# Note on Flecnodes

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**Abstract**

The flecnodes $F_i$ on a regular and non torsal ruling $R_0$ of a ruled surface $R$ are the points where $R$'s asymptotic tangents along $R_0$ hyperosculate the ruled surface. The name flecnode characterizes the intersection curve $c_i$ of the tangent plane $\tau_i$ with $R$ at $F_i$. It has a double point (a node) at $F_i$ and this node is an inflection point for both linear branches of $c_i$ at $F_i$.
We show a way to parameterize the smooth one-parameter family of flecnodes of $R$ which in general forms a curve with two branches. For that we derive the equation of the ruled quadric on three given lines in terms of Plücker coordinates of the given lines.

## 1 Introduction

The curve of *flecnodes* or *flecnodal curve* has attracted not that much interest to mathematicians compared to other geometric objects related to ruled surfaces. One reason for that maybe the mere absence of parameterizations. Another reason could be the following: Any curve can be considered as the striction curve of a ruled surface and the ruled surface is still not determined uniquely. So there is left plenty of freedom, which may be attractive for design porposes. This is not the case for the two curves of flecnodes. They cannot be chosen freely in order to find ruled surfaces passing through it, see [7] and so they are not that flexible. The curve of flecnodes is related to a ruled surface in a projectively invariant way [1, 2, 5, 8]. To the best of the author's knowledge the thesis [7] was the latest and maybe most exhaustive and comprehensive work on flecnodal curves.

In this note we want to show a way to parameterize the curves of flecnodes on a ruled surface. Firstly, we give a very brief introduction to the differential geometry of ruled surfaces in the Klein model of line space. In this section we try to keep things as short as possible. An introduction to projective differential geometry can be found for instance in the monographs [1, 2, 5, 10]. Secondly, we give the equation of a ruled quadric on three given lines in terms of the Plücker coordinates of these lines. Then we show that the Plücker coordinates of any three independent linear line complexes which carry the regulus of the ruled quadric can be used to derive the equation of the quadric carrying the regulus. Thereby we obtain a simple formula for the analytic solution to the problem of finding the lines intersecting four arbitrarily given lines in three-space. Thirdly, we are able to give a parameterization of the curve of flecnodes. Finally we point at an approximation of the curve of flecnodes, i.e., we show how to find a discrete model for it. This construction is justified by the following observation: If the lines of a discrete ruled surface are taken from a discretization of a sufficiently smooth ruled surface, then the discrete version of the flecnodal curve will converge to the smooth flecnodal curve, provided that the discretization of the ruled surface converges to the smooth surface.

# 2 Projective differential geometry of ruled surfaces

## 2.1 Klein's model of line space

In the following we use homogeneous Plücker coordinates $L = (\mathbf{l}, \bar{\mathbf{l}}) \in \mathbb{R}^6$ in order to describe a line $L$ in projective three-space $\mathbb{P}^3$. We remark that the coordinates $(\mathbf{l}, \bar{\mathbf{l}})$ for $L$ are unique only up to a non zero factor. Further they satisfy

$$\Omega(L, L) := 2\langle \mathbf{l}, \bar{\mathbf{l}} \rangle = 0, \tag{1}$$

with $\langle \cdot, \cdot \rangle$ being the canonical scalar product in $\mathbb{R}^3$. This identity will henceforth be referred to as the *Plücker identity.* Any vector from $\mathbb{R}^6 \setminus \{\mathbf{o}\}$ that satisfies Eq. (1) is a coordinate vector of a line in $\mathbb{P}^3$. For further reading on Plücker coordinates and their properties, we strongly recommend the study of [9, 14].

Since the coordinates $(\mathbf{l}, \bar{\mathbf{l}})$ of a line are homogeneous, we can interpret them as homogeneous coordinates of points in a projective space $\mathbb{P}^5$ of five dimensions. The mapping $\gamma$ that assigns to each line in $\mathbb{P}^3$ a point in $\mathbb{P}^5$ is usually called *Klein mapping* and it is one-to-one and onto, if considered as a mapping to the quadratic hypersurface $M_2^4 \subset \mathbb{P}^5$ given by the equation (1). The manifold $M_2^4$ is called *Klein's quadric* or *Plücker's quadric.* It carries two three-parameter families of planes corresponding to the stars of lines and ruled planes in $\mathbb{P}^3$. The lines in $M_2^4$ are the $\gamma$-images of pencils of lines in projective three-space [9, 14].

Intersecting lines $L$ and $M$ in $\mathbb{P}^3$ are mapped to points which are polar with regard to $M_2^4$. In terms of Plücker coordinates this is expressed by

$$\Omega(L, M) = \langle \mathbf{l}, \overline{\mathbf{m}} \rangle + \langle \bar{\mathbf{l}}, \mathbf{m} \rangle = 0, \tag{2}$$

i.e., the respective coordinate vectors of $L$ and $M$ annihilate the polarform $\Omega$ of $M_2^4$. Points $C = (\mathbf{c}, \overline{\mathbf{c}}) \notin M_2^4$ are the so called *extended Klein images of regular line complexes*, see [9, 14]. The Plücker coordinates of the lines of the complex $C$ fulfill $\Omega(C, X) = 0$. So their Klein images are contained in a hyperplanar section of $M_2^4$. A tangential hyperplane intersects $M_2^4$ in a three-dimensional quadratic cone $\Gamma$, whose vertex is the Klein image of a line $A$. $A$ is met by all the lines of the so called *singular linear line complex.* $A$ is said to be the axis of the complex. Further information on axes of linear line complexes, their computation, and their geometric meaning for singular as well as regular linear line complexes can be found in [9].

## 2.2 Differential geometric properties of ruled surfaces

If a curve $R \subset M_2^4$ is a $C^k$-curve in $M_2^4$ then its Klein preimage is said to be a *$C^k$-ruled surface.* An algebraic ruled surface $R$ is defined by an algebraic curve in $M_2^4$ and the algebraic degree of the curve and the ruled surface agree. In the following we denote the ruled surface as well as the curve in Plücker's quadric by the same letter, say $R$. Confusions will not occur.

Assume now $R: \ I \subset \mathbb{R} \to M_2^4$ is a $C^k$-ruled surface, where $k$ is at least 3. We can derive the first derivative points up to order 3 at a certain - and in the following not specified - value $t_0 \in I$ and denote them by $R_0 := R^{(0)}$, $\dot{R}$, $\ddot{R}$, and $\dddot{R}$. We keep in mind that the independency of points in projective space is equivalent to the linear independency of their respective coordinate vectors.

#### 2.2.1 Properties of first order

The point $R_0$ is either a regular or a singular point on $R \in M_2^4$, if $R_0$ and $\dot{R}$ are independent or not. Likewise we can say that $R_0$ is a *regular* or *singular ruling* on $R \subset \mathbb{P}^3$. In the following $[X_1, \ldots, X_k]$ denotes the projective subspace spanned by $k$ points $X_1, \ldots, X_k$. The line $T := [R_0, \dot{R}]$ is a tangent to both $R$ and $M_2^4$ at $R_0$. If now $T \subset M_2^4$, then $R_0$ is called a *torsal ruling* on $R$. For further details we refer the interested reader to [1, 5, 9, 10]. In the following we consider only those parts of ruled surfaces which are free of singular and torsal rulings.

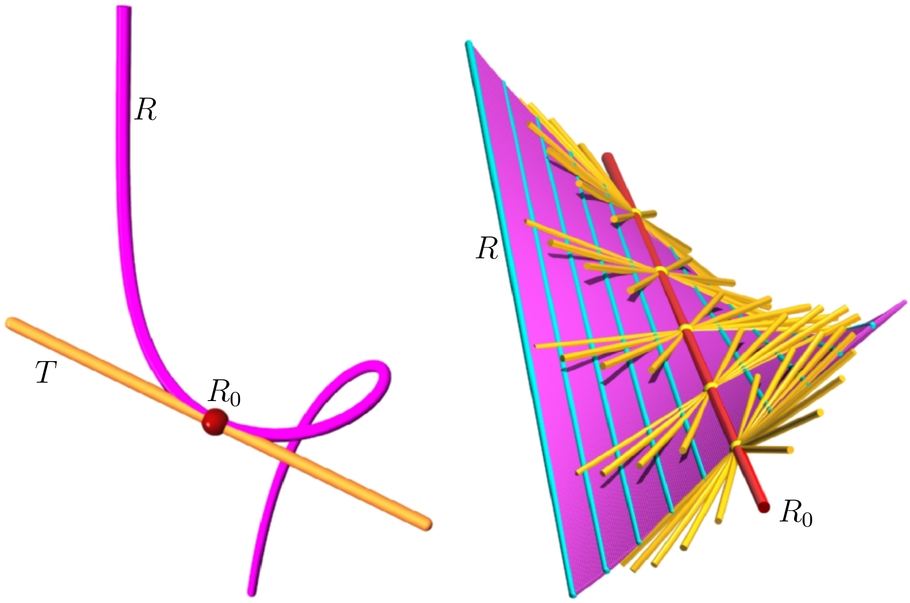


Figure 1: Left: Curve $R$ and a line element $(R_0, T)$ in the model space. Right: Parabolic linear line congruence of surface tangents of $R$ along $R_0$.

#### 2.2.2 Properties of second order

The plane $S = [R_0, \dot{R}, \ddot{R}]$ is the osculating plane of $R \subset M_2^4$ at $R_0$. In general $k := S \cap M_2^4$ is a conic section. Its $\gamma$-preimage is a regulus, i.e., one family of generators on a ruled quadric $L \subset \mathbb{P}^3$. If $S \subset M_2^4$, then $S$ is either a plane of the first kind (representing a star of lines in $\mathbb{P}^3$) or it is a plane of the second kind (representing a ruled plane). In the first case $\gamma^{-1}(k)$ is a quadratic cone and thus $R_0$ is a torsal generator. In the second case $\gamma^{-1}(k)$ is the set of tangents to a conic section. In both cases we have $T \subset M_2^4$ and $R_0$ is torsal which is excluded. So these two cases will not occur. Therefore in our case $L$ is a regular ruled quadric. It is called *Lie's osculating quadric*. $L$ and $R$ share the ruling $R_0$, the set of tangent planes along $R_0$, and the asymptotic tangents along $R_0$. The latter comprise the second family of rulings on Lie's osculating quadric $L$, cf. [1, 5, 9, 10].

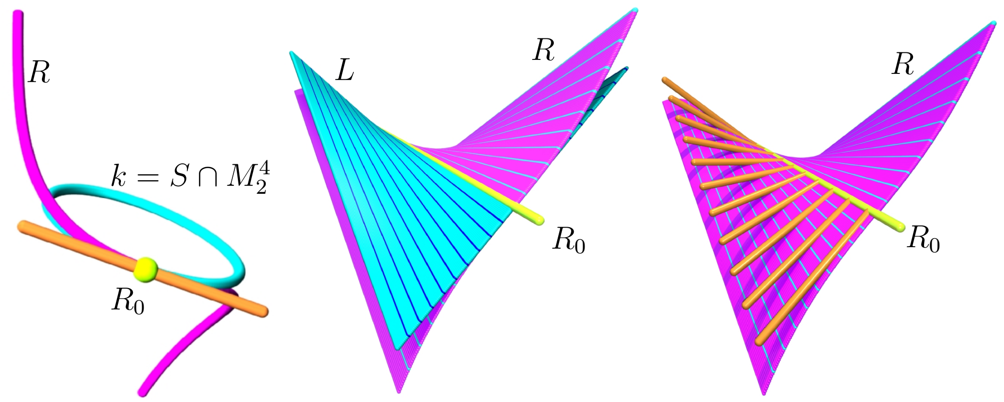


Figure 2: Left: Curve $R$ and the osculating conic section $k$ at $R_0$ in the Klein model. Middle: Osculating quadric $L \supset \gamma^{-1}(k)$ of $R$ at $R_0$. Right: Common asymptotic tangents of $L$ and $R$.

Fig. 2 shows a linear image of the curve $R \subset M_2^4$ together with the osculating conic section $k = S \cap M_2^4$. The respective $\gamma$-preimages are also shown. Further we see the common asymptotic tangents of $L$ and $R$ at $R_0$, which comprise the second family of lines on $L$.

As is the case for all ruled quadrics, Lie's osculating quadric carries two families of generators. The first family contains the line $R_0$. The second family consists of the set of *asymptotic tangents* of $L$ as well as $R$ at all points of $R_0$, see [1, 5, 6, 9, 10].

### 2.2.3 Properties of third order

The osculating three-space $O := [R_0, \dot{R}, \ddot{R}, \dddot{R}]$ of $R$ at $R_0$ meets Plücker's quadric $M_2^4$ in a two-dimensional quadric $Q$. This quadric is the Klein image of a *linear line congruence* $C$, see [9]. From $S \subset O$ we deduce $L \subset C$.

There are four types of linear line congruences to be distinguished: If $Q$ is a regular ruled quadric, then $C$ is usually called *hyperbolic*. Oval quadrics represent *elliptic linear line congruences*, which are sometimes called spread, see for example [3]. If $Q$ is a quadratic cone, then $C$ is known as *parabolic linear line congruence*. The case where $Q$ consists of two planes (which then intersect in a common line $\subset M_2^4$) belongs to the *singular linear line congruence*.

The hyperbolic as well as the elliptic linear line congruence can be generated as set of lines intersecting a pair of skew lines, called the *axes of the congruence*. In the hyperbolic case the axes are a pair of real and skew lines whereas the axes of an elliptic linear line congruence are a pair of (skew) conjugate complex lines. The parabolic linear line congruence somehow differs: There is only one axis (belonging to the linear line congruence, which is not the case for the other types). The lines of the parabolic linear line congruence can be arranged in pencils of lines whose vertices are located at the axis and whose planes (all of them passing through the axis) are mapped via a projective map to the vertices. The singular linear line congruence is the union of a star of lines with a ruled plane, where the star's vertex is contained in the plane in $\mathbb{P}^3$. Note that the surface tangents of any ruled surface $R$ behave that way at any regular and non-torsal ruling $R_0$. Further details on line congruences, especially linear ones can be found in [9, 14].

So $O$ contains the Klein images of lines in a linear line congruence. As outlined before, each linear line congruence, except the singular one, has at least one axis $A$. An axis $A$ has the property that it meets all the lines of the linear congruence. If we are looking for the lines, meeting all the lines in the congruence, we have to look for the intersection of $M_2^4$ with $O$'s polar space $P^1$ with regard to $M_2^4$. Obviously, these two points[1] are contained in the polar image $k^*$ of $k$, which is again a conic section of $M_2^4$. The conic section $k^*$ is the Klein image of $L$'s regulus of the second kind, i.e., the set of asymptotic tangents of $R$ along $R_0$. Consequently we have found *two osculating tangents* of $R$ along $R_0$ which are in *third order contact with* $R$ at certain points $F_i \in R_0$. The points $F_i$ of contact are called the *flecnodes* of $R_0$.

Figure 3 shows a part of a ruled surface $R$ together with parts of the curve of flecnodes $f_i$. The intersection curves $s_i$ of $R$ with the tangent planes $\tau_i$ at the flecnodes $F_i$ on a specific ruling are also shown.

The name *flecnode* is motivated by the following observation: The tangent plane $\tau_i$ to $R$ at $F_i$ intersects $R$ at a curve $c_i = s_i \cup R_0$. As $R_0$ is a straight line,

[1] These two points are obtained as the solutions of a quadratic equation. Therefore they can be real, conjugate complex, or they can coincide, and the case $P^1 \subset M_2^4$ will be ignored for the moment.

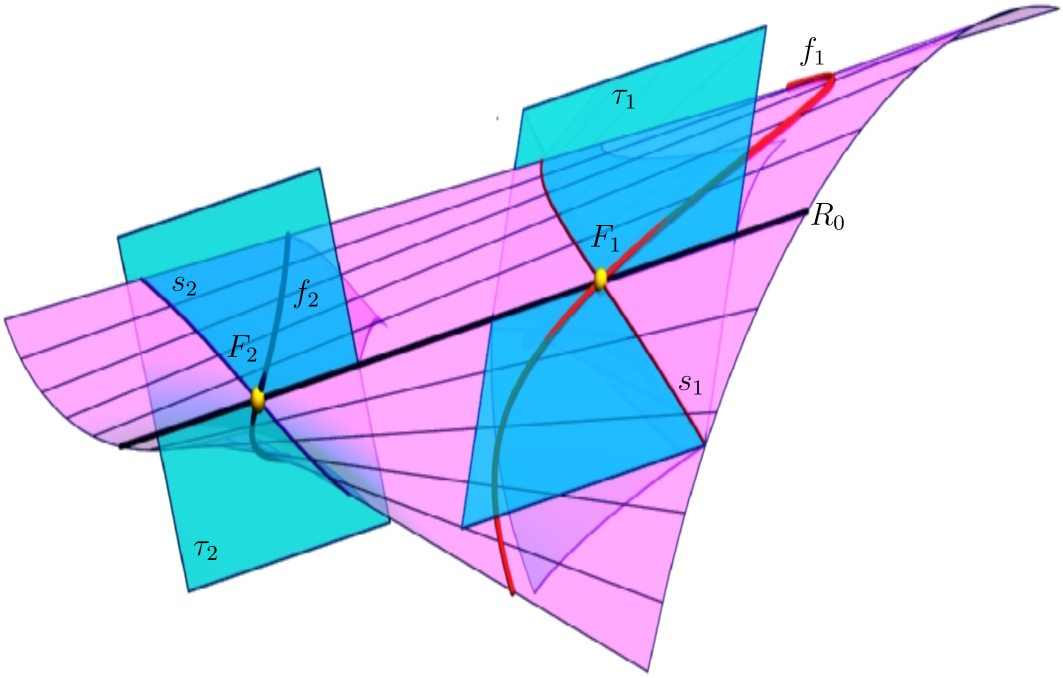


Figure 3: The two branches $f_i$ of the curve of flecnodes on $R$ in a neighbourhood of the ruling $R_0$ and the intersection curves $c_i = s_i \cup R_0$ of both tangent planes $\tau_i$ at the flecnodes $F_i$.

it carries only inflection points. The second branch $s_i$ touches the asymptotic tangent at $F_i$ and has a point of inflection, if and only if, $F_i$ is a flecnode. So $F_i$ is an inflection point for both linear branches of the planar intersection curves. So the name *flecnode* which is usually and originally used for planar curves (see e.g. [4, 12]) is carried over to the ruled surface. Fig. 4 displays an example of a flecnode.

# 3 The ruled quadric on three lines

Assume now we are given three independent lines $A = (\mathbf{a}, \overline{\mathbf{a}})$, $B = (\mathbf{b}, \overline{\mathbf{b}})$, and $C = (\mathbf{c}, \overline{\mathbf{c}})$, i.e., the respective points in $\mathbb{P}^5$ are independent and thus $P := [A, B, C] \subset \mathbb{P}^5$ is a plane. We exclude the cases where $P \subset M_2^4$, since then $A$, $B$, and $C$ do not span a uniquely defined regular ruled quadric. We state and proof the following:

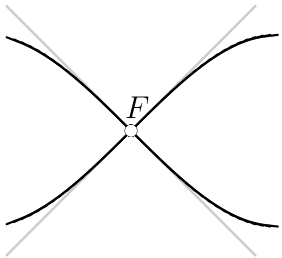


Figure 4: A planar curve with flecnode $F$.

**Lemma 3.1**
*Let $A = (\mathbf{a}, \overline{\mathbf{a}})$, $B = (\mathbf{b}, \overline{\mathbf{b}})$, and $C = (\mathbf{c}, \overline{\mathbf{c}})$ be three independent lines belonging to the same family of rulings on a regular ruled quadric $Q \subset \mathbb{P}^3$. Then the equation of $Q$ in terms of homogeneous point coordinates $(x_0, x_1, x_2, x_3) = (x_0, \mathbf{x})$ can be written in the form*

$$\begin{aligned}\langle \mathbf{x}, \overline{\mathbf{a}}\rangle \det(\mathbf{x}, \mathbf{b}, \mathbf{c}) + \langle \mathbf{x}, \overline{\mathbf{b}}\rangle \det(\mathbf{x}, \mathbf{c}, \mathbf{a}) + \langle \mathbf{x}, \overline{\mathbf{c}}\rangle \det(\mathbf{x}, \mathbf{a}, \mathbf{b})\\ +x_0(\langle \mathbf{x}, \overline{\mathbf{a}}\rangle(\langle \mathbf{b}, \overline{\mathbf{c}}\rangle - \langle \overline{\mathbf{b}}, \mathbf{c}\rangle) + \langle \mathbf{x}, \overline{\mathbf{b}}\rangle(\langle \mathbf{c}, \overline{\mathbf{a}}\rangle - \langle \overline{\mathbf{c}}, \mathbf{a}\rangle)\\ +\langle \mathbf{x}, \overline{\mathbf{c}}\rangle(\langle \mathbf{a}, \overline{\mathbf{b}}\rangle - \langle \overline{\mathbf{a}}, \mathbf{b}\rangle)) + x_0^2 \det(\overline{\mathbf{a}}, \overline{\mathbf{b}}, \overline{\mathbf{c}}) = 0.\end{aligned} \tag{3}$$

*Proof:* At first we note that $A$, $B$, and $C$ belong to one family of rulings on $Q$. Thus they are pairwise skew, i.e., no pair of Klein images is polar with regard to $M_2^4$. The other family of rulings is the set of lines $L = (\mathbf{l}, \overline{\mathbf{l}})$ intersecting all the three lines. So their Plücker coordinates have to fulfill

$$\Omega(A, L) = \Omega(B, L) = \Omega(C, L) = \Omega(L, L) = 0 \tag{4}$$

for any $L = \lambda A + \mu B + \nu C$ with $(\lambda : \mu : \nu) \neq (0 : 0 : 0)$. From these three intersection conditions we derive a point representation of $Q$. Assume a proper line $L \subset Q$ is spanned by two points $X = (x_0, \mathbf{x})$ and $L_u = (0, \mathbf{l})$, where $x_0 \neq 0$, which can always be achieved by chosing an apropriate coordinate frame. [2] The Plücker coordinates of $L$ now read $L = (x_0\mathbf{l}, \overline{\mathbf{l}})$. Then we can rewrite the intersection conditions (4) as

$$\langle \mathbf{a} \times \mathbf{x} + x_0\overline{\mathbf{a}}, \mathbf{l}\rangle = \langle \mathbf{b} \times \mathbf{x} + x_0\overline{\mathbf{b}}, \mathbf{l}\rangle = \langle \mathbf{c} \times \mathbf{x} + x_0\overline{\mathbf{c}}, \mathbf{l}\rangle = 0. \tag{5}$$

[2] At most two lines on $Q$ are ideal lines (spanned by two ideal points), but they also satisfy the intersection condition (4). Therefore it means no restriction to assume that $L$ is proper and the points $X$ and $L_u$ span $L$.

From that we conclude that the vectors $\mathbf{a}_\mathbf{x} := \mathbf{a} \times \mathbf{x} + x_0\overline{\mathbf{a}} \in \mathbb{R}^3$, $\mathbf{b}_\mathbf{x} := \mathbf{b} \times \mathbf{x} + x_0\overline{\mathbf{b}} \in \mathbb{R}^3$, and $\mathbf{c}_\mathbf{x} := \mathbf{c} \times \mathbf{x} + x_0\overline{\mathbf{c}} \in \mathbb{R}^3$ are linearly dependent. Thus $\det(\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}, \mathbf{c}_\mathbf{x}) = 0$ which leads to (3) and proves the lemma. $\square$

Remark: It is also possible to find the equation of the quadric on three lines in terms of Plücker coordinates as the determinant of a $6 \times 6$-matrix, see [15, Vol. 1, p. 332].

Obviously, Lemma 3.1 can also be used in order to determine the equation of the ruled quadric $Q$ carried by three independent linear line complexes, say $C_0$, $C_1$ and $C_2$. This is not clear from the above deduction of $Q$'s equation. But we are able to show:

**Lemma 3.2**
*Let $C_0 = (\mathbf{c}_0, \overline{\mathbf{c}}_0)$, $C_1 = (\mathbf{c}_1, \overline{\mathbf{c}}_1)$, and $C_2 = (\mathbf{c}_2, \overline{\mathbf{c}}_2)$ be three independent points in $\mathbb{P}^5$, which are not necessarily contained in $M_2^4$. Then the equation of the ruled quadric $Q$ whose generators of a certain kind are contained in all the three complexes $C_0$, $C_1$, and $C_2$ reads*

$$\begin{aligned}
&\langle \mathbf{x}, \overline{\mathbf{c}}_0 \rangle \det(\mathbf{x}, \mathbf{c}_1, \mathbf{c}_2) + \langle \mathbf{x}, \overline{\mathbf{c}}_1 \rangle \det(\mathbf{x}, \mathbf{c}_2, \mathbf{c}_0) + \langle \mathbf{x}, \overline{\mathbf{c}}_2 \rangle \det(\mathbf{x}, \mathbf{c}_0, \mathbf{c}_1) \\
&\quad + x_0(\langle \mathbf{x}, \overline{\mathbf{c}}_0 \rangle (\langle \mathbf{c}_1, \overline{\mathbf{c}}_2 \rangle - \langle \overline{\mathbf{c}}_1, \mathbf{c}_2 \rangle) + \langle \mathbf{x}, \overline{\mathbf{c}}_1 \rangle (\langle \mathbf{c}_2, \overline{\mathbf{c}}_0 \rangle - \langle \overline{\mathbf{c}}_2, \mathbf{c}_0 \rangle) \\
&\quad\quad + \langle \mathbf{x}, \overline{\mathbf{c}}_2 \rangle (\langle \mathbf{c}_0, \overline{\mathbf{c}}_1 \rangle - \langle \overline{\mathbf{c}}_0, \mathbf{c}_1 \rangle)) + x_0^2 \det(\overline{\mathbf{c}}_0, \overline{\mathbf{c}}_1, \overline{\mathbf{c}}_2) = 0.
\end{aligned} \tag{6}$$

Remark: The equation of a ruled quadric $Q$ whose rulings of one specific kind are contained in a two-parameter family of linear line complexes is given by Eq. (6) whether $Q$ carries real rulings or not. In the following we do not consider the case of an oval quadric, since later when we compute the osculating quadric $L$ of a ruled surface $R$ we can be sure that $L$ contains at least one ruling of $R$. Therefore it will be ruled or singular.

*Proof:* Following the remark we can assume that $C_0 \in M_2^4$ is a singular linear line complex, i.e., a straight line in $\mathbb{P}^3$. Once we have found one point $C_0 \in M_2^4$ which is a point on the conic section $k := [C_0, C_1, C_2] \cap M_2^4$ we can use it for a base point of a (rational) parameterization of $k$. Therefore we can assume that $C_1$ is a further point on $k$ and $C_2$ is the intersection of the tangents at $C_0$ and $C_1$, respectively. Now the two-parameter family of linear line complexes is given by

$$K(\lambda, \mu, \nu) = C_0\lambda + C_1\mu + C_2\nu. \tag{7}$$

From $\Omega(K, K) = 0$ we compute $(\lambda : \mu : \nu)$. For sake of simplicity we define $\Omega(C_i, C_j) := \Omega_{01}^C$. We observe $\Omega_{00}^C = \Omega_{11}^C = 0$ since $C_0$ and $C_1$ are points in

$M_2^4$. Further we have $\Omega_{02}^C = \Omega_{12}^C$ since $C_2$ is the common point of $k$'s tangents at $C_0$ and $C_2$, respectively. Now the equation of $k$ is $2\lambda\mu\Omega_{01}^C + \nu^2\Omega_{22}^C = 0$ and besides the points $C_0$ and $C_1$ we find $P = -2\Omega_{01}C_0 + \Omega_{22}C_1 + 2\Omega_{01}C_2$ for a further point on $k$.

Now we compute the ruled quadric on the three lines $C_0$, $C_1$, and $P$ according to Eq. (3). This yields Eq. (6) and completes the proof. □

Remark: The equation of Lie's osculating quadric $L$ of a ruled surface at $R_0$ can thus be derived by Eq. (6), if we let $C_0 = R_0$, $C_1 = \dot{R}$, and $C_2 = \ddot{R}$. In [15, Vol. 2, p. 51] it is shown how to write Lie's osculating quadric in terms of Plücker coordinates as the determinant of a $6 \times 6$-matrix.

The proof of Lemma 3.2 uses a rational parameterization of the conic section of $M_2^4$ and the plane spanned by three indpendent points $C_0$, $C_1$, and $C_2$ in order to make formula (3) applicable. An equivalent approach to rational representations of conic sections can be found in [13].

# 4 The lines meeting four arbitrary lines

## 4.1 The classical point of view

In this section we describe a classical problem in line geometry. We reformulate this problem in order to see that it is related to the problem of finding flecnodes. Then we see that solving this classical problem actually is the same as looking for flecnodes.

Assume we are given four arbitrary independent and pairwise skew lines $A$, $B$, $C$, and $D$ in projective three space. How to find the lines $L$ meeting the four given ones?

The solution to this problem sees the following considerations. Three of the lines, say $A$, $B$, and $C$ span a ruled quadric $Q$. The fourth line $D$ intersects $Q$ in two points $S_1$ and $S_2$, respectively. (Since $S_i$ are found as solutions of a quadratic equation, they can either be a pair of real points, or a pair of conjugate complex points, or one single real point with multiplicity two.) In any ruled quadric there are two lines passing through any point. So there are two generators $T_i$ of $Q$ passing through $S_i$ which are not from the same family of generators like the lines $A$, $B$, and $C$. The lines $T_i$ meet $A$, $B$, $C$, and by construction they also meet $D$, and thus they are the solutions to the problem. This is illustrated in Fig. 5.

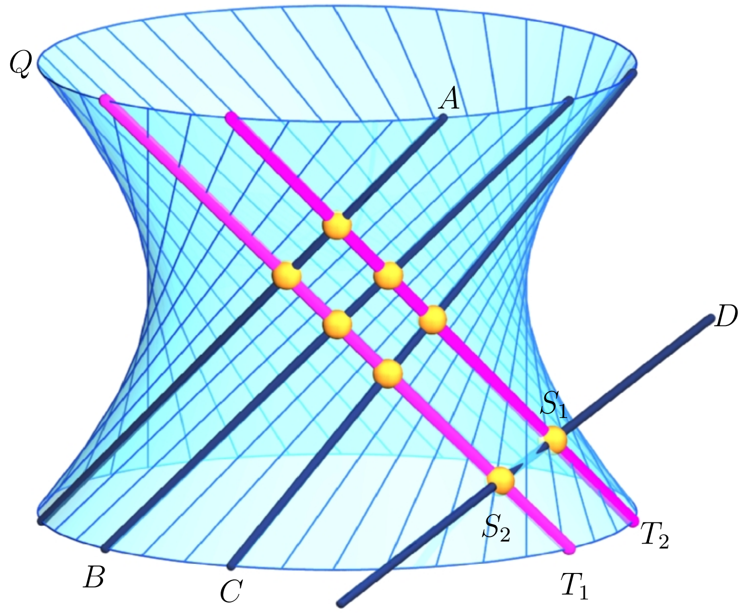


Figure 5: The lines $T_i$ meeting $A$, $B$, $C$, and $D$ and the ruled quadric on $A$, $B$, $C$.

The computation $T_i$ uses the following steps: Use Eq. (3) in order to derive the equation of the quadric $Q$ on the lines $A$, $B$, and $C$. Parameterize $D = (\mathbf{d}, \overline{\mathbf{d}})$ and insert it for $x = (x_0, \mathbf{x})$ into (3). This yields the intersection points of $D$ and $Q$, and from this place it is elementary to find the desired lines.

On the other hand one can intersect $M_2^4$ with the line $P \subset \mathbb{P}^5$ polar to $[A, B, C, D]$ with regard to $M_2^4$. This immediately gives the Plücker coordinates of the lines intersecting the four given ones.

## 4.2 Another point of view

Let $C_0$, $C_1$, $C_2$, and $C_3$ be four independent linear complexes of lines. They span a three-dimensional linear space of linear line complexes. Obviously this three-space intersects $M_2^4$ in a two-dimensional quadric whose points corresponds to the lines in a linear line congruence. Therefore a more general formulation of the above problem would read: Find the axes of a linear line congruence spanned by four independent linear line complexes. These complexes can be singular ones, i.e., straight lines in $\mathbb{P}^3$, but this is already discussed.

We find the axes of the linear line congruence $K(\kappa,\lambda,\mu,\nu) = \kappa C_0 + \lambda C_1 + \mu C_2 + \nu C_3$ as those lines whose Klein images are the intersection points of $M_2^4$ with $K$'s polar line with regard to $M_2^4$.

# 5 The parameterization of the curve of flecnodes

Let $R: \ I \subset \mathbb{R} \to M_2^4$ be a $C^3$-curve, i.e., the Klein image of a $C^3$-ruled surface in $\mathbb{P}^3$. Assume further that $R$ is free of singular and torsal rulings in $I$, and further that $\dim[R_0, \dot{R}, \ddot{R}, \dddot{R}] = 3$ in $I$.

Now we are able to compute the flecnodes on any ruling $R_0$ of $R$ in $I$. For that we solve the problem of finding axes of a linear line congruence spanned by the linear line complexes $R_0$, $\dot{R}$, $\ddot{R}$, and $\dddot{R}$. Note that $R_0$ is a singular linear line complex and $T = [R_0, \dot{R}]$ is tangent to $M_2^4$.

At first we determine the ruled quadric $Q$ contained in the linear line complexes $\dot{R}$, $\ddot{R}$, and $\dddot{R}$. According to Lemma 3.2 its equation is given by (6). Then we intersect with the ruling $R$ with $Q$. This gives:

**Theorem 5.1**
*Let $R: \ I \subset \mathbb{R} \to M_2^4$ be a $C^3$-curve being the Klein image of a ruled surface $R$ which is free of singular and torsal rulings in $I$. Then the flecnodes of the ruled surface $R$ at the ruling $R_0$ are given by*

$$\begin{aligned} &\langle \mathbf{x}, \dot{\bar{\mathbf{r}}}\rangle \det(\mathbf{x}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}) + \langle \mathbf{x}, \ddot{\bar{\mathbf{r}}}\rangle \det(\mathbf{x}, \dddot{\mathbf{r}}, \dot{\mathbf{r}}) + \langle \mathbf{x}, \dddot{\bar{\mathbf{r}}}\rangle \det(\mathbf{x}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) \\ &\quad + x_0(\langle \mathbf{x}, \dot{\bar{\mathbf{r}}}\rangle(\langle \ddot{\mathbf{r}}, \dddot{\bar{\mathbf{r}}}\rangle - \langle \ddot{\bar{\mathbf{r}}}, \dddot{\mathbf{r}}\rangle) + \langle \mathbf{x}, \ddot{\bar{\mathbf{r}}}\rangle(\langle \dddot{\mathbf{r}}, \dot{\bar{\mathbf{r}}}\rangle - \langle \dddot{\bar{\mathbf{r}}}, \dot{\mathbf{r}}\rangle) \\ &\quad\quad + \langle \mathbf{x}, \dddot{\bar{\mathbf{r}}}\rangle(\langle \dot{\mathbf{r}}, \ddot{\bar{\mathbf{r}}}\rangle - \langle \dot{\bar{\mathbf{r}}}, \ddot{\mathbf{r}}\rangle)) + \det(\dot{\bar{\mathbf{r}}}, \ddot{\bar{\mathbf{r}}}, \dddot{\bar{\mathbf{r}}}) = 0, \end{aligned} \tag{8}$$

*where $(x_0, \mathbf{x}) = \lambda(d_{1,0}, \mathbf{d}_1) + \mu(d_{2,0}, \mathbf{d}_2)$ is a parameterization of $R$ by means of two directrices*[3] *$d_1$ and $d_2$ in $\mathbb{P}^3$.*

Eq. (8) is a quadratic form in the homogeneous parameter $(\lambda : \mu)$. Its solutions $(\lambda : \mu)$ fix the flecnodes at $R_0$. We assume now that $R = R(t)$

[3] Directrices can easily be obtained from the Plücker representation of $R$, see [9].

depends on an affine parameter $t$. Consequently $(\lambda : \mu) = (\lambda(t) : \mu(t))$ depends on $t$. Thus

$$f(t) = \lambda(t)(d_{1,0}, \mathbf{d}_1) + \mu(t)(d_{2,0}, \mathbf{d}_2)$$

is a parameterization of either branch of the curve of flecnodes, since $(\lambda(t) : \mu(t))$ is obtained as solutions ot the quadratic equation (8).

Fig. 6 shows an example of an algebraic ruled surface with an algebraic curve of flecnodes. Both branches are shown.

# 6 A discrete version of the curve of flecnodes

In this final section we point at a discrete version of the curve of flecnodes. Assume we are given a smooth ruled surface $R: \ I \subset \mathbb{R} \to M_2^4$. We further want $R$ to be analytic, i.e., the Taylor expansions for all coordinate functions of the Plücker representation of $R$ as well as a parameterization by means of directrices converge in the interval $I$.

We evaluate $R = R(t)$ at $t_0 \in I$ and further at $t_0 - \varepsilon$, $t_0 + \varepsilon$, and $t_0 + 2\varepsilon$, where $\varepsilon > 0$ is sufficently small such that the Taylor expansions of $R$ converge in $[t_0 - 2\varepsilon, t_0 + 2\varepsilon]$.

The discrete analogues $F_{i,\varepsilon}$ of the flecnodes can now be defined as the intersection of the ruled quadric $Q$ on the lines $R_- := R(t_0 - \varepsilon)$, $R_+ := R(t_0 + \varepsilon)$, and $R_{++} := R(t_0 + 2\varepsilon)$ with the line $R_0 := R(t_0)$. Since the three-space $[R_0, R_-, R_+, R_{++}]$ converges to the osculating space $O = [R_0, \dot{R}, \ddot{R}, \dddot{R}]$ of $R$ at $R_0$ the points $F_{i,\varepsilon}$ converge to the flecnodes on $R_0$.

The convergence of $F_{i,\varepsilon} \to F_i$ is linear and can be improved. For that we symmetrize the process of computing the flecnodes at $R_0$. We let $Q_-$ be the ruled quadric on the lines $R_{-2\varepsilon}$, $R_{-\varepsilon}$, and $R_\varepsilon$. Further we define $Q_+$ be the ruled quadric on $R_{-\varepsilon}$, $R_\varepsilon$, and $R_{2\varepsilon}$. We determine the intersection points $F_{i,-}$ and $F_{i,+}$ of $Q_-$ and $Q_+$ with $R_0$. Now we define

$$F_i^{\mathrm{approx}} := \frac{1}{2}(F_{i,-} + F_{i,+}) \tag{9}$$

and claim:

**Corollary 6.1**
*Assume $R_0$ is a regulra non-torsal ruling on an analytic ruled surface and in*

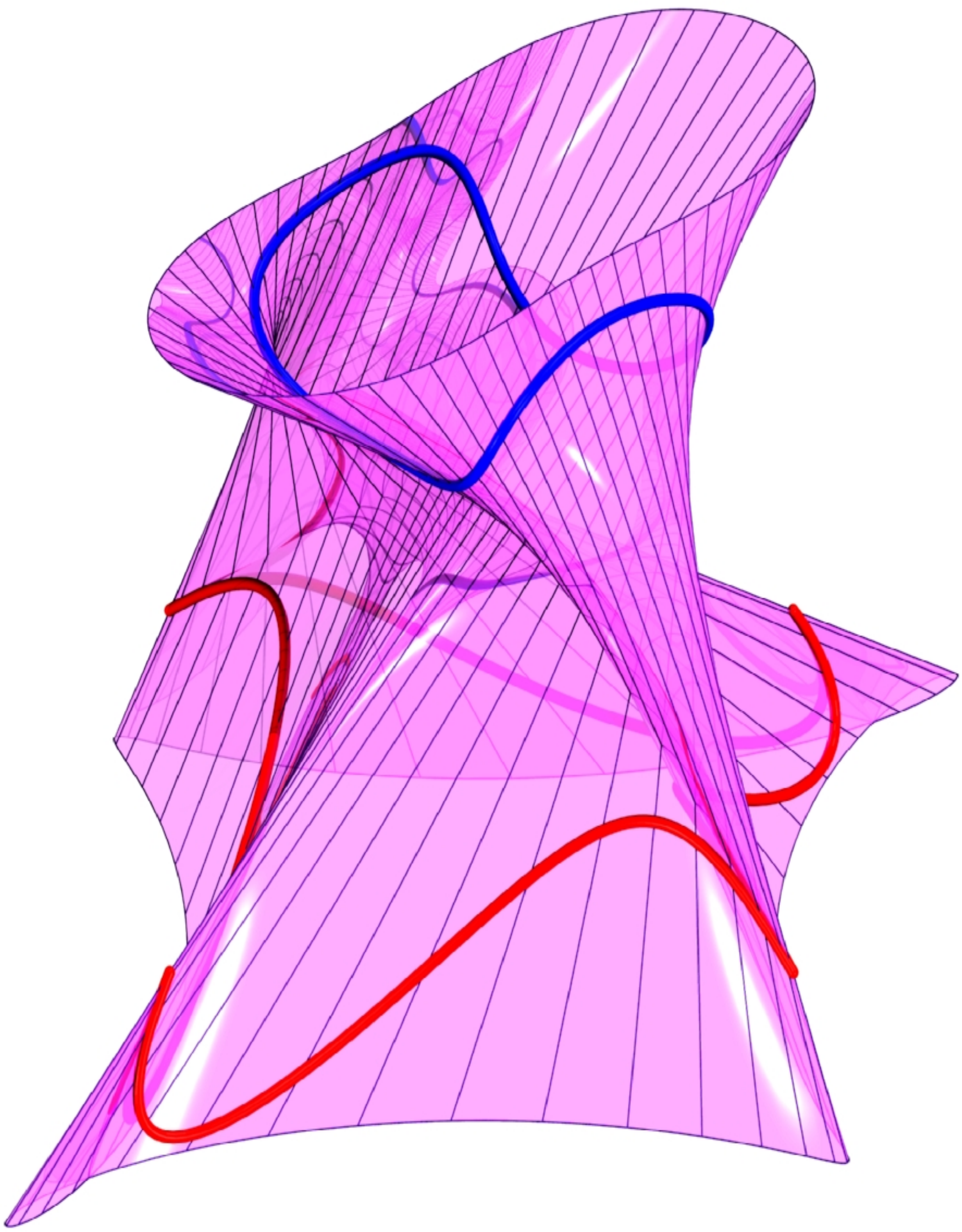

Figure 6: Closed algebraic ruled surface with its curve of flecnodes.

*a sufficiently large neighbourhood of $R_0$ the points $R_0$, $\dot{R}_0$, $\ddot{R}_0$, $\dddot{R}_0$, and $R_0^{(iv)}$ are independent.*

*The approximation of either flecnode on $R_0$ given by Eq.* (9) *has at least quadratic convergence.*

*Proof:* The points $F_{i,-}$ computed as the intersection of $Q_-$ with $R_0$ can be written in terms of Taylor series and read

$$F_{i,-} = F_i - \varepsilon \dot{F}_i + \frac{\varepsilon^2}{2} \ddot{F}_i - \dots. \tag{10}$$

Since $Q_+$ can be obtained from $Q_-$ by replacing $\varepsilon$ with $-\varepsilon$, the Taylor expansion of $F_{i,+}$ is obtained from $F_{i,-}$ by replacing $\varepsilon$ with $-\varepsilon$. Thus the arithmetic average of the series for $F_{i,-}$ and $F_{i,+}$ sum up to

$$F_{i,-} + F_{i,+} = F_i + \varepsilon^2 \ddot{F}_i + \dots. \tag{11}$$

Therefore the difference between $F_i$ and $F_i^{\text{approx}}$ converges towards 0 with quadratic precision, if $\varepsilon \to 0$. $\square$

Remark: The technique of creating better approximations by means of linear combinations of somehow symmetrized approximations allows further improvements. Any even order convergence rate can by achieved with sufficiently many Taylor series. This does not depend on the geometric problem to which it is applied to. It is more or less a property of Taylor series.

Let $f$ be real analytic in an $\varepsilon$-neighbourhood of $0$ and let further $S_k := \frac{1}{2}(f(k\varepsilon) + f(-k\varepsilon))$. Then we have

$$\begin{aligned}
S_2 - 4S_1 &= -3f + \frac{1}{2} f^{(iv)} + \dots, \\
S_3 - 6S_2 + 15S_1 &= 10f + \frac{1}{2} f^{(vi)} + \dots, \\
S_4 - 8S_3 + 28S_2 - 56S_1 &= -35f + \frac{1}{2} f^{(viii)} + \dots, \\
S_5 - 10S_4 + 45S_3 - 120S_2 + 210S_1 &= 126f + \frac{1}{2} f^{(x)} + \dots, \\
&\vdots \\
S_l - \binom{2l}{1} S_{l-1} + \binom{2l}{2} S_{l-1} + \dots &= (-1)^{l+1} \binom{2l-2}{l} f + \frac{1}{2} \cdot f^{(2l)} + \dots.
\end{aligned} \tag{12}$$

Generalizations and further improvements are straight forward, but the computational effort increases.

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