On homotopy continuation based singularity distance computations for 3-RPR manipulators

Aditya Kapilavai and Georg Nawratil

Institute of Discrete Mathematics and Geometry, TU Wien

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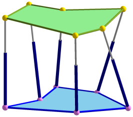
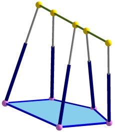
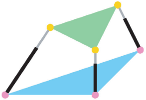
Introduction



Hexapod



Pentapod



3-RPR manipulator

- Parallel manipulators of Stewart-Gough type suffer from singular configurations.
- Evaluating the distance between a given non-singular configuration to the closest singular one is of interest for industrial applications (e.g. singularity free-path planning)

What is known

- For 3-RPR manipulator a geometric meaningful distance between two poses of the platform can be computed as

Distance function

$$d_3 := \sqrt{\frac{1}{3}\sum_{i=1}^{n}\langle \mathbf{P}_i^{\alpha} - \mathbf{P}_i^{\gamma}, \mathbf{P}_i^{\alpha} - \mathbf{P}_i^{\gamma}\rangle}$$

where $\mathbf{P}_i^{\gamma} = (x_i^{\gamma}, y_i^{\gamma})^T$ (resp. $\mathbf{P}_i^{\alpha} = ({x_i}^{\alpha}, {y_i}^{\alpha})^T$) denotes the ith platform anchor point in the given (resp. α-transformed) configuration w.r.t the fixed frame.

Singularity condition (V_3) for 3-RPR

α-transformed configuration is singular if and only if the carrier lines of the three legs intersect in a common point or are parallel.

General problem formulation

Optimization problem: Computation of the closest point (w.r.t. the metric d_3) on the singularity variety (V_3) to the given non-singular manipulator configuration.

The corresponding Lagrange function L reads

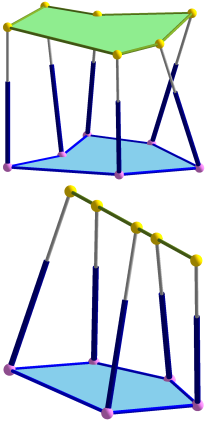
$$L: \; d_3^2 - \lambda V_3 = 0.$$

- The above formulation can be used to compute singularity distance as the global minimum.

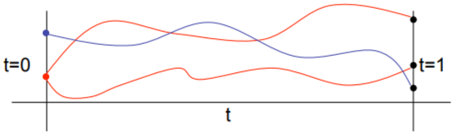
Common Procedure: Newton's method is simple and quick to solve a polynomial system.

Limitation: It only converges when a good initial guess is used, and even then this only yields one solution.

- Can we solve this system of equations in the case of hexapods and pentapods using symbolic approaches?
- The critical points of the corresponding polynomial Lagrange function cannot be found by the symbolic approach (Gröbner basis method) due to the degree and number of unknowns.



Another approach: Homotopy continuation method



- Start with “similar“ equations with known solutions.
- Continuously deform the equations towards the target system.
- Trace the solution paths as the deformation parameter t goes from 1 to 0.

Software for solving polynomial systems

Bertini uses homotopy continuation algorithms to produce all isolated solutions of a system.

Handling of Bertini

- Bertini homogenizes the polynomial equations on a grouping provided by the user.
- Bad grouping can lead to a tracking of solutions at infinity.
- The goal is to minimize the number of tracked paths $(=: B_{\min})$.

Key points with Bertini

- Partition of the variables into multiple groups lead to a multi-homogeneous homotopy in Bertini
- Bertini does not group the variables automatically
- User is responsible for choosing the good grouping of unknown variables that yields B_{min}

Example

Equations

$$f(x,y) = \left\{ \begin{array}{c} xy - 1 \\ x^2 - 1 \end{array} \right\} = 0$$

One Group

$$\{x,y\} \to \left\{ \frac{X}{W}, \frac{Y}{W} \right\}$$

$$F(W,X,Y) = \left\{ \begin{array}{c} XY - W^2 \\ X^2 - W^2 \end{array} \right\} = 0$$

Sol $= \{(1,\pm1,\pm1),(0,1,0)\}$

Two groups

$$\{x\},\{y\} \to \left\{ \frac{X}{U} \right\}, \left\{ \frac{Y}{V} \right\}$$

$$F(U,V,X,Y) = \left\{ \begin{array}{c} XY - UV \\ X^2 - U^2 \end{array} \right\} = 0$$

Sol $= \{([1,1],[1,1]),([1,-1],[1,-1])\}$

What exists so far in the literature

"there does not yet exist a truly efficient algorithm for finding optimal groupings, and the combinatorics are such that an exhaustive examination of all possible groupings becomes impractical as the number of variables grows much larger than 10".

- Wampler et.al studied multi-homogeneous homotopy by using isotropic coordiantes.
- No attempts have been made so far to compare this approach with other algebraic motion representations.

Bell number

The total number of all possible groupings of the n variables is given by the so-called *Bell number* $B(n)$ with $B(5) = 52$ and $B(6) = 203$.

Our initial goals with Bertini

By considering a 3-RPR planar maipulator, we formulate our singularity distance optimization problem w.r.t various algebraic motion representations of planar Euclidean/equiform kinematics.

- Divide algebraic motion representations into two classes (non-homogenous and homogenous)
- Find out which algebraic motion representation is suitable for Bertini?
- Verify the solutions obtained by Bertini with Maple 2018 using Grobner basis method.

Non-homogeneous representation

Point Based Representation (PBR): Transformation is formulated in terms of the first and second platform anchor points.

The transformed third platform point reads

$$\mathbf{P}_3^\alpha = \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \begin{pmatrix} x_2^\alpha - x_1^\alpha & y_1^\alpha - y_2^\alpha \\ y_2^\alpha - y_1^\alpha & x_2^\alpha - x_1^\alpha \end{pmatrix} \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} + \mathbf{P}_1^\alpha.$$

The Lagrange function L for equiform motion reads

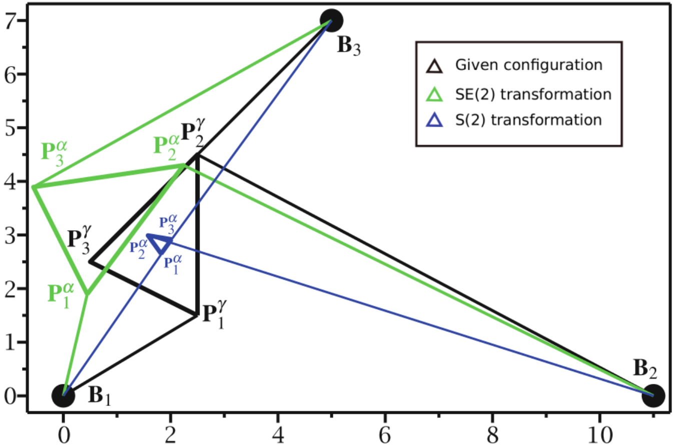
$$L: \; d_3^2 - \lambda V_3 = 0.$$

The Lagrange function L for Eulcedian motion reads

$$L: \; d_3^2 - \lambda V_3 - \mu M = 0, \quad M := \overline{\mathbf{P}_1^\alpha \mathbf{P}_2^\alpha}^2 - \overline{\mathbf{P}_1 \mathbf{P}_2}^2$$

where $\mathbf{P}_i = (x_i, y_i)^T$ denotes the coordinate vector of the i^{th} platform anchor point w.r.t the moving frame.

Computation of closest singular configuration



The given configuration is illustrated in black and the closest singular one under Euclidean/equiform transformations of the platform is displayed in green/blue.

Two other non-homogeneous representations

Planar Euler-Rodrigues Representation (PERR):

The transformation $\alpha: \mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$\alpha: \mathbf{P}_i \mapsto \mathbf{P}_i^{\alpha} := \begin{pmatrix} a_1^2 - a_2^2 & -2a_1a_2 \\ 2a_1a_2 & a_1^2 - a_2^2 \end{pmatrix} \mathbf{P}_i + \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$$

with $\quad a_1, \ldots, a_4 \in \mathbb{R} \quad \text{and} \quad M := a_1^2 + a_2^2 - 1$

Isotropic Coordinates Representation (ICR): $\mathbf{P}_i$ is represented by the pair $(z_i, \overline{z}_i)$ of conjugate complex numbers with $z_i = x_i + iy_i$.

The transformation $\alpha: \mathbb{C}^2 \to \mathbb{C}^2$ can be written as

$$\alpha: (z_i, \overline{z}_i) \mapsto (z_i^{\alpha}, \tilde{z}_i^{\alpha}) := (\kappa z_i + \tau, \tilde{\kappa}\overline{z}_i + \tilde{\tau})$$

with $\quad \kappa, \tau, \tilde{\kappa}, \tilde{\tau} \in \mathbb{C} \quad \text{and} \quad M := \kappa\tilde{\kappa} - 1$

Result: Comparison of Bézout numbers

Method	Best groupings	B_{min}	T_{avg}	Worst groupings	B_{max}	DOL
PBR	$\{(x_1^\alpha, y_1^\alpha, x_2^\alpha, y_2^\alpha), (\lambda)\}$	96	970	$\{(x_1^\alpha, y_1^\alpha), (x_2^\alpha, y_2^\alpha, \lambda)\}$	1 296	4
				$\{(x_1^\alpha, y_2^\alpha), (y_1^\alpha, x_2^\alpha, \lambda)\}$	1 296	
PERR	$\{(a_1, a_2), (a_3, a_4), (\lambda)\}$	828	73 000	$\{(a_1, a_2), (a_3, a_4, \lambda)\}$	14 025	6
				$\{(a_2, a_3), (a_1, a_3, \lambda)\}$	14 025	
ICR	$\{(\kappa, \tilde{\kappa}, \tau, \tilde{\tau}), (\lambda)\}$	96	61 870	$\{(\tilde{\kappa}, \tau), (\kappa, \tilde{\tau}, \lambda)\}$	1 296	4
				$\{(\kappa, \tilde{\tau}), (\tau, \tilde{\kappa}, \lambda)\}$	1 296	

Studied equiform motion representations

Method	Best groupings	B_{min}	T_{avg}	Worst groupings	B_{max}	DOL
PBR	$\{(x_1^\alpha, y_1^\alpha, x_2^\alpha, y_2^\alpha), (\lambda, \mu)\}$	144	1 800	$\{(x_1^\alpha), (x_2^\alpha, \mu), (y_1^\alpha, \lambda), (y_2^\alpha)\}$	8 448	4
				$\{(x_1^\alpha, \mu), (y_2^\alpha, \lambda), (x_2^\alpha), (y_1^\alpha)\}$	8 448	
PERR	$\{(a_1, a_2), (a_3, a_4), (\lambda, \mu)\}$	360	45 920	$\{(a_1, \lambda, a_3), (a_2, \mu, a_4)\}$	50 992	6
	$\{(a_1, a_2, a_4), (a_3), (\lambda, \mu)\}$	360	35 640	$\{(a_1, \mu, a_4), (a_2, \lambda, a_3)\}$	50 992	
ICR	$\{(\kappa), (\tilde{\kappa}), (\lambda), (\mu), (\tau, \tilde{\tau})\}$	136	249 880	$\{(\kappa, \tilde{\tau}), (\tilde{\kappa}, \mu, \tau, \lambda)\}$	2 187	4
				$\{(\kappa, \mu, \lambda, \tilde{\tau}), (\tilde{\kappa}, \tau)\}$	2 187	

Studied Euclidean motion representations

Homogeneous representations

Two homogeneous algebraic representations for each of the following motion groups

- Euclidean SE(2)
- Equiform S(2)

Two of them are based on on Study's kinematic mapping where each element of SE(3) is represented by a point $(e_0 : e_1 : e_2 : e_3 : t_0 : t_1 : t_2 : t_3)$ in the projective 7-dimensional space P^7 located on the so-called Study quadric.

Homogeneous representations

Blaschke-Grünwald Representation (BGR):

- Euclidean Motion
- Restricting Study's parametrization to planar motions by $e_1 = e_2 = t_0 = t_3 = 0$.

The transformation $\alpha: \ \mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$\alpha: \ \mathbf{P}_i \mapsto \mathbf{P}_i^\alpha := \frac{1}{\Delta}\left[\begin{pmatrix} e_0^2 - e_3^2 & -2e_0e_3 \\ 2e_0e_3 & e_0^2 - e_3^2 \end{pmatrix}\mathbf{P}_i + \mathbf{t}\right]$$

with $\mathbf{t} := [-2(e_0t_1 - e_3t_2), -2(e_0t_2 + e_3t_1)]^T$, $\Delta := e_0^2 + e_3^2$ and $e_0, e_3, t_1, t_2 \in \mathbb{R}$

The Lagrange function L reads

$$L: \ d_3^2 - \lambda V_3 = 0.$$

Homogeneous representations

Davidson-Hunt Representation (DHR):

- Equiform Motion
- Based on the analogy of the Study parameters to homogenous screw coordinates
- Interpreting the points in the ambient space P^7 of the Study quadric as spatial similarity transformations.

The transformation $\alpha: \mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$\alpha\colon \mathbf{P}_i \mapsto \mathbf{P}_i^{\alpha} := \frac{1}{\Delta^2}\left[(\Delta + e_0 t_0 + e_3 t_3)\begin{pmatrix} e_0^2 - e_3^2 & -2e_0e_3 \\ 2e_0e_3 & e_0^2 - e_3^2 \end{pmatrix}\mathbf{P}_i + \Delta\mathbf{t}\right]$$

with $e_0, e_3, t_0, t_1, t_2, t_3 \in \mathbb{R}$.

The Lagrange function L reads

$$L:\ d_3^2 - \lambda V_3 - \mu M = 0 \quad \text{with} \quad M := e_0 t_3 - e_3 t_0$$

Quaternion Based Representation (QBR):

- Equiform Motion

The transformation $\alpha:\ \mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$\alpha:\ \mathbf{P}_i \mapsto \mathbf{P}_i^\alpha := \frac{1}{\Delta}\left[\begin{pmatrix} e_0 f_0 - e_3 f_3 & -e_0 f_3 - e_3 f_0 \\ e_0 f_3 + e_3 f_0 & e_0 f_0 - e_3 f_3 \end{pmatrix}\mathbf{P}_i + \mathbf{t}\right]$$

with $e_0, e_3, f_0, f_3, t_1, t_2 \in \mathbb{R}$.

The Lagrange function L reads

$$L:\ d_3^2 - \lambda V_3 - \mu M = 0 \quad \text{with} \quad M := e_0 f_3 - e_3 f_0$$

Dual Cayley-Klein Representation (DCKR)

- Euclidean Motion
- Represented using isotropic point coordinates $(z_i, \overline{z}_i)$

The transformed by $\alpha: \ \mathbb{C}^2 \to \mathbb{C}^2$ can be written as

$$\alpha: \ (z_i, \overline{z}_i) \mapsto (z_i^\alpha, \tilde{z}_i^\alpha) := \frac{1}{\theta\tilde{\theta}} \Big(\theta(\theta z_i + 2\sigma), \tilde{\theta}(\tilde{\theta}\overline{z}_i + 2\tilde{\sigma}) \Big)$$

with $\theta, \sigma, \tilde{\theta}, \tilde{\sigma} \in \mathbb{C}$.

The Lagrange function L reads

$$L: \ d_3^2 - \lambda V_3 = 0.$$

Handling a homogeneous overdetermined system in Bertini

- System of n partial derivatives $\sum_{j=1}^{k} \frac{\partial L}{\partial m_j}$ $(j = 1, \ldots, n)$ of L results in a homogenous system of equations w.r.t $m_1, \ldots, m_k$

Relation between the partial derivatives $\frac{\partial L}{\partial m_j}$

$$\sum_{j=1}^{k} m_j \frac{\partial L}{\partial m_j} = 0$$

Solving overdetermined system in Bertini

One has to square up the system by replacing the k equations $\frac{\partial L}{\partial m_j} = 0$ for $j = 1, \ldots, k$ by $k - 1$ linear combinations of the form $\sum_{j=1}^{k} \square \frac{\partial L}{\partial m_j}$ where each $\square$ indicates a random complex number

Result: Comparison of Bézout numbers

- Only one possible grouping for BGR and DCKR
- Two possible grouping for DHR and QBR

Method	Best groupings	B_{min}	T_{avg}	Worst groupings	B_{max}	DOL
BGR	$\{(e_0, e_3, t_1, t_2), (\lambda)\}$	300	122 180	–	–	$\frac{5}{4}$
DCKR	$\{(\theta, \tilde{\theta}, \sigma, \tilde{\sigma}), (\lambda)\}$	300	115 283	–	–	$\frac{5}{4}$
DHR	$\{(e_0, e_3, t_0, t_1, t_2, t_3), (\lambda, \mu)\}$	165 240	–	$\{(e_0, e_3, t_0, t_1, t_2 t_3), (\lambda), (\mu)\}$	194 400	$\frac{11}{10}$
QBR	$\{(e_0, e_3, f_0, f_3, t_1, t_2), (\lambda, \mu)\}$	41 160	–	$\{(e_0, e_3, f_0, f_3, t_1, t_2), (\lambda), (\mu)\}$	82 320	$\frac{7}{6}$

Studied homogeneous representations

Discussion of solutions

Motion group	PBR and ICR	PERR	BGR and DCKR (Bertini)	BGR and DCKR (Maple)
SE(2)	32	64	162	32 & 1-dim set & 2-dim set
S(2)	19	86		

Total number of solutions (counted including multiplicity) for studied representations

For a detailed discussion of solutions we refer to:

A. Kapilavai and G. Nawratil: On homotopy continuation based singularity distance computations for 3-RPR manipulators. New Trends in Mechanism Science (D. Pisla, B. Corves eds.), pages 56-64, Springer (2020) [Extended version on arXiv:2004.08359]

Conclusions

- It can be observed that the B_{min} value obtained for ICR is the best one of all euclidean motion representations as suggested by Wampler et.al.
- For an equiform motion representation the lowest number of tracked paths is obtained by ICR and PBR.
- Surprisingly the PBR has in both motion groups by far the best computational performance with respect to T_{avg}.

Open issues and Future work

Due to the large Bézout numbers of DHR and QBR the question arises whether a computationally more efficient homogenous representation of S(2) exists?
All in all, this study suggests the usage of PBR for the future research on the spatial case (i.e. hexapods and pentapods) due to the good B_{min} values and the best results for T_{avg}.

Der Wissenschaftsfonds.

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Thank you for your attention

Questions?