On the spin surface of RSSR mechanisms with parallel rotary axes

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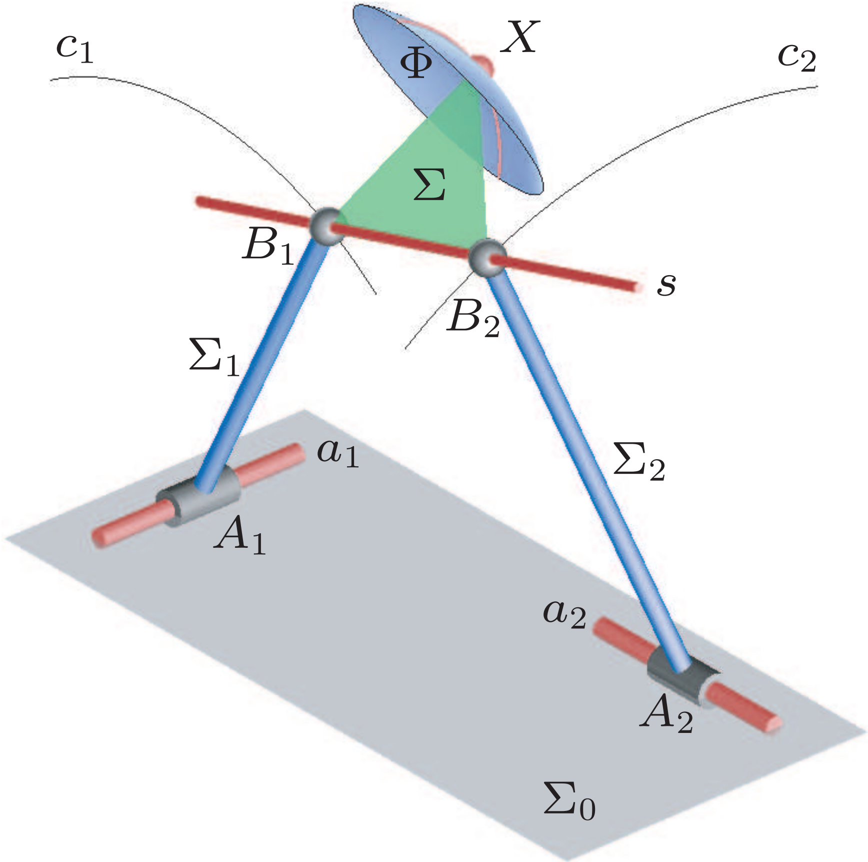
[1] RSSR mechanism

An RSSR mechanism is a closed kinematic chain, where two points B_i $(i = 1, 2)$ of the end-effector Σ are connected via spherical joints centered in B_i with the systems Σ_i, which themselves are coupled via rotary joints with the fixed system Σ_0.

Due to Grüblers formula the RSSR mechanism has two dofs, where one degree is the rotation of Σ about the line $s := [B_1, B_2]$.



[1] Cayley Theorems

The order of the ruled surface Γ generated by the line s can be determined by applying the two so called Cayley theorems given by **Fichter and Hunt [1]**.

1^{st} **Cayley Theorem**
If two points B_1 and B_2 a fixed distance apart on a line s are constrained to lie respectively on two curves c_1 and c_2 (no planar curves lying in parallel planes) then s generates a ruled surface whose degree is give by $2n_1(n_2 - p_2) + 2n_2(n_1 - p_1)$, where n_i denotes the order of c_i and p_i the circularity of c_i $(i = 1, 2)$. $\Rightarrow$ **8**

2^{nd} **Cayley Theorem**
In the special case that c_1 and c_2 are planar curves lying in parallel planes the order of the ruled surface is given by $2n_1(n_2 - p_2) + 2n_2(n_1 - p_1) - 2p_1p_2$. $\Rightarrow$ **6**

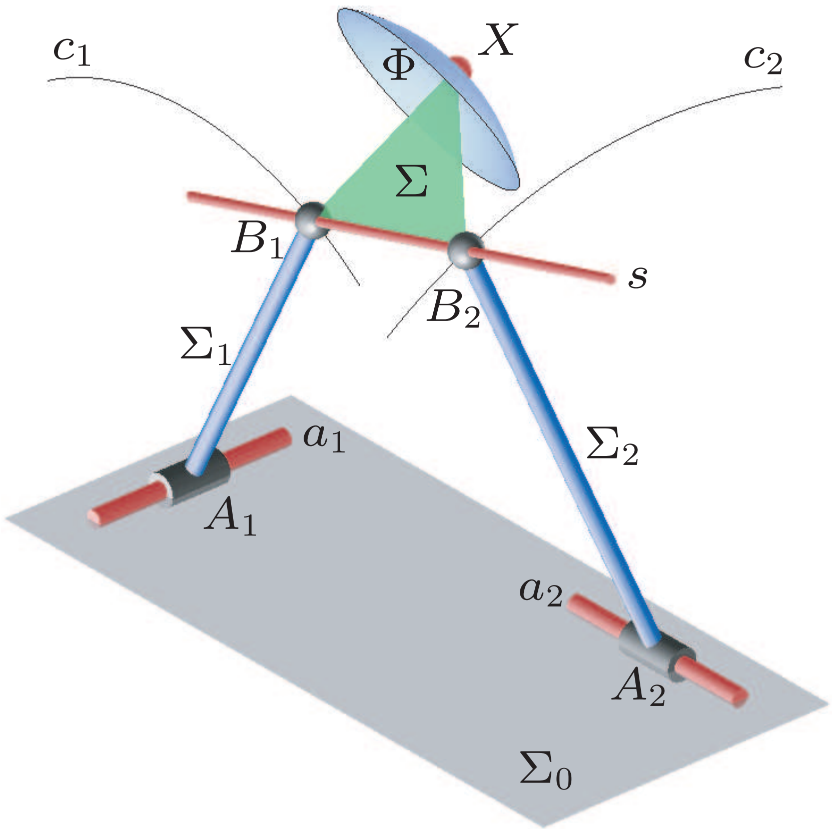
[2] Order and circularity of the spin surface

Hunt [2] proved that Φ is of order

$$2{\cdot}8 = 16 \ldots \text{general case}$$
$$2{\cdot}6 = 12 \ldots \text{special case}$$

Hunt [3] suggested that the circularity of Φ is 8 which was later proved by **Merlet [5]**.

For the special case **Lazard and Merlet [4]** stated that the circularity is 6.



Theorem 1
The spin surface of an RSSR mechanism with parallel rotary axes contains the imaginary spherical circle four times.

[2] Proof

W.l.o.g. we can choose a Cartesian coordinate system in Σ_0 as follows:

- c_i traced by the points B_i are located in planes parallel to the xy-plane.
- $A_1 = (0,0,0)^T$ and $A_2 = (t,0,u)^T$ with $t, u \geq 0$.

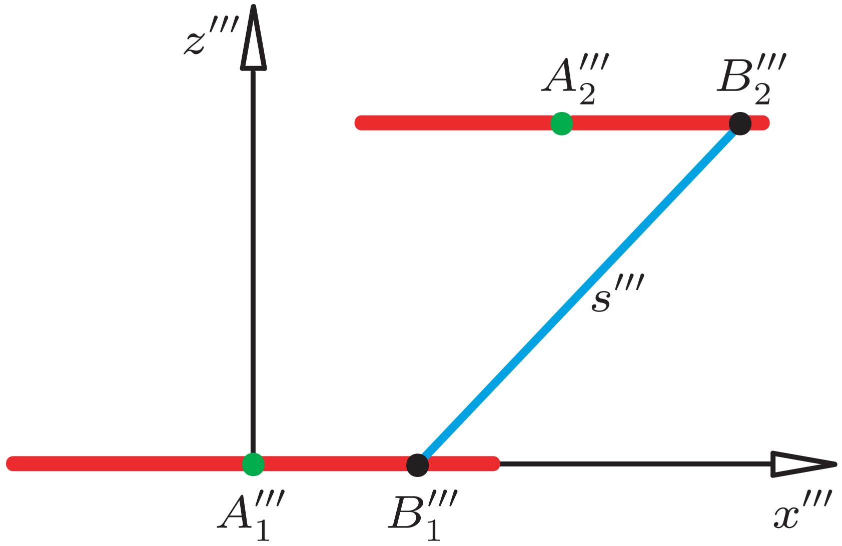
The paths of B_i parametrized by

$$B_i(\mu_i) = A_i + (r_i \cos\mu_i, r_i \sin\mu_i, 0)^T$$

with $r_i := \overline{A_i B_i}$ are coupled by

$$P_0 := \|B_1(\mu_1) - B_2(\mu_2)\|^2 - b^2 = 0$$

with $b := \overline{B_1 B_2} > 0$ and $b \geq u$.



[2] Proof

The locus of a point $X = (x, y, z)$ of Σ is determined by the conditions

$$P_i := \|B_i(\mu_i) - X\|^2 - d_i^2 = 0 \quad \text{with} \quad d_i := \overline{B_i X} \quad \text{for} \quad i = 1, 2.$$

After applying the half angle substitution

$$\cos \mu_i = \frac{1 - m_i^2}{1 + m_i^2} \quad \text{and} \quad \sin \mu_i = \frac{2m_i}{1 + m_i^2}$$

in P_i $(i = 1, 2, 3)$ to get algebraic expressions we start the elimination process:

$$P_{01} := Res(P_0, P_1, m_1) \quad \text{and} \quad Q := Res(P_{01}, P_2, m_2)$$

where Q is polynomial (order 12) of the spin surface Φ.

[2] Proof

After introducing homogeneous coordinates ($x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$) we intersect Φ with the plane at infinity ω determined by $x_0 = 0$ which yields $2^{16} r_1^4 r_2^4 F_0^4 F_1 F_2$

$$F_0 := x_1^2 + x_2^2 + x_3^2 \qquad k_0 : \; F_0 = 0$$

$$F_1 := (x_1^2 + x_2^2)(u-b)^2 + x_3 t \left[x_3 t - 2x_1(u-b)\right] \qquad k_1 : \; F_1 = 0$$

$$F_2 := (x_1^2 + x_2^2)(u+b)^2 + x_3 t \left[x_3 t - 2x_1(u+b)\right] \qquad k_2 : \; F_2 = 0$$

$\Rightarrow$ The intersection multiplicity of ω and Φ along k_0 is 4.

Remark
It is not clear that k_0 is a 4-fold curve of Φ because it could for example also be the case that k_0 is a 3-fold curve and that ω touches Φ along k_0.

[2] Proof

Therefore we intersect Φ with an arbitrary circle c_3. By using the abbreviation $C := \frac{1-h^2}{1+h^2}$, $S := \frac{2h}{1+h^2}$ and $D := 1 - C$ the circle c_3 can be parametrized as

$$\begin{bmatrix} (1-v_1^2)C + v_1^2 & v_1 v_2 D - v_3 S & v_1 v_3 D + v_2 S \\ v_1 v_2 D + v_3 S & (1-v_2^2)C + v_2^2 & v_2 v_3 D + v_1 S \\ v_1 v_3 D - v_2 S & v_2 v_3 D - v_1 S & (1-v_3^2)C + v_3^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

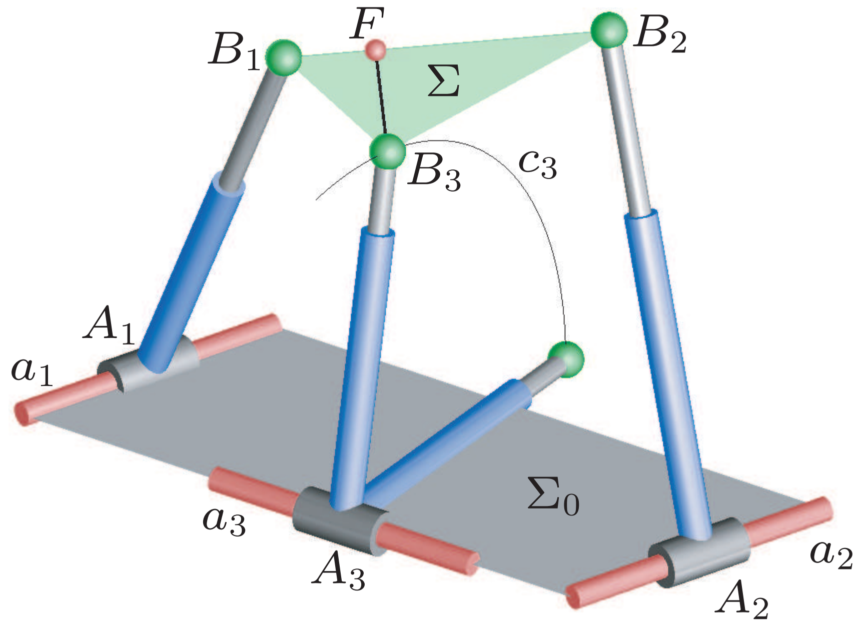
Plugging these expressions into the implicit representation of Φ yields a polynomial of degree 16 in the unknown h.

As the circle c_3 and Φ have $2 \times 12 = 24$ intersection points the two cyclic points of c_3 must be 4-fold points of Φ. $\square$

[3] Consequences for parallel manipulators

A parallel manipulator of TSSM type consists of a platform Σ which is connected via three SPR-legs with the base Σ_0, where the axes a_1, a_2, a_3 of the R-joints are coplanar.

If we skip the assumption of coplanarity we get a more generalized class of $GTSSM$ parallel manipulators.



Theorem 2
GTSSM manipulators with two parallel rotary axes cannot have more than 16 assembly modes except the degenerated cases with infinitely many solutions.

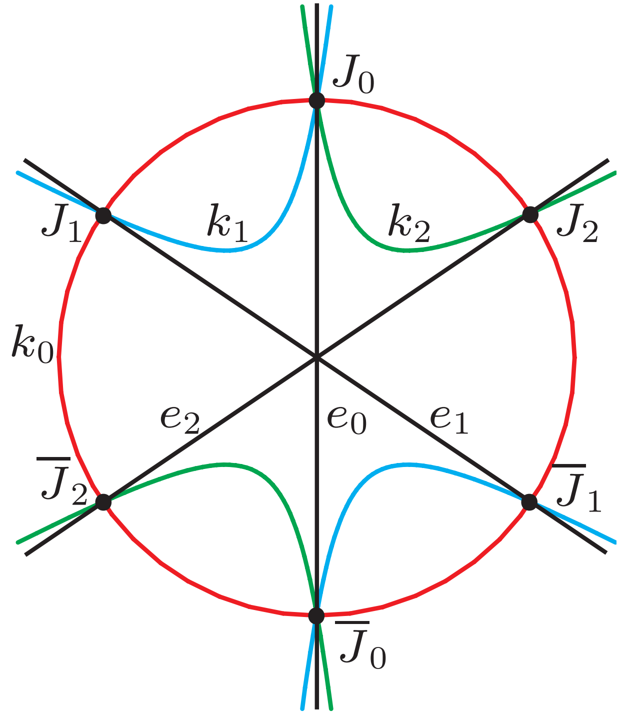
[3] Special Cases

We are interested in all cases with more than 8 common points of c_3 and Φ on k_0.
Necessary condition: Cyclic points of c_3 must also belong to $k_1 \in \omega$ and/or $k_2 \in \omega$.

$$k_0 \cap k_i : \quad J_0, \overline{J}_0, J_i, \overline{J}_i \quad \text{with} \quad J_0 = [1, i, 0]^T,$$

$$J_i = \left[(q_i^2 - t^2)I, q_i^2 + t^2, -2tq_iI\right]^T,$$

$$q_1 := u - b \quad \text{and} \quad q_2 := u + b$$



As the carrier plane ε of c_3 has to intersect k_0 in conjugate complex points there are three possibilities left for choosing $e := \varepsilon \cap \omega$:

[3] Special Case a)

$e_i := \left[J_i, \overline{J}_i\right]$ for $i = 1, 2$: Now e_i with projective line coordinates

$$J_i \times \overline{J}_i = [2q_i t : 0 : q_i^2 - t^2]^T$$

intersects Φ in the point J_i and $\overline{J}_i$ with multiplicity 5, but these points are not 5-fold points of Φ which can be shown as follows:

We intersect Φ with the line l spanned by the origin and the point J_i and $\overline{J}_i$, respectively, by inserting its parametric representation into Φ. Then the resulting equation splits up into $2^{16} r_1^4 r_2^4 F$ where F is a polynomial of degree 8 in the parameter of l.

$\Rightarrow$ e_i touches Φ in J_i and $\overline{J}_i$ which are still 4-fold points of Φ.

[3] Special Case b)

$e_0 := \left[J_0, \overline{J}_0\right]$: As J_0 and $\overline{J}_0$ are located on k_1 and k_2 the line e_0 with projective line coordinates

$$J_0 \times \overline{J}_0 = [0:0:1]^T \qquad (\Rightarrow\ a_1 \parallel a_2 \parallel a_3)$$

intersects Φ in these points with a multiplicity of 6.
Moreover J_0 and $\overline{J}_0$ are 6-fold points of Φ which can be proven as follows:

We set $v_1 = v_2 = 0$ and $v_3 = 1$ in the parametric representation of the circle c_3 and plug it into the equation of Φ. This yields a polynomial of degree 12 in the unknown h which finishes the proof.

Discuss the very special cases: $e_1 = e_2$, $e_0 = e_1$, $e_0 = e_2$ and $e_0 = e_1 = e_2$.

[3] $e_1 = e_2$

Now $e_1 = e_2$ intersects Φ in $J_1 = J_2$ and $\overline{J}_1 = \overline{J}_2$ with multiplicity 6. This happens if

$$det \begin{bmatrix} 2tq_1 & q_1^2 - t^2 \\ 2tq_2 & q_2^2 - t^2 \end{bmatrix} = 4tb(u^2 - b^2 + t^2) = 0.$$

- b must be greater than zero.
- $b^2 = u^2 + t^2$: It can be shown as in $a)$ that $J_1 = J_2$ and $\overline{J}_1 = \overline{J}_2$ are only 4-fold points of Φ.
- $t = 0$ will be discussed as last point of this case study.

[3] $e_0 = e_1$

e_0 equals e_1 for $(u-b)t = 0$. Therefore we set $u = b$ and assume $t \neq 0$.

It can be proven analogously to $b)$ that $J_0 = J_1$ and $\overline{J}_0 = \overline{J}_1$ are 7-fold points
$\Rightarrow$ The manipulator cannot have more than 10 assembly modes

But this threshold can be refined, because Γ can only have two real generators.
$\Rightarrow$ Φ degenerates into two coplanar circles which can be intersected by c_3 in a maximum of 4 real points.

Remark
Moreover it should be noted that such a manipulator has only 3 dofs, namely the translations in x and y direction as well as the rotation about s.

[3] $e_0 = e_2$ and $e_0 = e_1 = e_2$

e_0 equals e_2 for $(u+b)t = 0$.
As $u \geq 0$ and $b > 0$ holds this can only happen for $t = 0$ $(\Rightarrow a_3 \parallel a_1 = a_2)$.

But for $t = 0$ we get $e_0 = e_1 = e_2$ and the points $J_0 = J_1 = J_2$ and $\overline{J}_0 = \overline{J}_1 = \overline{J}_2$ are 8-fold points of Φ which can again be proven as in case $b)$.

Remark
If additionally $u = b$ holds the manipulator can only be assembled for $r_1 = r_2$
$\Rightarrow \Gamma$ is a cylinder of rotation and the point X can only be located in an annulus.
Such a manipulator has again only 3 dofs and a maximum of 4 real solutions.

End of all cases.

[3] Result of the case study

Theorem 3
GTSSM manipulators with two parallel rotary axes (a_1 and a_2) cannot have more than

$i)$ 12 assembly modes if the axis a_3 is parallel to a_1, a_2

$ii)$ 8 assembly modes if the axis a_3 is parallel to $a_1 = a_2$

$iii)$ 4 assembly modes if $u = b$ holds

except in the degenerated cases with infinitely many solutions.

[4] Maximum number of assembly modes

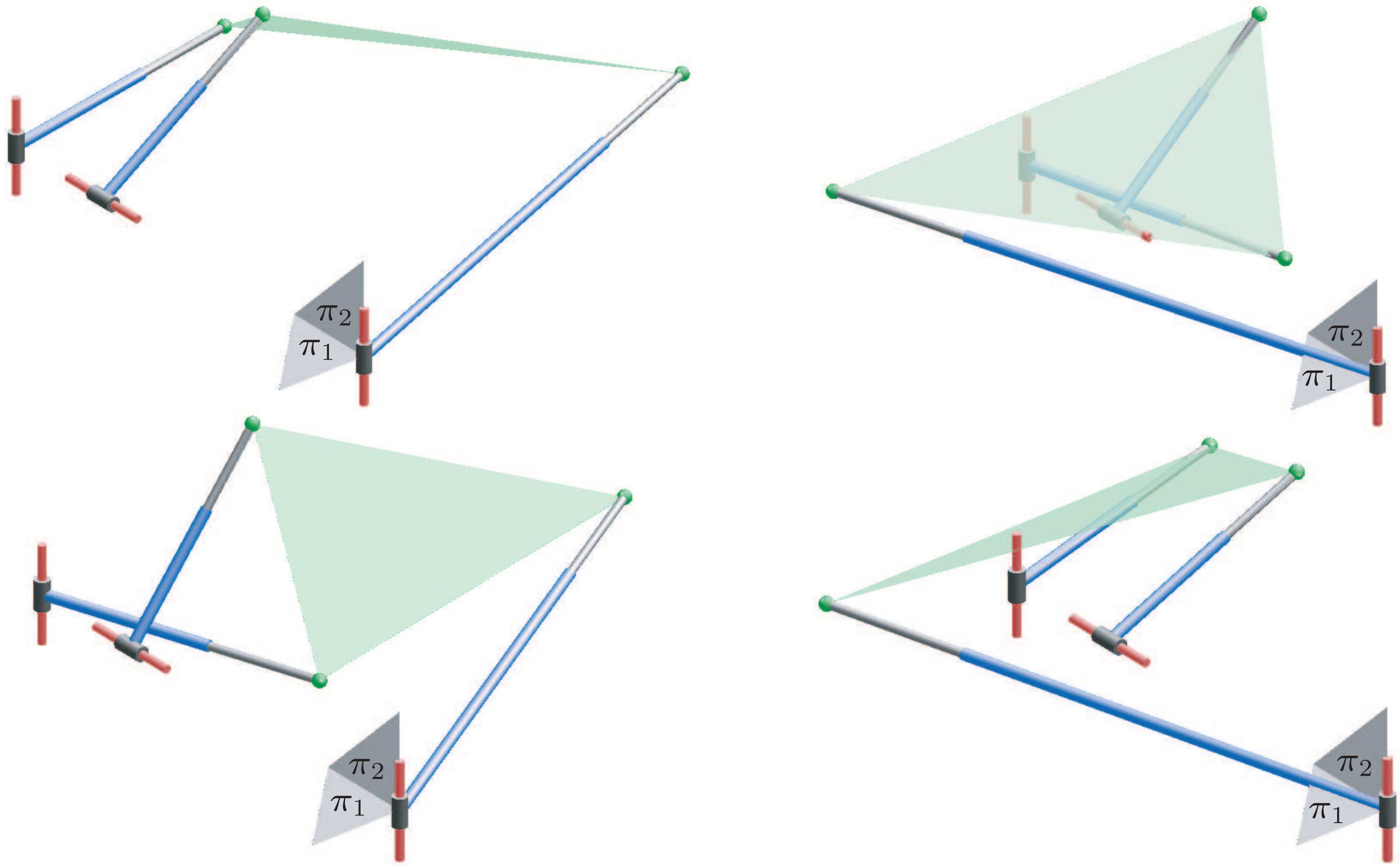
The upper bounds of assembly modes given in Theorem 2 and 3 cannot be improved due to the following examples:

- Theorem 3 $i)$: This manipulator corresponds to the original Stewart platform. An example with 12 real solutions was given by **Lazard and Merlet [4]**.

- Theorem 3 $ii)$: An example with 8 real solutions can easily be constructed by applying the same approach used in **[4]**.

- Theorem 3 $iii)$: The construction of an example with 4 real solutions is trivial.

- Theorem 2: An example of a GTSSM manipulator with two parallel rotary axes and 16 assembly modes was given by **Nawratil [8]**.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[5] References

[1] **Fichter E.F. and Hunt K.H. [1977]** Mechanical Couplings - A General Geometrical Theory, Trans. ASME B, Journal of Engineering for Industry **99** 77–81.

[2] **Hunt K.H. [1978]** Kinematic Geometry of Mechanisms, Clarendon Press, Oxford.

[3] **Hunt K.H. [1983]** Structural Kinematics of In-Parallel-Actuated Robot-Arms, Journal of Mechanisms, Transmissions, and Automation in Design **105** 705–712.

[4] **Lazard D. and Merlet J-P. [1994]** The (true) Stewart Platform has 12 configurations, In Proc. of IEEE International Conference on Robotics and Automation, 2160–2165.

[5] **Merlet J-P. [1989]** Manipulateurs paralléles, 4eme partie : mode d'assemblage et cinématique directe sous forme polynomiale, Technical Report 1135, INRIA.

[6] **Merlet J-P. [1992]** Direct Kinematics and Assembly modes of parallel manipulators, International Journal of Robotics Research **11** (2) 150–162.

[7] **Merlet J-P. [2006]** Parallel Robots, 2nd Edition, Springer.

[8] **Nawratil G. [to appear]** On the spin surface of RSSR mechanisms with parallel rotary axes, International Journal of Mechanisms and Robotics.