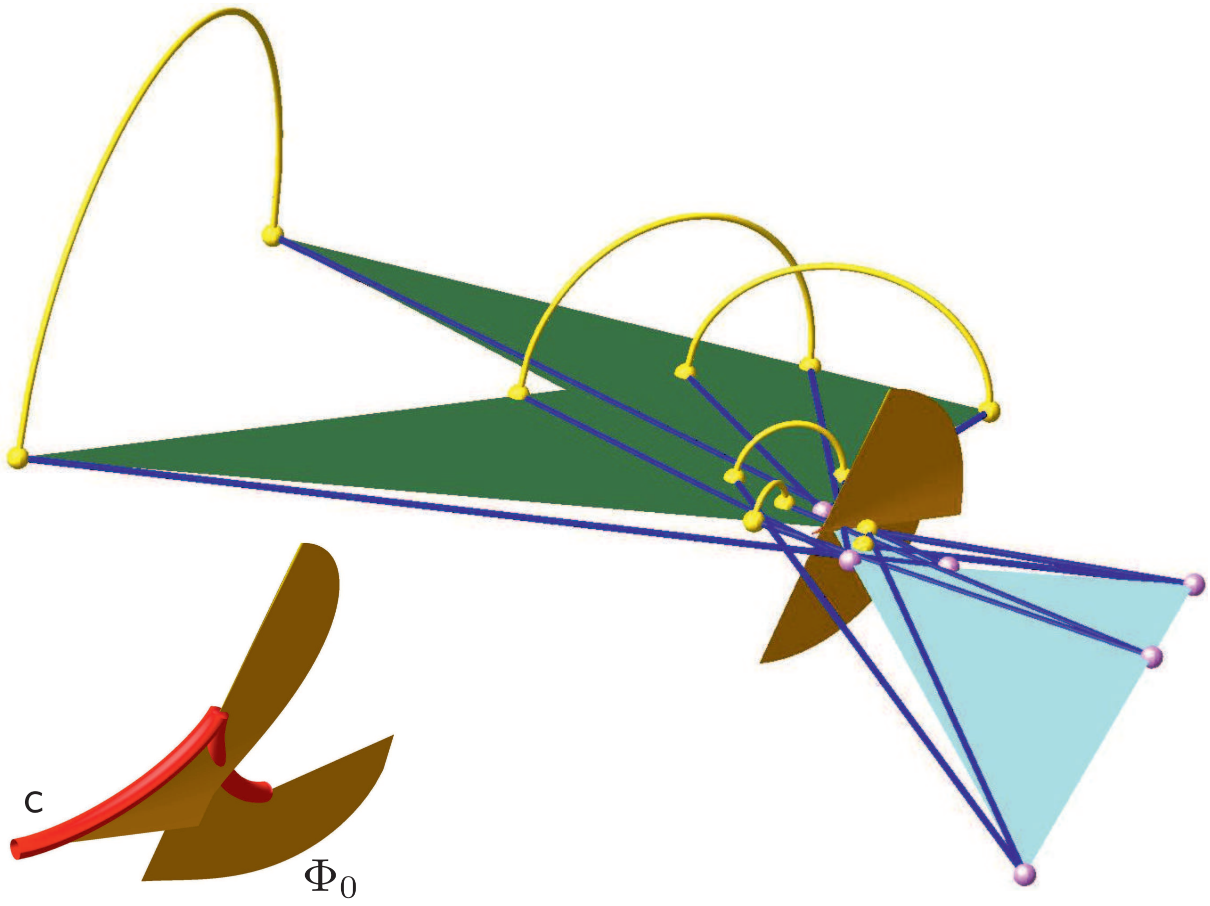
This curve can be parametrized (two branches).



5. Duporcq Hexapod

Trajectories of the platform points during the plane-symmetric self-motion between the totally flat transition poses 1 and 2.

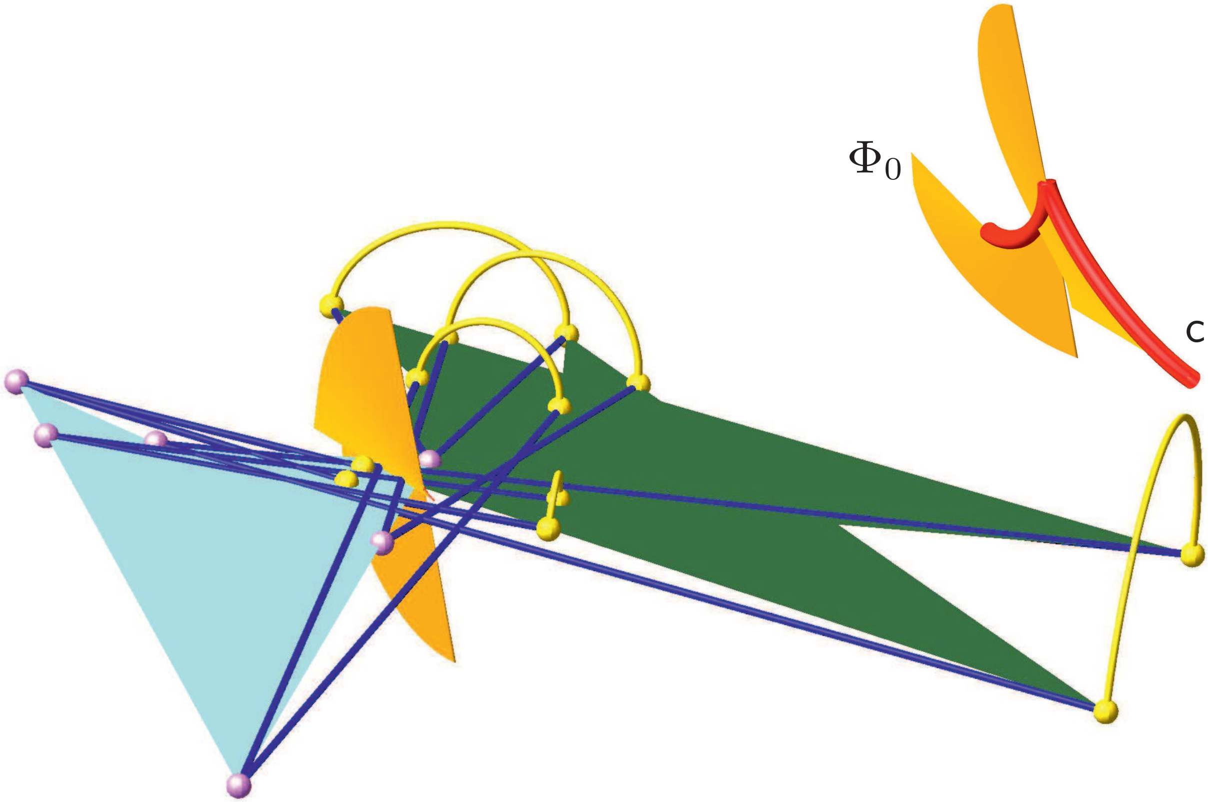
In the lower left corner a blow up of the fixed axode Φ_0 (including its line of regression c) is displayed.



5. Duporcq Hexapod

Trajectories of the platform points during the plane-symmetric self-motion between the totally flat transition poses 3 and 4.

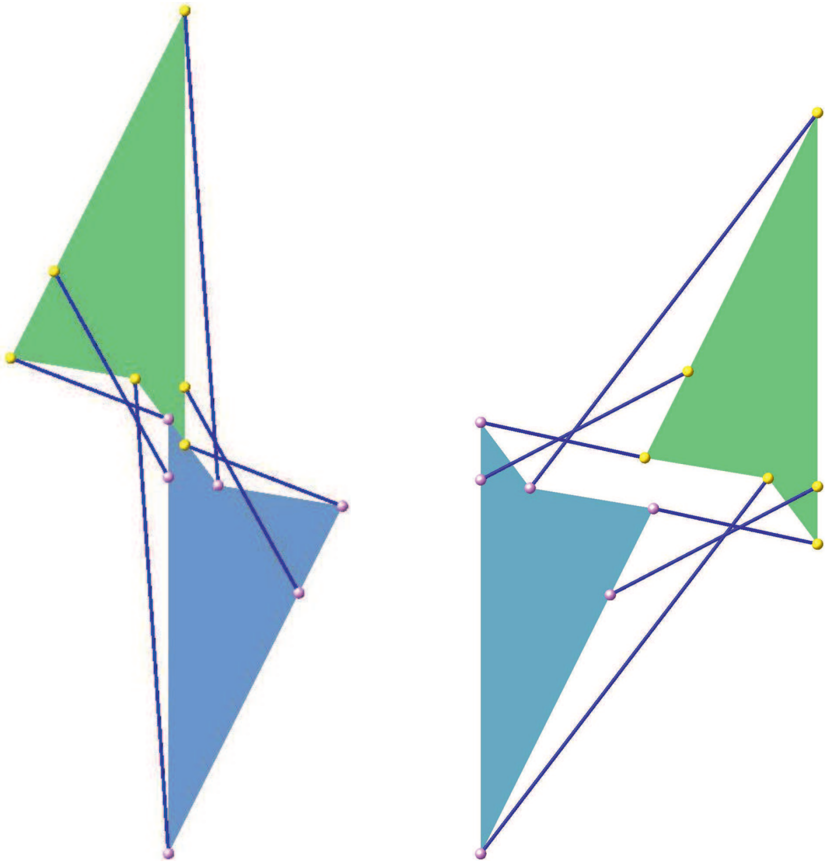
In the upper right corner a blow up of the fixed axode Φ_0 (including its line of regression c) is displayed.



5. Duporcq Hexapod

The Duporcq manipulator also has a 1-parametric translational self-motion (circular translation).

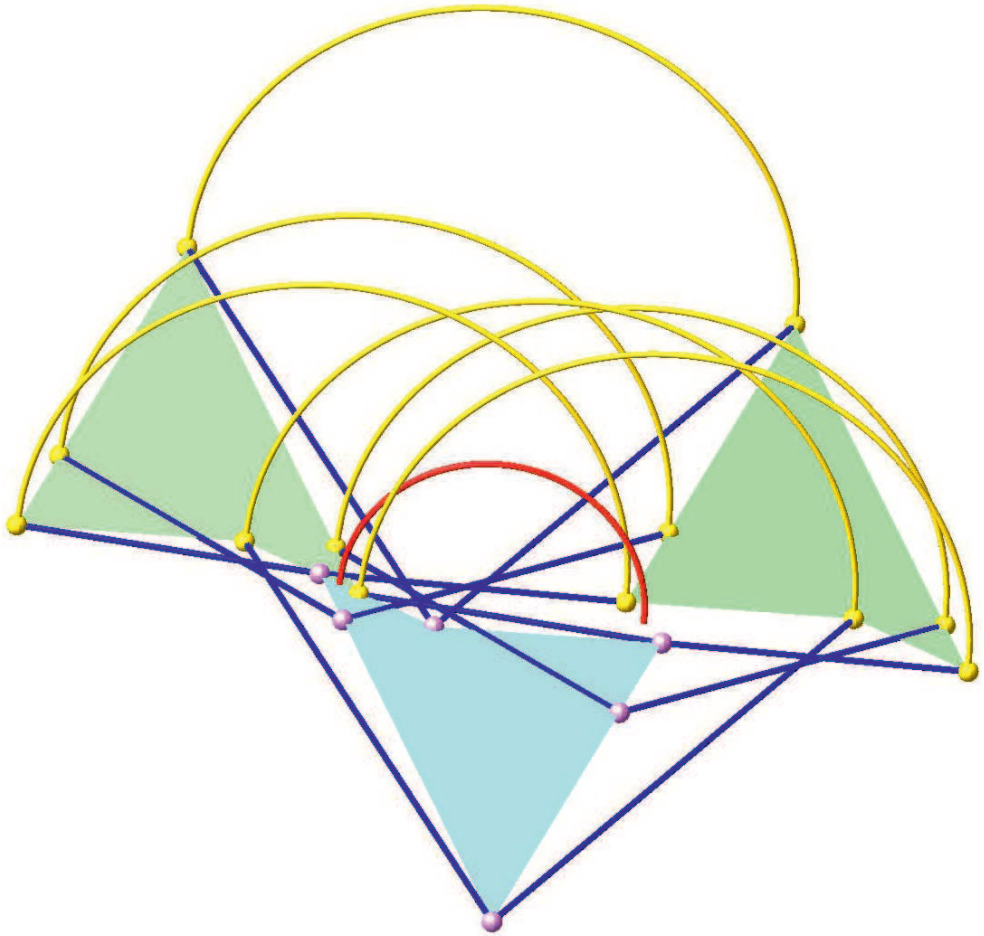
There are two transition poses between this translational self-motion and the 2-parametric line-symmetric self-motion. Again both branching singularities are totally flat configurations.



5. Duporcq Hexapod

This circular translation can also be seen as a point-symmetric motion, where the corresponding curve (half-circle) is illustrated in red.

Note that there is no branching singularity between plane-symmetric self-motions and point-symmetric self-motions.



Conclusion

Summed up one can say, that the Duporcq hexapod is a twofold kinematotropic mechanism, as there are branching singularities between the 2-dimensional line-symmetric self-motion and the 1-dimensional

- point-symmetric self-motion,
- plane-symmetric self-motion.

Due to its kinematotropic behavior and its total flat branching singularities the Duporcq manipulator is possibly of interest for the design of deployable structures.

Acknowledgements
The research was supported by Grant No. P 24927-N25 of the Austrian Science Fund FWF within the project "Stewart Gough Platforms with Self-Motions".