Composition of spherical four-bar-mechanisms

G. Nawratil and H. Stachel

Vienna University of Technology, Austria,
e-mail: `{nawratil,stachel}@dmg.tuwien.ac.at`

Abstract. We study the transmission by two consecutive four-bar linkages with aligned frame links. The paper focusses on so-called “reducible” examples on the sphere where the 4-4-correspondance between the input angle of the first four-bar and the output-angle of the second one splits. Also the question is discussed whether the components can equal the transmission of a single four-bar. A new family of reducible compositions is the spherical analogue of compositions involved at Burmester’s focal mechanism.

Key words: spherical four-bar linkage, overconstrained linkage, Kokotsakis mesh, Burmester’s focal mechanism, 4-4-correspondance

1 Introduction

Let a spherical four-bar linkage be given by the quadrangle $I_{10}A_1B_1I_{20}$ (see Fig. 1) with the frame link $I_{10}I_{20}$, the coupler A_1B_1 and the driving arm $I_{10}A_1$. We use the output angle φ_2 of this linkage as the input angle of a second coupler motion with vertices $I_{20}A_2B_2I_{30}$. The two frame links are assumed in aligned position as well as the driven arm $I_{20}B_1$ of the first four-bar and the driving arm $I_{20}A_2$ of the second one. This gives rise to the following

Questions:
(i) Can it happen that the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible so that the composition admits two one-parameter motions? In this case we call the composition *reducible*.
(ii) Can one of these components produce a transmission which equals that of a single four-bar linkage ?

A complete classification of such reducible compositions is still open, but some examples are known (see Sect. 3). For almost all of them exist planar counterparts. We focus on a case where the planar analogue is involved at Burmester’s focal mechanism [2, 5, 11, 4] (see Fig. 3a). It is not possible to transfer the complete focal mechanism onto the sphere as it is essentially based on the fact that the sum of interior angles in a planar quadrangle equals 2π, and this is no longer true in spherical geometry. Nevertheless, algebraic arguments show that the reducibility of the included four-bar compositions can be transferred.

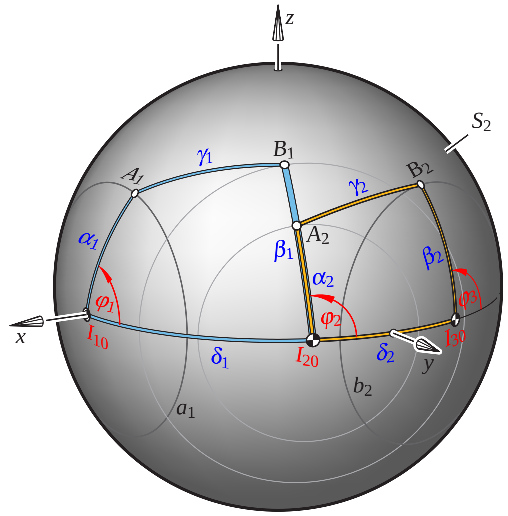

Fig. 1 Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2$

Remark: The problem under consideration is of importance for the classification of flexible Kokotsakis meshes [7, 1, 10]. This results from the fact that the spherical image of a flexible mesh consists of two compositions of spherical four-bars sharing the transmission $\varphi_1 \mapsto \varphi_3$. All the examples known up to recent [6, 10] are based on reducible compositions.

The geometry on the unit sphere S^2 contains some ambiguities. Therefore we introduce the following *notations and conventions:*

1. Each point A on S^2 has a diametrically opposed point $\overline{A}$, its *antipode*. For any two points A, B with $B \neq A, \overline{A}$ the *spherical segment* or *bar* AB stands for the shorter of the two connecting arcs on the great circle spanned by A and B. We denote this great circle by $[AB]$.
2. The *spherical distance* $\overline{AB}$ is defined as the arc length of the segment AB on S^2. We require $0 \leq \overline{AB} \leq \pi$ thus including also the limiting cases $B = A$ and $B = \overline{A}$.
3. The *oriented angle* $\sphericalangle ABC$ on S^2 is the angle of the rotation about the axis OB which carries the segment BA into a position aligned with the segment BC. This angle is oriented in the mathematical sense, if looking from outside, and can be bounded by $-\pi < \sphericalangle ABC \leq \pi$.

2 Transmission by a spherical four-bar linkage

We start with the analysis of the first spherical four-bar linkage with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 (Fig. 1). We set $\alpha_1 = \overline{I_{10}A_1}$ for the length of the driving arm, $\beta_1 = \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$, and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi\,.$$

The movement of the coupler remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi-\alpha_1$ and $\pi-\gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a cartesian coordinate frame with I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see Fig. 1). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}. This results in the following coordinates:

$$A_1 = \begin{pmatrix} \mathrm{c}\alpha_1 \\ \mathrm{s}\alpha_1\,\mathrm{c}\varphi_1 \\ \mathrm{s}\alpha_1\,\mathrm{s}\varphi_1 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} \mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{s}\beta_1\,\mathrm{s}\delta_1\,\mathrm{c}\varphi_2 \\ \mathrm{c}\beta_1\,\mathrm{s}\delta_1 + \mathrm{s}\beta_1\,\mathrm{c}\delta_1\,\mathrm{c}\varphi_2 \\ \mathrm{s}\beta_1\,\mathrm{s}\varphi_2 \end{pmatrix}.$$

Herein s and c are abbreviations for the sine and cosine function, respectively. In these equations the lengths α_1, β_1 and δ_1 are signed. The coordinates would also be valid for negative lengths. The constant length γ_1 of the coupler implies

$$\begin{aligned} &\mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1\,\mathrm{c}\varphi_2 + \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1\,\mathrm{c}\varphi_1 \\ &+ \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1\,\mathrm{c}\varphi_1\,\mathrm{c}\varphi_2 + \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\varphi_1\,\mathrm{s}\varphi_2 = \mathrm{c}\gamma_1. \end{aligned} \tag{1}$$

In comparison to [3] we emphasize algebraic aspects of this transmission. Hence we express $\mathrm{s}\varphi_i$ and $\mathrm{c}\varphi_i$ in terms of $t_i := \tan(\varphi_i/2)$ since t_1 is a *projective coordinate* of point A_1 on the circle a_1. The same is true for t_2 and $B_1 \in b_1$. From (1) we obtain

$$\begin{gathered} -K_1(1+t_1^2)(1-t_2^2) + L_1(1-t_1^2)(1+t_2^2) + M_1(1-t_1^2)(1-t_2^2) \\ +4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,t_1t_2 + N_1(1+t_1^2)(1+t_1^2) = 0\,, \\ \begin{aligned} K_1 &= \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1\,, & M_1 &= \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1\,, \\ L_1 &= \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1\,, & N_1 &= \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1\,. \end{aligned} \end{gathered} \tag{2}$$

This biquadratic equation describes a *2-2-correspondence* between points A_1 on circle $a_1 = (I_{10};\alpha_1)$ and B_1 on $b_1 = (I_{20};\beta_1)$. It can be abbreviated by

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{3}$$

setting

$$\begin{gathered} c_{00} = -K_1 + L_1 + M_1 + N_1\,, \quad c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,, \quad c_{02} = K_1 + L_1 - M_1 + N_1\,, \\ c_{20} = -K_1 - L_1 - M_1 + N_1\,, \qquad c_{22} = K_1 - L_1 + M_1 + N_1 \end{gathered} \tag{4}$$

under $c_{11} \neq 0$. Alternative expressions can be found in [10].

Remark: Also at planar four-bar linkages mechanisms there is a 2-2-correspondance of type (3).

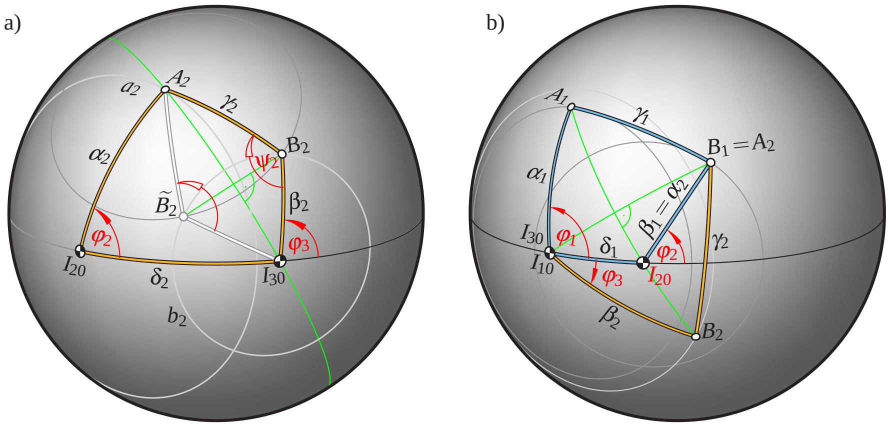

Fig. 2 a) Opposite angles φ_2 and ψ_2 at the second spherical four-bar $I_{20}A_2B_2I_{30}$.
b) Composition of two orthogonal four-bar linkages with $I_{30} = I_{10}$.

There are two particular cases:

Spherical isogram: Under the conditions $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ opposite sides of the quadrangle $I_{10}A_1B_1I_{20}$ have equal lengths. In this case we have $c_{00} = c_{22} = 0$ in (3), and eq. (1) converts into $[\mathrm{s}(\alpha_1-\gamma_1)t_2 - (\mathrm{s}\alpha_1+\mathrm{s}\gamma_1)t_1]\,[\mathrm{s}(\alpha_1-\gamma_1)t_2 - (\mathrm{s}\alpha_1-\mathrm{s}\gamma_1)t_1]$ (for details see [10]). *The 2-2-correspondance splits into two projectivities*[1] $t_1 \mapsto t_2 = \dfrac{\mathrm{s}\alpha_1 \pm \mathrm{s}\gamma_1}{\mathrm{s}(\alpha_1-\gamma_1)}\,t_1$, provided $\alpha_1 \neq \gamma_1, \pi-\gamma_1$. Both projectivities keep $t_1 = 0$ and $t_1 = \infty$ fixed. These parameters belong to the two aligned positions of coupler A_1B_1 and frame link $I_{10}I_{20}$. In these positions a bifurcation is possible between the two one-parameter motions of the coupler against the frame link.

Orthogonal case: For a given point $A_1 \in a_1$ the corresponding $B_1, \widetilde{B}_1 \in b_1$ are the points of intersection between the circles $(A_1;\gamma_1)$ and $b_1 = (I_{20};\beta_1)$ (compare Fig. 2a). Hence, the corresponding B_1 and $\widetilde{B}_1$ are located on a great circle perpendicular to the great circle $[A_1I_{20}]$. Under the condition $\cos\alpha_1\cos\beta_1 = \cos\gamma_1\cos\delta_1$ which according to [10] is equivalent to $\det\begin{pmatrix} c_{22} & c_{02} \\ c_{20} & c_{00}\end{pmatrix} = 0$, the diagonals of the spherical quadrangle $I_{10}A_1B_1I_{20}$ are orthogonal (Fig. 2b) as each of the products equals the products of cosines of the four segments on the two diagonals. Hence, B_1 and $\widetilde{B}_1$ are always aligned with I_{10}, but also conversely, the two points A_1 and $\widetilde{A}_1$ corresponding to B_1 are aligned with I_{20}.

Note that the 2-2-correspondence (3) depends only on the ratio of the coefficients $c_{22} : \cdots : c_{00}$. With the aid of a CA-system we can prove:

Lemma 1 *For any spherical four-bar linkage the coefficients c_{ik} defined by* (4) *obey*

$$c_{11}^6 + 16\,(K^2+L^2-2M^2-1)\,c_{11}^4 + 256\,[(M^2-K^2)(M^2-L^2)+2M^2]\,c_{11}^2 - 4096\,M^4 = 0\,.$$

[1] Since the vertices of the moving quadrangle can be replaced by their antipodes whithout changing the motion, this case is equivalent to $\beta_1 = \pi-\alpha_1$ and $\delta_1 = \pi-\gamma_1$. We will not mention this in the future but only refer to an ‘appropriate choice of orientations’ of the hinges.

Conversely, in the complex extension any biquadratic equation of type (3) *defines the spherical four-bar linkage uniquely — up to replacement of vertices by their antipodes. However, the vertices need not be real.*

At the end of our analysis we focus on opposite angles in the spherical quadrangle $I_{20}A_2B_2I_{30}$: The diagonal A_2I_{30} divides the quadrangle into two triangles, and we inspect the interior angles φ_2 at I_{20} and ψ_2 at B_2 (Fig. 2a). Also for non-convex quadrangles, the spherical Cosine Theorem implies

$$\cos\overline{A_2I_{30}} = \mathrm{c}\beta_2\,\mathrm{c}\gamma_2 + \mathrm{s}\beta_2\,\mathrm{s}\gamma_2\,\mathrm{c}\psi_2 = \mathrm{c}\alpha_2\,\mathrm{c}\delta_2 + \mathrm{s}\alpha_2\,\mathrm{s}\delta_2\,\mathrm{c}\varphi_2\,.$$

Hence there is a linear function

$$\mathrm{c}\psi_2 = k_2 + l_2\mathrm{c}\varphi_2 \quad \text{with} \quad k_2 = \frac{\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2}{\mathrm{s}\beta_2\,\mathrm{s}\gamma_2}\,, \quad l_2 = \frac{\mathrm{s}\alpha_2\,\mathrm{s}\delta_2}{\mathrm{s}\beta_2\,\mathrm{s}\gamma_2}\,. \tag{5}$$

For later use it is necessary to define also ψ_2 as an oriented angle, hence

$$\psi_2 = \sphericalangle I_{30}B_2A_2, \quad \varphi_2 = \sphericalangle I_{30}I_{20}A_2 \quad \text{under} \quad -\pi < \psi_2, \varphi_2 \le \pi\,.$$

We note that in general for given φ_2 there are two positions B_2 and $\widetilde{B}_2$ on the circle b_1 obeying (5) (Fig. 2a). They are placed symmetrically with respect to the diagonal A_2I_{30}; the signs of the corresponding oriented angles ψ_2 are different.

Remark: Also Eq. (5) describes a 2-2-correspondance of type (3) between φ_1 and φ_2, but with $c_{11} = 0$. A parameter count reveals that this 2-2-correspondance does not characterize the underlying four-bar uniquely.

3 Composition of two spherical four-bar linkages

Now we use the output angle φ_2 of the first four-bar linkage as input angle of a second coupler motion with vertices $I_{20}A_2B_2I_{30}$ and consecutive side lengths α_2, γ_2, β_2, and δ_2 (Fig. 1). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission.

Due to (3) the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$\begin{aligned} c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} &= 0\,, \\ d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} &= 0\,. \end{aligned} \tag{6}$$

The d_{ik} are defined by equations analogue to eqs. (4) and (2). We eliminate t_2 by computing the *resultant* of the two polynomials with respect to t_2 and obtain

$$\det\begin{pmatrix} c_{22}t_1^2+c_{02} & c_{11}t_1 & c_{20}t_1^2+c_{00} & 0 \\ 0 & c_{22}t_1^2+c_{02} & c_{11}t_1 & c_{20}t_1^2+c_{00} \\ d_{22}t_3^2+d_{20} & d_{11}t_3 & d_{02}t_3^2+d_{00} & 0 \\ 0 & d_{22}t_3^2+d_{20} & d_{11}t_3 & d_{02}t_3^2+d_{00} \end{pmatrix} = 0\,. \qquad (7)$$

This biquartic equation expresses a *4-4-correspondance* between points A_1 and B_2 on the circles a_1 and b_2, respectively (Fig. 1).

Up to recent, to the authors' best knowledge the following examples of reducible compositions are known. Under appropriate notation and orientation these are:

1. **Isogonal** type [7, 1]: At each four-bar opposite sides are congruent; the transmission $\varphi_1 \to \varphi_3$ is the product of two projectivities and therefore again a projectivity. Each of the 4 possibilities can be obtained by one single four-bar linkage. This is the spherical image of a flexible octahedron of Type 3 (see, e.g., [8]):
2. **Orthogonal** type [10]: We combine two orthogonal four-bars such that they have one diagonal in common (see Fig. 2b), i.e., under $\alpha_2 = \beta_1$ and $\delta_2 = -\delta_1$, hence $I_{30} = I_{10}$. Then the 4-4-correspondance between A_1 and B_2 is the square of a 2-2-correspondance.
3. **Symmetric** type [10]: We specify the second four-bar linkage as mirror of the first one after reflection in an angle bisector at I_{20} (see [10, Fig. 5b]). Thus φ_3 is congruent to the angle opposite to φ_1 in the first quadrangle. Hence the 4-4-correspondance is reducible; the components are expressed by the linear relation $\mathrm{c}\varphi_3 = \pm(k_1 + l_1\mathrm{c}\varphi_1)$ in analogy to (5).

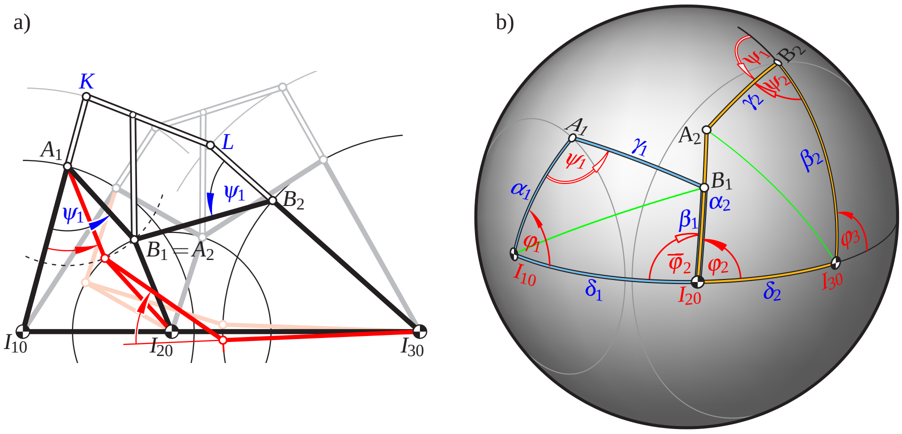

Fig. 3 a) Burmester's focal mechanism and the second component of a four-bar composition. b) Reducible spherical composition obeying Dixon's angle condition for ψ_1 — equally oriented

At the end we present a new family of reducible compositions: In Fig. 3a Burmester's focal mechanism is displayed, an overconstrained planar linkage (see [2, 5, 11, 4]). The full lines in this figure show a planar composition of two four-bar linkages with the additional property that the transmission $\varphi_1 \to \varphi_3$ equals that of one single four-bar linkage with the coupler KL. Due to Dixon and Wunderlich this composition is characterized by congruent angles $\psi_1 = \sphericalangle I_{10}A_1B_1$ and $\sphericalangle LB_2A_2$ which is adjacent to $\psi_2 = \sphericalangle I_{30}B_2A_2$.[2] However, this defines only one component of the full motion of this composition. The second component is defined by $\psi_1 = \sphericalangle I_{10}A_1B_1 = -\sphericalangle LB_2A_2$ (see Fig. 3a). For the sake of brevity, we call the overall condition $\sphericalangle I_{10}A_1B_1 = \pm \sphericalangle LB_2A_2$ *Dixon's angle condition* and prove in the sequel that also at the spherical analogue this defines reducible compositions.

Lemma 2 *For the composition of two spherical four-bars Dixon's angle condition* $\sphericalangle I_{10}A_1B_1 = \pm \sphericalangle \bar{I}_{30}B_2A_2$ *is equivalent to*

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \pm\mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2).$$

In terms of c_{ik} and d_{ik} it is equivalent to proportional polynomials

$$D_1 = (c_{11}t_2)^2 - 4(c_{22}t_2^2 + c_{20})(c_{02}t_2^2 + c_{00}), \quad D_2 = (d_{11}t_2)^2 - 4(d_{22}t_2^2 + d_{02})(d_{20}t_2^2 + d_{00}).$$

Proof. In the notation of Fig. 3b Dixon's angle condition is equivalent to $\mathrm{c}\psi_1 = \mathrm{c}(\pi - \psi_2) = -\mathrm{c}\psi_2 = -k_2 - l_2\,\mathrm{c}\varphi_2$ by (5). At the first four-bar we have analogously

$$\mathrm{c}\psi_1 = -k_1 - l_1\,\mathrm{c}\varphi_2\,, \quad k_1 = \frac{\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1}{\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1}\,, \quad l_1 = \frac{\mathrm{s}\beta_1\,\mathrm{s}\delta_1}{\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1}\,. \tag{8}$$

Hence, $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2$ for all $\mathrm{c}\varphi_2$ is equivalent to $k_1 = k_2$ and $l_1 = l_2$. This gives the first statement in Lemma 2. The $\pm$ results from the fact that changing the sign of γ_2 has no influence on the 2-2-correspondance $\varphi_2 \mapsto \varphi_3$, but replaces ψ_2 by $\psi_2 - \pi$.

If the angle condition holds and $\psi_1 = 0$ or π, the distances $\overline{I_{10}B_1}$ and $\overline{I_{30}A_2}$ are extremal. For the corresponding angles φ_2 there is just one corresponding φ_1 and one φ_3. Hence, when for any t_2 the corresponding t_1-values by (3) coincide, then also the corresponding t_3-values by (6) are coincident. Hence, the discriminants D_1 and D_2 of the two equations in (6) — when solved for t_2 — have the same real or pairwise complex conjugate roots.

Conversely, proportional polynomials D_1 and D_2 have equal zeros. Hence the linear functions in (5) and (8) give the same $\mathrm{c}\varphi_2$ for $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2 = \pm 1$. Therefore $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2$ is true in all positions, and the composition of the two four-bars fulfills Dixon's angle condition. □

The second characterization in Lemma 1 is also valid in the planar case. So, the algebraic essence is the same on the sphere and in the plane. Since in the plane the reducibility is guaranteed, the same must hold on the sphere. This can also be confirmed with the aid of a CA-system: The resultant splits into two biquadratic polynomials like the left hand side in (3). By Lemma 1 each component equals the transmission by a spherical four-bar, but the length of the frame link differs from the distance $\overline{I_{10}I_{30}}$ because otherwise this would contradict the classification of flexible octahedra. General results on conditions guaranteeing real four-bars have not yet been found. We summarize:

[2] This condition is invariant against exchanging the input and the output link. The compositions along the other sides of the four-bar $I_{10}KLI_{30}$ in Fig. 3a obey analogous angle conditions.

Theorem 3 *Any composition of two spherical four-bar linkages obeying Dixon's angle condition* $\psi_1 = \sphericalangle I_{10}A_1B_1 = \pm \sphericalangle \overline{I}_{30}B_2A_2$ *(see Fig. 3b) is reducible. Each component equals the transmission* $\varphi_1 \to \varphi_3$ *of a single, but not necessarily real spherical four-bar linkage.*

Example: The data $\alpha_1 = 38.00°$, $\beta_1 = 26.00°$, $\gamma_1 = 41.50°$, $\delta_1 = 58.00°$, $\alpha_2 = -40.0400°$, $\beta_2 = 123.1481°$, $\gamma_2 = -123.3729°$, $\delta_2 = 82.0736°$ yield a reducible 4-4-correspondence according to Theorem 3. The components define spherical four-bars with lengths $\alpha_3 = 60.2053°$, $\beta_3 = 53.5319°$, $\gamma_3 = 8.6648°$, $\delta_3 = 14.5330°$ or $\alpha_4 = 24.7792°$, $\beta_4 = 157.1453°$, $\gamma_4 = 160.4852°$, $\delta_4 = 33.8081°$.

4 Conclusions

We studied compositions of two spherical four-bar linkages where the 4-4-correspondance between the input angle φ_1 and output angle φ_3 is reducible. We presented a new family of reducible compositions. However, a complete classification is still open. It should also be interesting to apply the principle of transference (e.g., [9]) in order to study dual extensions of these spherical mechanisms.

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