Stewart Gough platforms with non-cubic singularity surface

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Abstract

In the general case the singularity locus of a parallel manipulator of Stewart Gough type is a cubic surface in the space of translations. In this article we determine all Stewart Gough platforms which possess only a linear or quadratic singularity surface. These manipulators have the advantage, that their singularity surface can easily be visualized because it is a plane or a quadric for any orientation of the platform. Moreover we also give a geometric interpretation of these manipulators.

Key words: parallel manipulators, Stewart Gough platforms, singularity surface, design

1 Introduction

A parallel manipulator of Stewart Gough type consists of two systems, namely the platform Σ and the base Σ_0, which are connected via six Spherical-Prismatic-Spherical (or Spherical-Prismatic-Universal) joints. The geometry of such a manipulator is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ and by the six platform anchor points $\mathsf{m}_i \in \Sigma$ with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$. By using Euler parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group $\mathsf{SO}(3)$, the coordinates $\mathbf{m}'_i$ of m_i with respect to the fixed space can be written as $\mathbf{m}'_i = K^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$

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with

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}, \quad (1)$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and the homogenizing factor $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

1.1 Review on the singularity analysis of SG platforms

It is well known (cf. Merlet [1]) that a Stewart Gough platform (SG platform) is singular if and only if the carrier lines of the prismatic legs belong to a linear line complex. This is the case if $Q := det(\mathbf{Q}) = 0$ holds, whereas the i^{th} row of the 6×6 matrix $\mathbf{Q}$ equals the Plücker coordinates $\underline{\mathbf{l}}_i := (\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}_i' - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the i^{th} carrier line.

Due to the complexity of Q for the general SG platform, early studies on singularities focused on special SG platforms, namely those of TSSM, SSM or MSSM architecture. Work on these manipulators were done by several researchers (e.g. [1–7]). Moreover based on Grassmann-Cayley algebra the geometric explanations for the singularities of 31 combinatorial classes of SG platforms were given by Ben-Horin and Shoham [8,9].

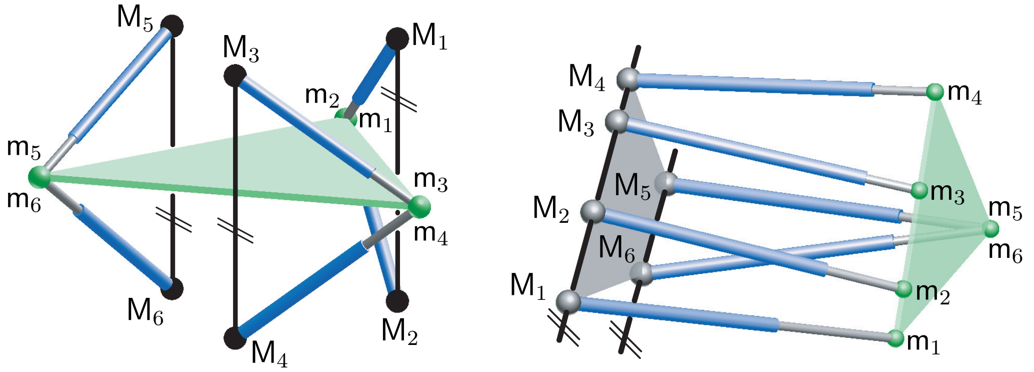
At the turn of the millennium the first analytic expressions of the singularity locus of the general SG platform for a fixed orientation was obtained (cf. [10,11]). Based on a cascaded expansion of the determinant of $\mathbf{Q}$ Li et al. [12] were able to derive the first analytic expression of Q explicitly. A modified version of this method was given in [13]. Moreover base on [12,13] it was possible to give algorithms for the computation of the maximal singularity-free sphere in the workspace of a general SG platform [14] and the maximal singularity-free workspace for a given orientation [15].

In this context it should be mentioned that the most efficient algorithm to check if a given workspace is singularity-free is based on interval analysis (cf. [16]) and does not use the analytic form of Q. Moreover note that in practice it is not only desirable to identify the singularity itself, because the loss of controllability and may very large forces/torques in the actuators can also appear in its vicinity. Therefore also different measures indicating the closeness of a given pose to the next singularity are used. Two such measures/indices and a detailed review on this topic was given by the author in [17].

Due to Li et al. [12] it is desirable for designers to have a graphical representation of the singularity set of SG platforms, because then it is easy to identify the singular locations within the given workspace and determining whether the singularities can be avoided.

As it is impossible to represent the singularity surface in 6-dimensional space graphically, only the visualization of 3-dimensional subspaces make sense. This can be done for a general SG platform according to [12] (for special designs see also [18,19]).

Usually one fixes the orientational part and visualizes the singularity surface which is in the general case a cubic surface in the translation parameters t_1, t_2, t_3. In this case the analytic expression of Q allows designers to visualize interactively the singularity locus on the given workspace for a fixed orientation. But the drawback of surfaces of degree 3 is that they can have very complicated shapes. Therefore it is desirable from the design point of view to handle only with SG platforms which have a simple singularity surface for any orientation.



(a) (b)

Fig. 1. The determination of the whole set of non-planar parallel manipulators with a cylindrical singularity surface is still an open problem, but there are good reasons to conjecture that this set consists of only one element, namely manipulator (*i*) without planar base given in (a). An example for type (*ii*) is illustrated in (b).

The best, one can think of in this context, are non-architecturally singular parallel manipulators those singularity set for any orientation of the platform is a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations. In this case the singularity set can easily be visualized as curve by choosing p as projection direction. In addition the computation of singularity-free zones reduces to a 5-dimensional task. In Nawratil [20] it was proven that there only exist two planar manipulator designs (see Fig. 1) with a cylindrical surface, namely:

(*i*) $m_1 = m_2$, $m_3 = m_4$, $m_5 = m_6$ and $[M_1, M_2] \parallel [M_3, M_4] \parallel [M_5, M_6] \parallel p$,

(*ii*) $[M_5, M_6] \parallel [M_1, \ldots, M_4] \parallel p$, $m_5 = m_6$, M_1, M_2, M_3, M_4 and m_1, m_2, m_3, m_4 are collinear.

Unfortunately, the set of SG platforms with a cylindrical singularity surface has only a very limited variety due to the above given conditions on the design parameters. Therefore one has to look for manipulators with other simple singularity surfaces. KARGER [21] suggested to use SG platforms with a quadratic singularity surface. These manipulators have the following advantages:

- All types of quadrics have well known and rather simple shapes.
- Due to the degree reduction it becomes easier to obtain closed form information about singular positions.
- The computations of singularity free zones in the space of translations (for a fixed orientation) reduces to the minimization of a quadratic function under a quadratic constraint.

Until now the following results about SG platforms with non-cubic singularity surface are known:

Planar case: KARGER [21] stated that the problem can be solved explicitly as a linear system without giving any details about the structure of the solution set. Moreover he proved that the singularity surface of manipulators with affine equivalent platform and base is a plane. NAWRATIL [22] showed that the set of Schönflies-singular SG platforms, whereas the axis of the Schönflies group is orthogonal to the carrier plane of the platform or base anchor points, also possesses this property. This is also true for the manipulators (*i*) and (*ii*) given above (cf. [20]).

Non-planar case: KARGER [21] showed that manipulators with affine equivalent base and platform have a non-cubic singularity surface. In addition, he remarked that these manipulators seem to be not the only possibility, but the general solution is a difficult task.

Nevertheless, in this article we present a surprising simple method how all manipulators with a non-cubic singularity surface can be determined and characterized. The geometric interpretation of the result shows that there do not exist further non-planar SG platforms with a quadratic singularity surface (beside those with affine equivalent platform and base) as conjectured by KARGER.

Moreover this article also contributes to the field of classifying SG platforms with respect to their geometrical structure (cf. [23]). In this context also the work of the Spanish research group around THOMAS should be mentioned which is also interested in the classification of SG platforms in families sharing the same singularity structure (cf. [24–26]).

In Section 2 we describe how the coefficients of Q are manipulated in order to get a simple set $\mathcal{C}$ of necessary and sufficient conditions for a non-cubic singularity surface. Based on this set $\mathcal{C}$ we prove in Section 4 the main theorem on SG platforms with a non-cubic singularity surface. The preparatory work for this main theorem (cf. Theorem 6) is done in Section 3. Moreover we also give a geometric interpretation of these manipulators in Section 5.

Without loss of generality (w.l.o.g.) we can choose special coordinate systems in the platform and the base such that $A_1 = B_1 = B_2 = C_1 = C_2 = C_3 = a_1 = b_1 = b_2 = c_1 = c_2 = c_3 = 0$ holds. Under consideration of this choice of coordinate systems we are able to give the set $\mathcal{C}$ of conditions explicitly, as we use the following compact notation which was introduced in [22]. This notation is similar to the one used in [12] to obtain the analytic expression of Q for the general SG platform: We denote the determinant of certain 5×5 matrices as follows:

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ab}| := det(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ab}) \quad \text{with} \tag{2}$$

$$\mathbf{a}=\begin{bmatrix}a_2\\ \vdots\\ a_6\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}b_2\\ \vdots\\ b_6\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}A_2\\ \vdots\\ A_6\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}B_2\\ \vdots\\ B_6\end{bmatrix},\ \mathbf{Ab}=\begin{bmatrix}A_2b_2\\ \vdots\\ A_6b_6\end{bmatrix}. \tag{3}$$

Moreover we define the following 10 vectors:

$$\mathbf{c}=\begin{bmatrix}c_2\\ \vdots\\ c_6\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}C_2\\ \vdots\\ C_6\end{bmatrix},\ \mathbf{Aa}=\begin{bmatrix}A_2a_2\\ \vdots\\ A_6a_6\end{bmatrix},\ \mathbf{Ac}=\begin{bmatrix}A_2c_2\\ \vdots\\ A_6c_6\end{bmatrix},\ \mathbf{Ba}=\begin{bmatrix}B_2a_2\\ \vdots\\ B_6a_6\end{bmatrix},$$

$$\mathbf{Bb}=\begin{bmatrix}B_2b_2\\ \vdots\\ B_6b_6\end{bmatrix},\ \mathbf{Bc}=\begin{bmatrix}B_2c_2\\ \vdots\\ B_6c_6\end{bmatrix},\ \mathbf{Ca}=\begin{bmatrix}C_2a_2\\ \vdots\\ C_6a_6\end{bmatrix},\ \mathbf{Cb}=\begin{bmatrix}C_2b_2\\ \vdots\\ C_6b_6\end{bmatrix},\ \mathbf{Cc}=\begin{bmatrix}C_2c_2\\ \vdots\\ C_6c_6\end{bmatrix}.$$

Two determinants are called *conjugated* if they result from each other by exchanging the platform and the base. For example the conjugate determinant to $|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Ab}|$ is given by $|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ba}|$. It should be noted that there also exist so-called *self-conjugated* determinants, e.g. $|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Bb}|$.

Moreover we denote the coefficients of $t_1^i t_2^j t_3^k$ of Q by Q^{ijk}. The coefficients of $e_0^a e_1^b e_2^c e_3^d$ of Q^{ijk} are denoted by Q^{ijk}_{abcd}.

2 Computing the set $\mathcal{C}$ of equations

We start by computing Q in dependency of t_1, t_2, t_3 and the r_{ij}'s of Eq. (1). Then we extract all 10 cubic coefficients:

$$Q^{300},\ Q^{030},\ Q^{003},\ Q^{210},\ Q^{201},\ Q^{120},\ Q^{021},\ Q^{102},\ Q^{012},\ Q^{111}. \quad (4)$$

At this stage we replace the r_{ij}'s by the Euler Parameters according to Eq. (1) and simplify the resulting expressions. We end up with 10 homogeneous equations $Q^{ijk} = 0$ of degree 4 in the unknowns e_0, e_1, e_2 and e_3. In the following we describe how the coefficients Q^{ijk}_{abcd} of these equations $Q^{ijk} = 0$ have been manipulated in order to get the simplest (shortest) conditions. The given procedure is the same for all 10 cubic coefficients Q^{ijk} of Eq. (4). We compute the following 28 linear combinations of 4 coefficients of Q^{ijk}:

$$Q^{ijk}_{4000} + Q^{ijk}_{0400} + Q^{ijk}_{0040} + Q^{ijk}_{0004} = 0 \quad Q^{ijk}_{4000} - Q^{ijk}_{0400} + Q^{ijk}_{0040} - Q^{ijk}_{0004} = 0 \quad (5)$$
$$Q^{ijk}_{4000} + Q^{ijk}_{0400} - Q^{ijk}_{0040} - Q^{ijk}_{0004} = 0 \quad Q^{ijk}_{4000} - Q^{ijk}_{0400} - Q^{ijk}_{0040} + Q^{ijk}_{0004} = 0 \quad (6)$$
$$Q^{ijk}_{3100} + Q^{ijk}_{1300} + Q^{ijk}_{0031} + Q^{ijk}_{0013} = 0 \quad Q^{ijk}_{1210} + Q^{ijk}_{0121} + Q^{ijk}_{2101} + Q^{ijk}_{1012} = 0 \quad (7)$$
$$Q^{ijk}_{3100} + Q^{ijk}_{1300} - Q^{ijk}_{0031} - Q^{ijk}_{0013} = 0 \quad Q^{ijk}_{1210} + Q^{ijk}_{0121} - Q^{ijk}_{2101} - Q^{ijk}_{1012} = 0 \quad (8)$$
$$Q^{ijk}_{3100} - Q^{ijk}_{1300} + Q^{ijk}_{0031} - Q^{ijk}_{0013} = 0 \quad Q^{ijk}_{1210} - Q^{ijk}_{0121} + Q^{ijk}_{2101} - Q^{ijk}_{1012} = 0 \quad (9)$$
$$Q^{ijk}_{3100} - Q^{ijk}_{1300} - Q^{ijk}_{0031} + Q^{ijk}_{0013} = 0 \quad Q^{ijk}_{1210} - Q^{ijk}_{0121} - Q^{ijk}_{2101} + Q^{ijk}_{1012} = 0 \quad (10)$$
$$Q^{ijk}_{3010} + Q^{ijk}_{0301} + Q^{ijk}_{1030} + Q^{ijk}_{0103} = 0 \quad Q^{ijk}_{2110} + Q^{ijk}_{0112} + Q^{ijk}_{1201} + Q^{ijk}_{1021} = 0 \quad (11)$$
$$Q^{ijk}_{3010} + Q^{ijk}_{0301} - Q^{ijk}_{1030} - Q^{ijk}_{0103} = 0 \quad Q^{ijk}_{2110} + Q^{ijk}_{0112} - Q^{ijk}_{1201} - Q^{ijk}_{1021} = 0 \quad (12)$$
$$Q^{ijk}_{3010} - Q^{ijk}_{0301} + Q^{ijk}_{1030} - Q^{ijk}_{0103} = 0 \quad Q^{ijk}_{2110} - Q^{ijk}_{0112} + Q^{ijk}_{1201} - Q^{ijk}_{1021} = 0 \quad (13)$$
$$Q^{ijk}_{3010} - Q^{ijk}_{0301} - Q^{ijk}_{1030} + Q^{ijk}_{0103} = 0 \quad Q^{ijk}_{2110} - Q^{ijk}_{0112} - Q^{ijk}_{1201} + Q^{ijk}_{1021} = 0 \quad (14)$$
$$Q^{ijk}_{3001} + Q^{ijk}_{0310} + Q^{ijk}_{0130} + Q^{ijk}_{1003} = 0 \quad Q^{ijk}_{2011} + Q^{ijk}_{0211} + Q^{ijk}_{1120} + Q^{ijk}_{1102} = 0 \quad (15)$$
$$Q^{ijk}_{3001} + Q^{ijk}_{0310} - Q^{ijk}_{0130} - Q^{ijk}_{1003} = 0 \quad Q^{ijk}_{2011} + Q^{ijk}_{0211} - Q^{ijk}_{1120} - Q^{ijk}_{1102} = 0 \quad (16)$$
$$Q^{ijk}_{3001} - Q^{ijk}_{0310} + Q^{ijk}_{0130} - Q^{ijk}_{1003} = 0 \quad Q^{ijk}_{2011} - Q^{ijk}_{0211} + Q^{ijk}_{1120} - Q^{ijk}_{1102} = 0 \quad (17)$$
$$Q^{ijk}_{3001} - Q^{ijk}_{0310} - Q^{ijk}_{0130} + Q^{ijk}_{1003} = 0 \quad Q^{ijk}_{2011} - Q^{ijk}_{0211} - Q^{ijk}_{1120} + Q^{ijk}_{1102} = 0 \quad (18)$$

In addition we compute the following 6 linear combinations of 2 coefficients of Q^{ijk}:

$$Q^{ijk}_{2200} + Q^{ijk}_{0022} = 0 \qquad Q^{ijk}_{2020} + Q^{ijk}_{0202} = 0 \qquad Q^{ijk}_{2002} + Q^{ijk}_{0220} = 0 \qquad (19)$$
$$Q^{ijk}_{2200} - Q^{ijk}_{0022} = 0 \qquad Q^{ijk}_{2020} - Q^{ijk}_{0202} = 0 \qquad Q^{ijk}_{2002} - Q^{ijk}_{0220} = 0 \qquad (20)$$

The set $\mathcal{C}^{ijk}$ of equations is completed by $Q^{ijk}_{1111} = 0$. As the 34 linear combinations of Eqs. (5) – (20) are linearly independent the vanishing of $\mathcal{C}^{ijk}$ equals the vanishing of all 35 coefficients of Q^{ijk}.

If we compute these 35 conditions for each of the 10 cubic coefficients Q^{ijk} of Eq. (4), we recognize that 75 conditions are fulfilled identically. Moreover the remaining 275 conditions can be written as linear combinations of 5×5 determinants of the form:

$$|\mathbf{u}, \mathbf{v}, \mathbf{X}, \mathbf{Y}, \mathbf{Zw}| \quad \text{with} \quad \mathbf{u}, \mathbf{v} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}, \mathbf{X}, \mathbf{Y} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}, \\ \text{and} \quad \mathbf{Zw} \in \{\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}, \mathbf{Ca}, \mathbf{Cb}, \mathbf{Cc}\} \tag{21}$$

according to the notation introduced in Section 1.3. This follows from the linear decomposition of the Jacobian's determinant (cf. Li et al. [12]).

Moreover one can find the following 40 conditions within the remaining 275 conditions, whereas the following four are self-conjugated:

$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Aa}| = 0 \qquad |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}| = 0 \tag{22}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Aa}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Cc}| = 0 \tag{23}$$

The remaining 36 conditions are given in 18 pairs of conjugated conditions:

$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Ab}| = 0 \qquad |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}| = 0 \tag{24}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{C}, \mathbf{Aa}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Aa}| = 0 \tag{25}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{C}, \mathbf{Ab}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}| = 0 \tag{26}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{C}, \mathbf{Ca}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Ac}| = 0 \tag{27}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{C}, \mathbf{Cb}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Bc}| = 0 \tag{28}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Ba}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Ab}| = 0 \tag{29}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Bb}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}| = 0 \tag{30}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Bc}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Cb}| = 0 \tag{31}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Ca}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Ac}| = 0 \tag{32}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Cb}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Bc}| = 0 \tag{33}$$
$$|\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{C}, \mathbf{Cc}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{Cc}| = 0 \tag{34}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Ac}| = 0 \qquad |\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Ca}| = 0 \tag{35}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Ba}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Ab}| = 0 \tag{36}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Bb}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Bb}| = 0 \tag{37}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Bc}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Cb}| = 0 \tag{38}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Ca}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Ac}| = 0 \tag{39}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Cb}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Bc}| = 0 \tag{40}$$
$$|\mathbf{a}, \mathbf{c}, \mathbf{B}, \mathbf{C}, \mathbf{Cc}| = 0 \qquad |\mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{C}, \mathbf{Cc}| = 0 \tag{41}$$

If we set these 40 determinants equal to zero we see that our system of equations reduces to 102 conditions, which are linear combinations of further 33 determinants of the form given in Eq. (21). If we simplify the ideal spanned by these 102 equations, we end up with 20 equations which can be broken up into nine independent systems. Eight systems consist of two equations in 3 determinants. In addition these 8 systems can be grouped into 4 pairs of conjugate systems. In the following these 4 pairs are given, whereas conjugate conditions are again written in the same line:

$$|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ab}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Ac}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ba}|+|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ca}|=0 \tag{42}$$

$$|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ab}|+|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Aa}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ba}|+|\mathbf{a},\mathbf{c},\mathbf{B},\mathbf{C},\mathbf{Aa}|=0 \tag{43}$$

$$|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Cb}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Cc}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Bc}|+|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Cc}|=0 \tag{44}$$

$$|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Cb}|+|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ca}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Bc}|+|\mathbf{a},\mathbf{c},\mathbf{B},\mathbf{C},\mathbf{Ac}|=0 \tag{45}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Ba}|+|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{C},\mathbf{Aa}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ab}|+|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Aa}|=0 \tag{46}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Ba}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ca}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ab}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Ac}|=0 \tag{47}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bb}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Cb}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Bb}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Bc}|=0 \tag{48}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bb}|+|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{C},\mathbf{Ab}|=0 \quad |\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Bb}|+|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ba}|=0 \tag{49}$$

The ninth system consists of the following 4 equations, whereas the last one is self-conjugated:

$$|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{C},\mathbf{Ac}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bc}|-|\mathbf{b},\mathbf{c},\mathbf{B},\mathbf{C},\mathbf{Aa}|-|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ba}|=0 \tag{50}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Cc}|+|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bc}|-|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ca}|-|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ba}|=0 \tag{51}$$

$$|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bc}|+|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{C},\mathbf{Ac}|+|\mathbf{a},\mathbf{c},\mathbf{B},\mathbf{C},\mathbf{Ab}|+|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Bb}|=0 \tag{52}$$

$$\begin{aligned}&|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{C},\mathbf{Ba}|+|\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Ca}|+|\mathbf{a},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{Cb}|-\\&|\mathbf{a},\mathbf{c},\mathbf{B},\mathbf{C},\mathbf{Ab}|-|\mathbf{a},\mathbf{b},\mathbf{B},\mathbf{C},\mathbf{Ac}|-|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{C},\mathbf{Bc}|=0\end{aligned} \tag{53}$$

Moreover the conjugated conditions to the other 3 equations of this system can be obtained by adding the respective equation to the self-conjugated one.

Therefore we reduced the system of 275 equations to the above given set $\mathcal{C}$ of 60 equations. In the following we refer to these equations by giving the respective number plus the letter l (for $l$eft) or r (for $r$ight); e.g. by Eq. (22l) the equation $|\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Aa}|=0$ is meant.

3 Preparatory work for the main theorem

Under consideration of the special coordinate systems introduced in Section 1 the following lemmata and theorems of Section 3 and 4 hold:

Lemma 1 *W.l.o.g. we can assume for a non-architecturally singular SG platform that* $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 1$ *and* $rk(\mathbf{A}, \mathbf{B}, \mathbf{C}) > 1$ *hold. Moreover we can assume that* $\mathbf{c}$ *resp.* $\mathbf{C}$ *equals the zero vector if* $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 2$ *resp.* $rk(\mathbf{A}, \mathbf{B}, \mathbf{C}) = 2$ *holds. Therefore we can assume* $rk(\mathbf{a}, \mathbf{b}) = rk(\mathbf{A}, \mathbf{B}) = 2$.

Proof: If $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}) < 2$ resp. $rk(\mathbf{A}, \mathbf{B}, \mathbf{C}) < 2$ holds all platform resp. base anchor points are located on a line or collapse into one point. Such manipulators are architecturally singular.

The condition $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 2$ resp. $rk(\mathbf{A}, \mathbf{B}, \mathbf{C}) = 2$ implies the planarity of the platform resp. base. For such a manipulator with planar platform and/or base we can assume that the anchor points are located in the xy-plane of the coordinate system of the moving and/or the fixed frame. □

W.l.o.g. we can assume for the rest of this article that Lemma 1 holds.

Theorem 1 *The platform and the base of a manipulator can be scaled independently without loosing the property of possessing a non-cubic singularity surface. Therefore this property is invariant with respect to Euclidean similarity transformations.*

Proof: The proof of Theorem 1 follows immediately from the fact that all 60 conditions of $\mathcal{C}$ can be written as linear-combinations of determinants of the structure given in Eq. (21). □

Theorem 2 *There does not exist any SG platform possessing a non-cubic singularity surface with* $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 4$.

Proof: In the following we prove that the 60 conditions of the set $\mathcal{C}$ plus the assumption $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 4$ imply

$$rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}, \mathbf{Ca}, \mathbf{Cb}, \mathbf{Cc}) = 4. \tag{54}$$

It was shown by NAWRATIL [27] that this rank condition is sufficient (but not necessary) for a manipulator to be architecturally singular.

1. $rk(\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}) = 2$: As a consequence $\mathbf{a}$ and $\mathbf{b}$ can be written as a linear combination of $\mathbf{A}$ and $\mathbf{B}$, i.e.

$$\mathbf{a} = \lambda_a \mathbf{A} + \mu_a \mathbf{B}, \qquad \mathbf{b} = \lambda_b \mathbf{A} + \mu_b \mathbf{B}. \tag{55}$$

Moreover $rk(\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{C})=4$ has to hold. Therefore we have to show that $\mathbf{Aa},\ldots,\mathbf{Cc}$ can be written as linear combinations of $\mathbf{c},\mathbf{A},\mathbf{B}$ and $\mathbf{C}$. If one views the equations in the correct order, one can see that this is the case. Table 1 contains the respective numbers of equations, whereas the order of these equations is given by the indices. Depending on the two cases which have to be distinguished, we need 2 resp. 3 steps:

Tab. 1	**Aa**	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
$\lambda_a\mu_b\neq 0$	$(43l)_2$	$(36r)_1$	$(39r)_1$	$(26l)_1$	$(37l)_1$	$(38l)_1$	$(39l)_1$	$(38r)_1$	$(41l)_1$
$\mu_a\lambda_b\neq 0$	$(23l)_1$	$(42l)_2$	$(35l)_1$	$(42r)_2$	$(52)_3$	$(44r)_2$	$(35r)_1$	$(44l)_2$	$(23r)_1$

Table 1
This table has to be read as follows: E.g. the vector **Aa** has to be a linear combinations of the vectors $\mathbf{c},\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ due to Eq. (23l) for the case $\mu_a\lambda_b\neq 0$.

2. $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B})=4$: In this case we distinguish the following three subcases:
   a. $rk(\mathbf{a},\mathbf{b},\mathbf{c})=rk(\mathbf{A},\mathbf{B},\mathbf{C})=2$: From Lemma 1 we get $\mathbf{C}=\mathbf{c}=\mathbf{0}$, i.e. a planar SG platform. In this case we are left with the 4 conditions Eqs. (22) and (24). As $\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B}$ are linearly independent, these 4 conditions imply
   $$rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Aa},\mathbf{Ab},\mathbf{Ba},\mathbf{Bb})=4 \qquad (56)$$
   which is the planar version of Eq. (54) given by Röschel and Mick [28].
   b. $rk(\mathbf{a},\mathbf{b},\mathbf{c})=2$ and $rk(\mathbf{A},\mathbf{B},\mathbf{C})=3$: Due to Lemma 1 we can assume $\mathbf{c}=\mathbf{0}$. Moreover we can set
   $$\mathbf{C}=\lambda_C\mathbf{a}+\mu_C\mathbf{b}+\rho_C\mathbf{A}+\tau_C\mathbf{B}. \qquad (57)$$
   Due to Eqs. (22) and (24), the rank condition of Eq. (56) is also valid in this case. As $\mathbf{Ac}=\mathbf{Bc}=\mathbf{Cc}=\mathbf{0}$ holds, we only have to show that $\mathbf{Ca}$ and $\mathbf{Cb}$ are linear combinations of $\mathbf{a},\mathbf{b},\mathbf{A}$ and $\mathbf{B}$. This follows directly from Eq. (47l) and Eq. (48l), respectively.
   c. $rk(\mathbf{a},\mathbf{b},\mathbf{c})=rk(\mathbf{A},\mathbf{B},\mathbf{C})=3$: In this case we get Eq. (57) and
   $$\mathbf{c}=\lambda_c\mathbf{a}+\mu_c\mathbf{b}+\rho_c\mathbf{A}+\tau_c\mathbf{B}. \qquad (58)$$
   Clearly, Eq. (56) holds again due to Eqs. (22) and (24). $\mathbf{Ac}$ and $\mathbf{Ca}$ resp. $\mathbf{Bc}$ and $\mathbf{Cb}$ are linear combinations of $\mathbf{a},\mathbf{b},\mathbf{A}$ and $\mathbf{B}$ due to the equations given in Eq. (47) resp. Eq. (48). Finally $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{Cc})=4$ follows from Eq. (51) which implies Eq. (54).
3. $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B})=3$: W.l.o.g. we can assume that $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{C})=4$ holds. As non of the vectors $\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B}$ equal the zero vector and due to $rk(\mathbf{a},\mathbf{b})=rk(\mathbf{A},\mathbf{B})=2$ (cf. Lemma 1), we can always assume w.l.o.g. that $rk(\mathbf{j},\mathbf{A},\mathbf{B},\mathbf{C})=4$ and
   $$\mathbf{i}=\lambda_i\mathbf{j}+\rho_i\mathbf{A}+\tau_i\mathbf{B}, \quad \mathbf{c}=\lambda_c\mathbf{j}+\rho_c\mathbf{A}+\tau_c\mathbf{B}+\mu_c\mathbf{C} \qquad (59)$$

with $\mathbf{i}, \mathbf{j} \in \{\mathbf{a}, \mathbf{b}\}$ and $\mathbf{i} \neq \mathbf{j}$ hold. Therefore we have to show that $\mathbf{Aa}, \ldots, \mathbf{Cc}$ can be written as linear combinations of $\mathbf{j}, \mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. In Table 2 the respective numbers of equations are given which yield this fact, whereas we have to distinguish two cases. Moreover Table 2 finishes the proof of Theorem 2. □

Tab. 2	**Aa**	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
$\tau_i \neq 0$	$(25l)_1$	$(26l)_1$	$(42l)_2$	$(46l)_2$	$(48l)_2$	$(52)_3$	$(27l)_1$	$(28l)_1$	$(44l)_2$
$\rho_i \neq 0$	$(46l)_2$	$(49l)_2$	$(52)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$

Table 2
This table has to be read as Table 1.

Now the question remains open, if there exist non-architecturally singular manipulators with $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$ possessing a non-cubic singularity surface. In order to reduce the number of cases which have to be discussed for the answering of this question, we specify the coordinate systems in the platform and the base in the following way:

Lemma 2 *W.l.o.g. we can specify the special coordinate systems introduced in Section 1 such that $a_2 = A_2 = 1$ holds.*

Proof: We only have to prove that any non-architecturally singular manipulator possesses at least one pair of anchor points $(\mathsf{m}_i, \mathsf{m}_j)$ and $(\mathsf{M}_i, \mathsf{M}_j)$, whereas neither m_i and m_j nor M_i and M_j coincide; i.e. $\mathsf{m}_i \neq \mathsf{m}_j \wedge \mathsf{M}_i \neq \mathsf{M}_j$. This can be done as follows: As not all platform points can collapse into a point there exist two different platform anchor points m_i and m_j. If the corresponding base anchor points do not coincide we are done. In the other case there has to exist another base anchor point M_k which differs from $\mathsf{M}_i = \mathsf{M}_j$. In the worst case m_k coincides with m_i but then $(\mathsf{m}_j, \mathsf{m}_k)$ and $(\mathsf{M}_j, \mathsf{M}_k)$ is this point pair.

Now we can renumerate the platform and base anchor points in such a way that this point pair has the indices 1 and 2 and therefore $A_2 a_2 \neq 0$ holds. W.l.o.g. we can set $A_2 = a_2 = 1$ due to Theorem 1. □

Lemma 3 *A manipulator with non-planar platform and base ($\mathbf{C} \neq \mathbf{0} \wedge \mathbf{c} \neq \mathbf{0}$) and $\mathbf{Bb} = \mathbf{Bc} = \mathbf{Cb} = \mathbf{Cc} = \mathbf{0}$ possesses 4 collinear anchor points.*

Proof: As $\mathbf{c} \neq \mathbf{0}$ holds, we can assume $c_6 \neq 0$. As a consequence we get $B_6 = C_6 = 0$ due to $\mathbf{Bc} = \mathbf{Cc} = \mathbf{0}$. Moreover $\mathbf{C} \neq \mathbf{0}$ holds and therefore we can assume $C_5 \neq 0$ which implies $b_5 = c_5 = 0$. We are left with the equations $B_4 b_4 = B_4 c_4 = C_4 b_4 = C_4 c_4 = 0$, which imply $C_4 = B_4 = 0$ or $c_4 = b_4 = 0$. In both cases 4 anchor points are collinear $(\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_4, \mathsf{M}_6$ or $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_4, \mathsf{m}_5)$. □

Theorem 3 *All parallel manipulators of SG type with $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$ and $rk(\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}) = 3$, possessing a non-cubic singularity surface, have 4* *collinear anchor points.*

Proof: As the two rank conditions $rk(\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}) = 3$ and $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$ imply a non-planar platform and base, we only have to prove that $\mathbf{Bb} = \mathbf{Bc} = \mathbf{Cb} = \mathbf{Cc} = \mathbf{0}$ holds (cf. Lemma 3). Due to Lemma 2 the following five cases have to be distinguished:

1. $\mathbf{b} = \lambda_b(\mathbf{a} - \mathbf{A}) + \mu_b\mathbf{B}$ with $\lambda_b\mu_b \neq 0$: As $rk(\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$ holds, the vectors $\mathbf{Aa}, \ldots, \mathbf{Cc}$ can be written as linear combinations of $\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ or they equal the zero vector. If one views the equations in the correct order, one can see that $\mathbf{Bb} = \mathbf{Bc} = \mathbf{Cb} = \mathbf{Cc} = \mathbf{0}$ holds. The equations (and their order) which yield this result are given in Table 3:

Tab. 3	**Bb**	**Bc**	**Cb**	**Cc**
a	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
c	$(30l)_1$	$(31l)_1$	$(33l)_1$	$(34l)_1$
A	$(37l)_1$	$(38l)_1$	$(40l)_1$	$(41l)_1$
B	$(37r)_3$	$(40r)_3$	$(38r)_3$	$(23r)_1$
C	$(30r)_1$	$(28r)_1$	$(31r)_1$	$(34r)_1$

Table 3

This table has to be read as follows: E.g. the vector **Bb** has to be independent of **C** due to Eq. $(30r)$, because otherwise the vanishing of this equation would imply $rk(\mathbf{a}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) < 5$, which is a contradiction. The abbreviation $(L.2)$ means that e.g. the vector **Bb** has to be independent of **a**, because otherwise a_2 is equal to zero but this contradicts Lemma 2. Moreover the correct order of the equations is given by the indices.

2. $\mathbf{b} = \lambda_b(\mathbf{a} - \mathbf{A})$ with $\lambda_b \neq 0$: For this case we also refer to Table 3.
3. $\mathbf{b} = \mu_b\mathbf{B}$ with $\mu_b \neq 0$: This case is more complicated. For its solution we refer to Table 4.

Tab. 4	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(36r)_1$	$(39r)_1$	$(L.2)_2$	$(37r)_1$	$(40r)_1$	$(L.2)_2$	$(38r)_1$	$(41r)_1$
c	$(26l)_1$		$(46l)_1$	$(48l)_1$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_4$
A	$(L.2)_2$	$(L.2)_2$	$(36l)_1$	$(37l)_1$	$(38l)_1$	$(39l)_1$	$(40l)_1$	$(41l)_1$
B		$(35l)_1$		$(52)_5$	$(45r)_3$	$(35r)_1$	$(45l)_3$	$(23r)_1$
C	$(46r)_1$	$(27r)_1$	$(26r)_1$	$(48r)_1$	$(28r)_1$		$(53)_5$	$(44r)_4$

Table 4

This table has to be read as Table 3.

4. $\mathbf{a} = \lambda_a\mathbf{A} + \mu_a\mathbf{B}$ with $\lambda_a\mu_a \neq 0$: For this case we refer to Table 5.
5. $\mathbf{a} = \lambda_a\mathbf{A}$ with $\lambda_a \neq 0$: For this case we also refer to Table 5. □

Tab. 5	**Bb**	**Bc**	**Cb**	**Cc**
b	$(37l)_1$	$(38l)_1$	$(40l)_1$	$(41l)_1$
c	$(30l)_1$	$(31l)_1$	$(33l)_1$	$(34l)_1$
A	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
B	$(37r)_1$	$(40r)_1$	$(38r)_1$	$(41r)_1$
C	$(30r)_1$	$(33r)_1$	$(31r)_1$	$(34r)_1$

Table 5
This table has to be read as Table 3.

Theorem 4 *All parallel manipulators of SG type with non-planar platform and base, $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B}) = 4$ and $rk(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{A},\mathbf{B},\mathbf{C}) = 5$, possessing a non-cubic singularity surface, have 4 collinear anchor points.*

Proof: W.l.o.g. we can assume $rk(\mathbf{a},\mathbf{b},\mathbf{A},\mathbf{B},\mathbf{C}) = 5$. Due to Lemma 3 we only have to show that $\mathbf{Bb} = \mathbf{Bc} = \mathbf{Cb} = \mathbf{Cc} = \mathbf{0}$ holds.

1. $\mathbf{c} = \lambda_c(\mathbf{a} - \mathbf{A}) + \mu_c\mathbf{b} + \beta_c\mathbf{B} + \gamma_c\mathbf{C}$ with $\lambda_c \neq 0$: For this case we refer to Table 6.

Tab. 6	**Bb**	**Bc**	**Cb**	**Cc**
a	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b	$(37l)_3$	$(38l)_3$	$(40l)_3$	$(41l)_3$
A	$(30l)_1$	$(31l)_1$	$(33l)_1$	$(34l)_1$
B	$(37r)_4$	$(40r)_4$	$(28l)_1$	$(41r)_4$
C	$(22r)_1$	$(48r)_5$	$(31r)_4$	$(34r)_5$

Table 6
This table has to be read as Table 3.

2. $\mathbf{c} = \mu_c\mathbf{b} + \beta_c\mathbf{B}$ with $\mu_c\beta_c \neq 0$: For this case we refer to Table 7.

Tab. 7	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(36r)_2$	$(39r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b				$(52)_5$	$(45r)_5$	$(35r)_3$	$(45l)_4$	$(23r)_6$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$			$(49l)_4$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_5$
C	$(24l)_1$	$(27r)_2$	$(24r)_1$	$(22r)_1$	$(28r)_2$		$(48l)_5$	$(51)_6$

Table 7
This table has to be read as Table 3.

Tab. 8	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(36r)_2$	$(39r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b				$(49r)_3$	$(28r)_5$	$(35r)_3$	$(45l)_4$	$(23r)_6$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$			$(49l)_4$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_5$
C	$(24l)_1$		$(24r)_1$	$(22r)_1$	$(48r)_4$		$(48l)_5$	$(51)_6$

Table 8
This table has to be read as Table 3.

3. $\mathbf{c} = \mu_c\mathbf{b} + \beta_c\mathbf{B} + \gamma_c\mathbf{C}$ with $\mu_c\beta_c\gamma_c \neq 0$: For this case we refer to Table 8.
4. $\mathbf{c} = \mu_c\mathbf{b} + \gamma_c\mathbf{C}$ with $\mu_c\gamma_c \neq 0$: For this case we refer to Table 9.

Tab. 9	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(29r)_2$	$(32r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b			$(26r)_3$	$(49r)_4$	$(28r)_5$		$(53)_6$	$(44r)_6$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$			$(49l)_4$	$(45r)_4$	$(27l)_1$	$(28l)_1$	$(23r)_3$
C	$(24l)_1$		$(24r)_1$	$(22r)_1$	$(48r)_5$		$(48l)_5$	$(51)_5$

Table 9
This table has to be read as Table 3.

5. $\mathbf{c} = \gamma_c\mathbf{C}$ with $\gamma_c \neq 0$: For this case we refer to Table 10.

Tab. 10	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(29r)_2$	$(32r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b		$(27r)_2$	$(26r)_2$	$(49r)_3$	$(28r)_2$	$(42r)_2$	$(53)_6$	$(44r)_2$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$	$(42l)_2$		$(49l)_3$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_2$
C	$(24l)_1$		$(24r)_1$	$(22r)_1$	$(48r)_4$		$(48l)_4$	$(51)_5$

Table 10
This table has to be read as Table 3.

6. $\mathbf{c} = \beta_c\mathbf{B} + \gamma_c\mathbf{C}$ with $\beta_c\gamma_c \neq 0$: For this case we refer to Table 11.
7. $\mathbf{c} = \beta_c\mathbf{B}$ with $\beta_c \neq 0$: For this case we refer to Table 12. Moreover this case also finishes the proof of Theorem 4. □

In the following we discuss the remaining special case:

Theorem 5 *All parallel manipulators of SG type with planar platform* ($\Leftrightarrow$ $\mathbf{c} = \mathbf{0}$) *and* $rk(\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$, *possessing a non-cubic singularity surface, have 4 collinear anchor points.*

Tab. 11	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(29r)_2$	$(32r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b		$(27r)_2$	$(26r)_2$	$(49r)_3$	$(28r)_2$	$(35r)_2$	$(45l)_3$	$(23r)_2$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$			$(49l)_4$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_4$
C	$(24l)_1$		$(24r)_1$	$(22r)_1$	$(48r)_4$		$(48l)_5$	$(51)_5$

Table 11
This table has to be read as Table 3.

Tab. 12	**Ab**	**Ac**	**Ba**	**Bb**	**Bc**	**Ca**	**Cb**	**Cc**
a	$(36r)_2$	$(39r)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b		$(35l)_2$	$(42r)_2$	$(52)_5$	$(44r)_2$	$(35r)_2$	$(45l)_3$	$(23r)_2$
A	$(L.2)_3$	$(L.2)_3$	$(29l)_1$	$(30l)_1$	$(31l)_1$	$(32l)_1$	$(33l)_1$	$(34l)_1$
B	$(26l)_1$			$(49l)_4$	$(50)_4$	$(27l)_1$	$(28l)_1$	$(44l)_4$
C	$(24l)_1$	$(47r)_2$	$(24r)_1$	$(22r)_1$	$(48r)_2$		$(48l)_5$	$(51)_5$

Table 12
This table has to be read as Table 3.

Proof: The proof is again done by contradiction. Due to Table 13 we get:

Tab. 13	**Aa**	**Ab**	**Ba**	**Bb**	**Ca**	**Cb**
a			$(L.2)_2$	$(L.2)_2$	$(L.2)_2$	$(L.2)_2$
b						
A			$(29l)_1$	$(30l)_1$	$(32l)_1$	$(33l)_1$
B	$(25l)_1$	$(26l)_1$			$(27l)_1$	$(28l)_1$
C	$(22l)_1$	$(24l)_1$	$(24r)_1$	$(22r)_1$		

Table 13
This table has to be read as Table 3.

$$\begin{aligned} \mathbf{Aa} &= \lambda_{Aa}\mathbf{a} + \mu_{Aa}\mathbf{b} + \alpha_{Aa}\mathbf{A}, \quad \mathbf{Ba} = \mu_{Ba}\mathbf{b} + \beta_{Ba}\mathbf{B}, \quad \mathbf{Ca} = \mu_{Ca}\mathbf{b} + \gamma_{Ca}\mathbf{C}, \\ \mathbf{Ab} &= \lambda_{Ab}\mathbf{a} + \mu_{Ab}\mathbf{b} + \alpha_{Ab}\mathbf{A}, \quad \mathbf{Bb} = \mu_{Bb}\mathbf{b} + \beta_{Bb}\mathbf{B}, \quad \mathbf{Cb} = \mu_{Cb}\mathbf{b} + \gamma_{Cb}\mathbf{C}, \end{aligned}$$

with $\lambda_{Aa}+\alpha_{Aa} = 1$ and $\lambda_{Ab}+\alpha_{Ab} = 0$. Moreover the four equations Eqs. ($46l$), ($47l$), ($48l$) and ($49l$) remain, which imply

$$X := \alpha_{Aa} = \beta_{Ba} = \gamma_{Ca} \quad \text{and} \quad Y := \alpha_{Ab} = \beta_{Bb} = \gamma_{Cb}. \tag{60}$$

Therefore we get:

$$\begin{aligned} \mathbf{Aa} &= (1-X)\mathbf{a} + \mu_{Aa}\mathbf{b} + X\mathbf{A}, & \mathbf{Ba} &= \mu_{Ba}\mathbf{b} + X\mathbf{B}, & \mathbf{Ca} &= \mu_{Ca}\mathbf{b} + X\mathbf{C}, \\ \mathbf{Ab} &= \quad -Y\mathbf{a} + \mu_{Ab}\mathbf{b} + Y\mathbf{A}, & \mathbf{Bb} &= \mu_{Bb}\mathbf{b} + Y\mathbf{B}, & \mathbf{Cb} &= \mu_{Cb}\mathbf{b} + Y\mathbf{C}. \end{aligned}$$

Due to **Ca** and **Cb** we have to distinguish the following two cases:

1. $b_3 = 0$: We get immediately $0 = YB_3$ and $B_3a_3 = XB_3$. We distinguish two cases:
   a. $B_3 \neq 0$: Therefore we get $Y = 0$ which implies $\mathbf{Ab} = \mu_{Ab}\mathbf{b}$, $\mathbf{Bb} = \mu_{Bb}\mathbf{b}$ and $\mathbf{Cb} = \mu_{Cb}\mathbf{b}$. As $b_4b_5b_6 \neq 0$ has to hold (otherwise 4 platform anchor points are collinear) the above expressions can only vanish for $\mu_{Ab} = A_4 = A_5 = A_6$, $\mu_{Bb} = B_4 = B_5 = B_6$ and $\mu_{Cb} = C_4 = C_5 = C_6$. This is a contradiction as $\mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6$ holds.
   b. $B_3 = 0$: In this case we consider the expression for **Ab** which yields $Ya_3 = YA_3$. As for $Y = 0$ we get a special case of item 1a, we set $a_3 = A_3$. Now the expression for **Aa** implies $a_3 = 1$, which is a contradiction as the second and third leg coincide.
2. $b_3 \neq 0$: Therefore $\mu_{Ca} = \mu_{Cb} = 0$ holds. As the base is non-planar we can assume w.l.o.g. that $C_6 \neq 0$. Then we get immediately $a_6 = X$ and $b_6 = Y$ from **Ca** and **Cb**, respectively. Moreover the formulas for **Ba** and **Bb** imply $\mu_{Ba}b_6 = 0$ and $\mu_{Bb}b_6 = 0$, respectively. Now we distinguish the following two cases:
   a. $b_6 = 0$: Therefore we get $C_4 = C_5 = 0$ from $\mathbf{Cb} = \mathbf{0}$, because otherwise 4 platform anchor points are already collinear. As $b_3b_4b_5 \neq 0$ has to hold, the conditions $\mathbf{Bb} = \mu_{Ba}\mathbf{b}$ and $\mathbf{Ab} = \mu_{Ab}\mathbf{b}$ can only be fulfilled for $\mu_{Ba} = B_3 = B_4 = B_5$ and $\mu_{Ab} = A_3 = A_4 = A_5$. This already implies $\mathsf{M}_3 = \mathsf{M}_4 = \mathsf{M}_5$, a contradiction.
   b. $b_6 \neq 0$: Now $\mu_{Ba} = \mu_{Bb} = 0$ holds and we end up with $\mathbf{Bb} = b_6\mathbf{B}$ and $\mathbf{Cb} = b_6\mathbf{C}$. The respective equations imply $b_i = b_6$ or $B_i = C_i = 0$ for $i = 2, \ldots, 5$. For all possible combinatorial cases we get 4 collinear points either in the platform or in the base. □

4 The main theorem

Theorem 6 (Main Theorem)
A non-architecturally singular SG platform possesses a non-cubic singularity surface if and only if $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) < 4$ *holds.*

Proof: The proof of the sufficiency follows immediately from the fact that all 60 conditions can be written as linear-combinations of determinants of the structure given in Eq. (21).

The proof for the necessity is done as follows: Due to Theorems 2 to 5 of Section 3 we only have to show that all manipulators with 4 collinear anchor points and $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) = 5$ fulfilling the 60 equations of the set $\mathcal{C}$ are architecturally singular. Moreover due to Corollary 1 of Nawratil [27] we can only get the five types of architecturally singular manipulators those legs belong in each configuration to a singular linear line complex.

The proof is done by contradiction. W.l.o.g. we can assume $\mathsf{m}_1, \ldots, \mathsf{m}_4$ collinear, i.e. $b_3 = b_4 = c_4 = c_5 = 0$. We split the proof up into the following cases, whereas the case study breaks down if we get $rk(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{A}, \mathbf{B}, \mathbf{C}) < 5$.

We consider the Eqs. (31l), (34l), (37l) and (40l) which yield

$$FB_6 = 0, \quad FC_6 = 0, \quad FB_5 = 0, \quad FC_5 = 0, \tag{61}$$

with $F := b_5 c_6 B_3 C_4$. Therefore we distinguish the following cases:

4.1 5 platform anchor points are collinear

W.l.o.g. we can assume that $\mathsf{m}_1, \ldots, \mathsf{m}_5$ are collinear. Therefore we set $b_5 = c_6 = 0$. Moreover we can assume $b_6 \neq 0$ because otherwise we get solution (a) of Corollary 1, [27]. If 4 of these 5 platform anchor points coincide we get solution (d) of Corollary 1, [27]. Therefore we only have to distinguish the following three cases:

3 of the 5 collinear points coincide: W.l.o.g. we can set $\mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$. Now Eqs. (29$l$) and (32$l$) can only vanish without contradiction (w.c.) for:

1. $C_4 = 0$: Now Eq. (53) yields the contradiction.
2. $B_3 = 0$, $C_4 \neq 0$: Then Eqs. (25l), (27l) and (46l) cannot vanish w.c..
3. $B_5 = C_5 = 0$, $B_3 C_4 \neq 0$: Now Eq. (25l) can only vanish w.c. for $A_5(A_3 - 1) = 0$. As $A_5 = 0$ ($\Rightarrow \mathsf{M}_1 = \mathsf{M}_5$) implies solution (c) of Corollary 1, [27] we set $A_3 = 1$. Then Eq. (46l) yields the contradiction.

2 of the 5 collinear points coincide: W.l.o.g. we can set $a_3 = 0$, i.e. $\mathsf{m}_1 = \mathsf{m}_3$. Due to Eqs. (27$l$) and (32$l$) we have to distinguish two cases:

1. $\mathsf{m}_4 = \mathsf{m}_5$: Now Eq. (46$l$) can only vanish w.c. for:
   a. $B_3 = 0$: As $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear we can set $C_4 = 0$ w.l.o.g.. Now Eqs. (22l), (25l) and (46l) cannot vanish w.c..

b. $B_4C_5 - B_5C_4 = 0$, $B_3 \neq 0$: As $C_4 = C_5 = 0$ already yields a contradiction, we can assume $C_4 \neq 0$. Then we can express B_5 from $B_4C_5 - B_5C_4 = 0$ and A_5 from Eq. (46l) w.l.o.g.. Finally Eq. (47l) yields the contradiction.

2. $C_4 = 0$, $\mathsf{m}_4 \neq \mathsf{m}_5$: Now Eq. (25$l$) can only vanish w.c. for:
   a. $A_4 = 1$: Now Eqs. (29l) and (46l) imply $B_4 = 0$ ($\Rightarrow \mathsf{M}_2 = \mathsf{M}_4$). Then Eqs. (22$l$) and (24$r$) cannot vanish w.c..
   b. $A_3 = 0$, $A_4 \neq 1$: Now Eq. (46l) already yields the contradiction.

5 collinear points are pairwise distinct: Due to Eq. (32l) we have to distinguish the following two cases:

1. $B_3 = 0$: As $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear we can set $C_4 = 0$ w.l.o.g.. Now Eq. (47l) cannot vanish w.c..
2. $C_4 = 0$, $B_3B_4 \neq 0$: In this case Eq. (47l) already yields the contradiction.

For the remaining discussion we can assume that no 5 platform or base anchor points are collinear.

4.2 4 anchor points of the planar platform are collinear

In this case we set $c_6 = 0$. Moreover due to Subsection 4.1 we can assume $b_5b_6 \neq 0$. Eqs. (28l) and (33l) can only vanish w.c. in the following two cases:

$\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ **are coplanar:** Therefore we set $C_4 = 0$. Then Eq. (29l) yields $B_3B_4(a_3 - a_4)(C_5b_6 - C_6b_5)$:

1. $B_3B_4 = 0$: W.l.o.g. we can set $B_3 = 0$. Now Eqs. (22r) and (24l) can only vanish w.c. for $B_5 = B_6$ and $A_5 = A_6$. Then Eq. (48l) yields $C_5 = C_6$ ($\Rightarrow \mathsf{M}_5 = \mathsf{M}_6$). Now Eq. (47$l$) implies $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6) = 0$ [1]. Then Eqs. (25l) and (46l) can only vanish w.c. for solution (e) of Corollary 1, [27].
2. $\mathsf{m}_3 = \mathsf{m}_4$, $B_3B_4 \neq 0$: Now Eqs. (22r), (24l) and (48l) imply $\mathsf{M}_5 = \mathsf{M}_6$. Then Eqs. (25$r$) and (46$l$) can only vanish w.c. for $a_4 = 0$ or $a_4 = 1$. In both cases Eq. (47l) yields the contradiction.
3. $C_5b_6 - C_6b_5 = 0$: This condition already yields the contradiction.

$\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ **are not coplanar:** Now Eqs. (28l) and (33l) can only vanish w.c. for $C_5 = C_6$. Due to Eqs. (26l) and (48l) we distinguish two cases:

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[1] $coll(\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k) = 0$ denotes the algebraic condition for the three points $\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k$ to be collinear.

1. $A_3 = a_3$: We have to distinguish two cases due to Eqs. (22r) and (24l):
   a. $\mathsf{M}_5 = \mathsf{M}_6$: Now Eq. (32$l$) can only vanish w.c. for:
      i. $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6) = 0$: W.l.o.g. we can solve the collinearity condition for a_6. Then Eq. (47l) implies $\mathsf{m}_3 = \mathsf{m}_4$. Finally Eq. (25$l$) yields the contradiction.
      ii. $C_6 = 0$, $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6) \neq 0$: Now Eq. (47$l$) implies $\mathsf{m}_3 = \mathsf{m}_4$. Then Eqs. (25$l$) and (46$l$) can only vanish w.c. for solution (b) of Corollary 1, [27].
   b. $A_4 = a_4$, $\mathsf{M}_5 \neq \mathsf{M}_6$: Now Eqs. (30$l$) and (49$l$) already yield the contradiction.
2. $\mathsf{M}_5 = \mathsf{M}_6$, $A_3 \neq a_3$: Due to Eq. (32l) we distinguish the following cases:
   a. $B_3 = 0$: As $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear we can set $C_6 = 0$ w.l.o.g.. Then Eq. (47l) implies $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6) = 0$. From Eq. (46$l$) we get $a_4 = a_3(A_3 - 1)/(A_3 - a_3)$. Now Eq. (46$l$) can only vanish w.c. for $A_3 = 0$ or $A_3 = 1$. In both cases we get solution (e) of Corollary 1, [27].
   b. $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6) = 0$, $B_3 \neq 0$: W.l.o.g. we can solve the collinearity condition for a_6. Then Eq. (46l) can only vanish for:
      i. $\mathsf{m}_3 = \mathsf{m}_4$: From Eq. (25$l$) we get $A_6 = [A_3(C_4 - C_6) + A_4C_6]/C_4$. Finally Eq. (46$l$) yields the contradiction.
      ii. $A_6 = [a_4(C_4 - C_6) + A_4C_6]/C_4$, $\mathsf{m}_3 \neq \mathsf{m}_4$: Now Eq. (29$l$) yields the contradiction.
   c. $C_6 = 0$, $coll(\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6)B_3 \neq 0$: Now Eq. (29$l$) can only vanish for:
      i. $B_6 = 0$: Then Eq. (47l) implies $\mathsf{m}_3 = \mathsf{m}_4$. W.lo.g. we can express A_6 from Eq. (46l). Now Eq. (25l) can only vanish w.c. for $a_4 = 0$ or $a_4 = 1$. In both cases we get solution (b) of Corollary 1, [27].
      ii. $coll(\mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6) = 0$, $B_6 \neq 0$: Now Eq. (47l) cannot vanish w.c..

4.3 4 anchor points of the non-planar platform are collinear

Due to Subsection 4.1 and 4.2 we can assume $b_5c_6 \neq 0$. Moreover due to Subsection 4.2 we can assume that there do not exist 4 collinear anchor points in a planar base. Due to Eq. (61) we distinguish the following three subcases:

1. $B_3 = 0$: As $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear we can set $C_4 = 0$ w.l.o.g.. Now the Eqs. (47r), (48r), (49r) and (53) imply $A_5 = A_6$, $B_5 = B_6 = 0$ and $C_5 = 0$. Then Eq. (51) yields the contradiction.
2. $C_4 = 0$, $B_3B_4 \neq 0$: In this case Eqs. (51) and (53) imply $C_5 = C_6 = 0$ ($\Rightarrow$ base is planar). Now Eqs. (47r) and (48r) can only vanish w.c. for $\mathsf{M}_5 = \mathsf{M}_6$. As $B_3(a_4 - A_4) - B_4(a_3 - A_3) = 0$ yields a contradiction we can express A_6 and B_6 from Eqs. (46r) and (49r), respectively.
   a. $a_3 + a_4 \neq 1$: Under this assumption we can compute A_4 from the only non-contradicting factor of Eq. (26r). Then Eq. (25r) yields the contradiction.

b. $a_3 = a_4 - 1$: Now Eq. (26r) can only vanish w.c. for $a_4(a_4-1)(B_3-B_4) = 0$. In all 3 cases Eq. (25r) yields the contradiction.

3. $B_5 = B_6 = C_5 = C_6 = 0$, $B_3C_4 \neq 0$: Now Eqs. (51) and (52) imply $\mathsf{m}_3 = \mathsf{m}_4$ and $\mathsf{M}_5 = \mathsf{M}_6$, respectively. From Eq. (50) we get $A_6 = 1 - a_4$. Now Eq. (43r) can only vanish w.c. for $a_4(a_4-1) = 0$. In both cases we get solution (b) of Corollary 1, [27]. □

5 Geometric interpretation

Under consideration of the special coordinate systems introduced in Section 1 the main theorem (Theorem 6) characterizes all SG platforms with a non-cubic singularity surface. For arbitrary coordinate systems in the platform and the base the condition of Theorem 6 reads as $rk(\mathbf{M}) < 5$ (cf. [28]) with

$$\mathbf{M} = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & A_1 & B_1 & C_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & c_6 & A_6 & B_6 & C_6 \end{pmatrix}. \tag{62}$$

If we interpret the entries of each row as homogeneous coordinates $(1 : a_i : b_i : c_i : A_1 : B_i : C_i)$ of a point X_i of a 6-dimensional projective space P^6 the characterization can also be done as follows:

If the six points $\mathsf{X}_1, \ldots, \mathsf{X}_6$ are located in a 3-dimensional projective subspace of P^6 the corresponding manipulator has a non-cubic singularity surface (assumed that the manipulator is not architecturally singular). Moreover the following theorem holds:

Theorem 7 *A non-architecturally singular SG platform possesses a non-cubic singularity surface if and only if there exists a affine correspondence between the platform and the base or if the manipulator is planar with* $rk(\mathbf{M}) = 4$.

Proof: In the following we assume that the manipulator is non-planar. W.l.o.g. we can assume that the platform is non-planar and that the anchor points $\mathsf{m}_1, \ldots, \mathsf{m}_4$ form a tetrahedron. Therefore the four points $\mathsf{X}_1, \ldots, \mathsf{X}_4$ already span the 3-dimensional projective subspace.

Moreover for the proof we choose again the special coordinate systems in the platform and in the base introduced in Section 1, whereas $a_2b_3c_4 \neq 0$ has to hold. Now the Matrix $\mathbf{M}$ reads as follows:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & a_2 & 0 & 0 & A_2 & 0 & 0 \\ 1 & a_3 & b_3 & 0 & A_3 & B_3 & 0 \\ 1 & a_4 & b_4 & c_4 & A_4 & B_4 & C_4 \\ \hline 1 & a_5 & b_5 & c_5 & A_5 & B_5 & C_5 \\ 1 & a_6 & b_6 & c_6 & A_6 & B_6 & C_6 \end{pmatrix}. \tag{63}$$

Then we consider the rank of the 3×3 submatrix $\mathbf{W}$ containing the coordinates of $\mathsf{M}_2, \mathsf{M}_3$ and M_4.

- $rk(\mathbf{W}) < 2$: All base anchor points collapse into a point or are located on a line, respectively. These are architecturally singular manipulators.
- $rk(\mathbf{W}) = 2$: The base is planar and we get a singular affinity which maps the platform anchor points onto the corresponding base anchor points.
- $rk(\mathbf{W}) = 3$: The base is non-planar and we have a regular affinity between corresponding anchor points.

The case study for the planar case was already done by KARGER [23] in another context. Therefore we only sum up the results of the cited publication:

- $rk(\mathbf{M}) < 3$: This determines architecturally singular manipulators.
- $rk(\mathbf{M}) = 3$: There exists a regular affinity.
- $rk(\mathbf{M}) = 4$: This is the remaining special case. According to KARGER [23] no special geometric properties for these planar parallel manipulators of SG type were known so far. □

Examples *Many examples for planar SG platforms with* $rk(\mathbf{M}) = 4$ *were already illustrated by the author in [22]. A further example was given by* KARGER *in [21]. Moreover the manipulator of Fig. 1(b) is also of this type.*
It should also be noted that the manipulator of Fig. 1(a) has a quadratic singularity surface as there exists a singular affinity between the base anchor points and the corresponding platform anchor points (cf. [20]). ◇

6 Conclusion and future work

In this article we determined the whole set of parallel manipulators of Stewart Gough type which possesses a non-cubic singularity surface in the space of translations. It turns out that this set can easily be characterized by the rank condition given in the stated main theorem (cf. Theorem 6). Moreover we presented a geometric characterization of these manipulators in Section 5.

Based on the given results we have already determined all manipulators with a linear singularity surface (cf. [29]). In a further step we want to determine all manipulators with a quadratic singularity surface, which splits up into two planes for any orientation. Such SG platforms exist because the manipulator (*ii*) of Subsection 1.2 has this property (cf. NAWRATIL [20]).

Indeed in this context it would also be interesting to determine all parallel manipulators of SG type possessing a cubic singularity surface which splits up into a plane and a quadric (or even into 3 planes) for any orientation of the platform (cf. [24,25]).

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