ISSAC 2003 Tutorial 1

Classical Geometry for Symbolic Geometric Computing

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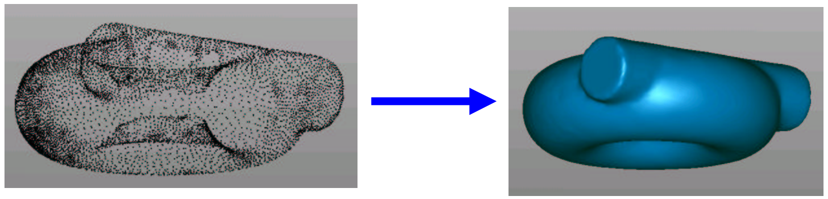
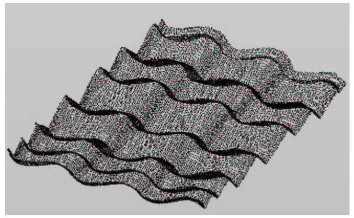
  - Rational offsets
  - Parametrization of special surfaces
- Minkowski sums
- Line geometry: some basics
  - Line geometry for reverse engineering

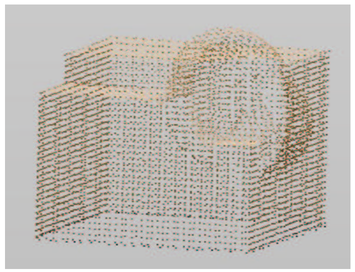
Laguerre sphere geometry Part1



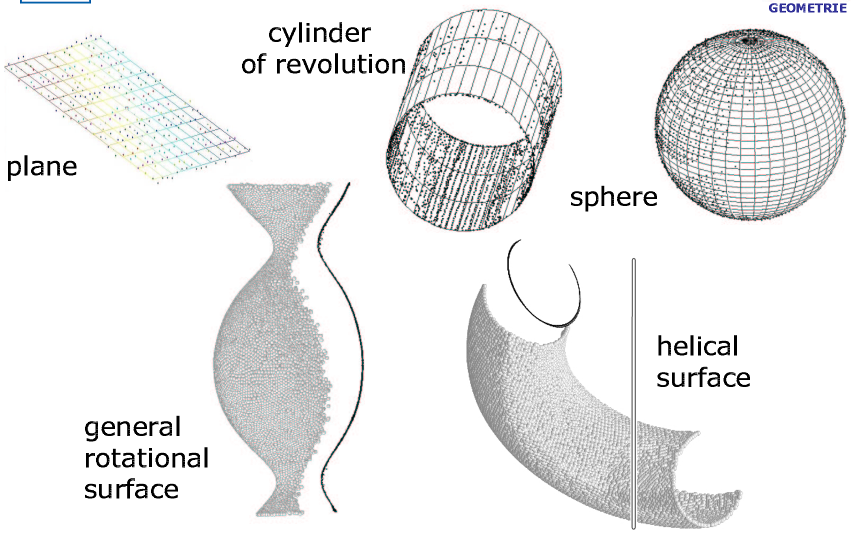
Reverse engineering

- Reverse engineering deals with the reconstruction of a computer model from an existing object
- Automatic detection of special surfaces in the reconstruction process of the CAD model is important for a precise CAD representation

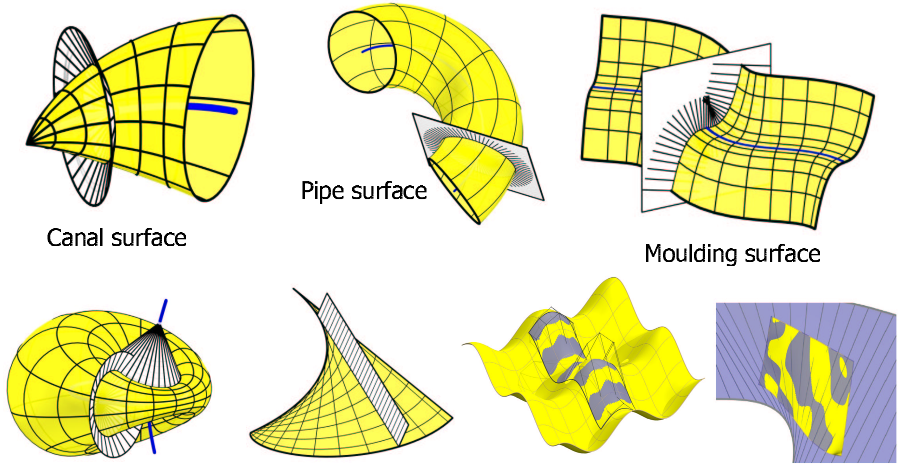




RE of **simple** surfaces

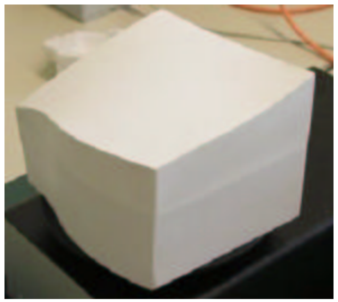


RE of **special** surfaces



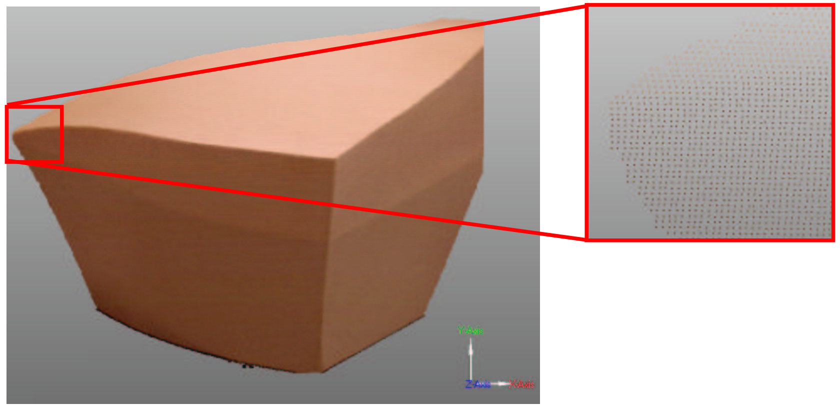
Data acquisition

- Minolta VIVID 900 optical laser scanner
- Object to be scanned:



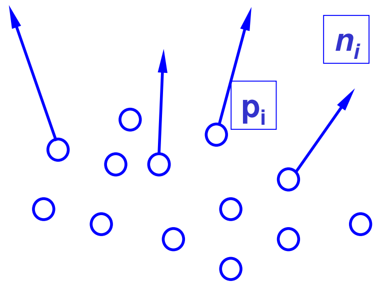
Data acquisition

- One shot of the object: 215 203 points

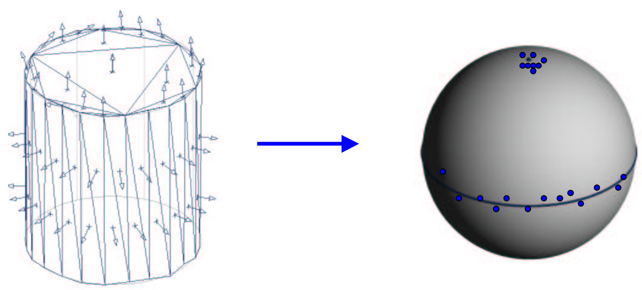


Surface normal estimation

- Given: Point cloud $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, …, $\boldsymbol{p}_N$ representing a surface
- Estimate surface normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, …, $\boldsymbol{n}_N$ (→ tangent planes)
- Various methods:
  - Local regression plane
  - Local quadric surfaces
  - …
- Problems: Edges in data

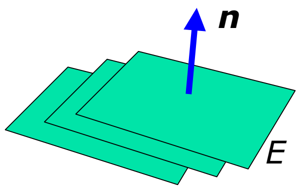


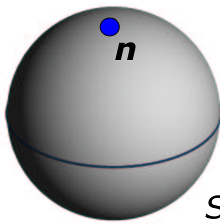
Gaussian Sphere Methods



Gaussian image of an oriented plane

- Given: oriented plane E: $\boldsymbol{nx}+d=0$ in R^3 with unit normal vector $\boldsymbol{n}$.
- Gaussian image of E is the point $\boldsymbol{n}$ on the unit sphere S^2 in R^3.
- Gaussian pre-image of a point $\boldsymbol{n}$ of S^2 is the pencil of parallel planes $\boldsymbol{nx}+d=0$ with varying d.



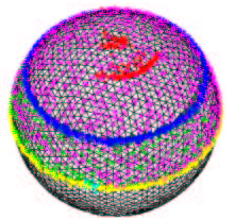


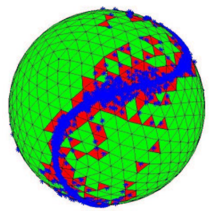
Recognition of special surfaces

- Algorithms are used that investigate the Gaussian image of a triangulated data point cloud for occurrence of special clusters:

triangulated data, point cloud ⇒	Gaussian image
planar region	point-like cluster
cylindrical region	curve-like cluster along great circle
region of a right circular cone	curve-like cluster along small circle
region of developable surf.	curve-like cluster

⇍





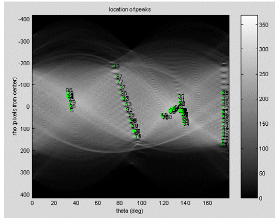
Problems of Gaussian sphere methods

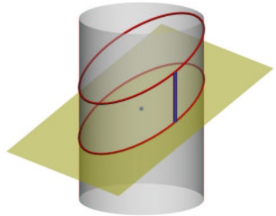
Restriction of tangent planes ***nx****+d=0* to normals ***n*** results in the following problems:

- Translated objects (e.g. parallel planes) have the same Gaussian image
- Great circle (small circle) as Gaussian image does not characterize a cylinder or cone of revolution, respectively.

This loss of information can be avoided by working in the space of planes (leads to *Laguerre geometry*)

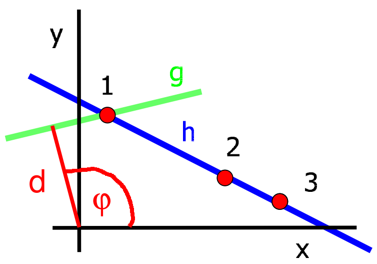
Laguerre Geometric and Hough Transform Methods



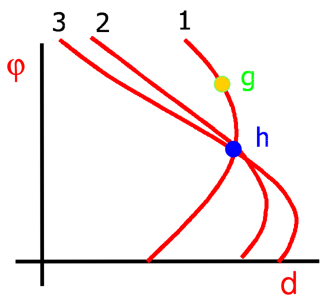


2D-Hough transform

- *Lines* are mapped to *points* of the Hough plane.
- *Lines through a point* appear as *curve* in the Hough plane.
- *Points of a Line h* are mapped to *curves passing through a point h* in the Hough plane.



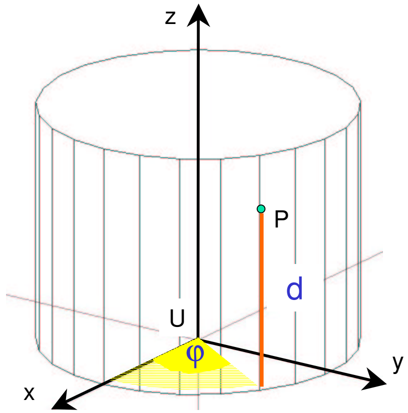
Original plane



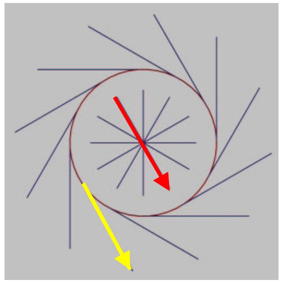
Hough plane

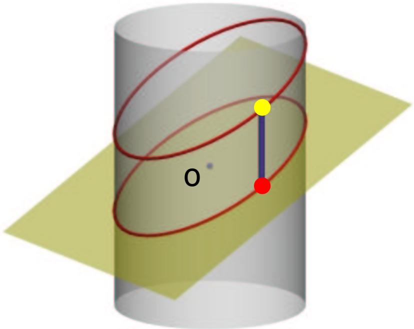
2D-Blaschke cylinder

- Map oriented line with distance *d* from the origin and directional angle φ onto a cylinder (*Blaschke cylinder*)
- Standard Hough plane is the planar development of this cylinder



2D-Blaschke model of 2D-Laguerre geometry

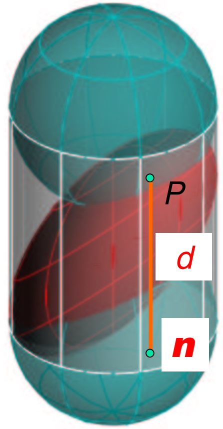




Lines of a pencil ⟶ ellipse; center *o*
tangents of oriented circle ⟶ ellipse; center *(0,0,r)*
Dilation by distance *d* ⟶ translation by *(0,0,d)*

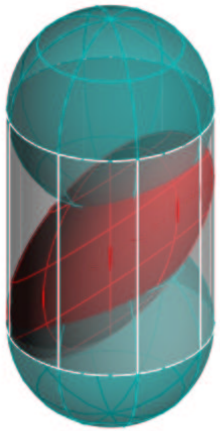
3D Laguerre geometry

- Given or. plane E with unit normal vector $\mathbf{n} = (n_1,n_2,n_3)$ and distance d from the origin
  $E: n_1x+n_2y+n_3z+d= \mathbf{nx}+d=0$
- Plane E is mapped onto the point $E^b = (\mathbf{n},d)$ in R^4;
- 3D-Blaschke cylinder
  $B: u_1^2 + u_2^2 + u_3^2 = 1$
- Cross sections of B are copies of the Gaussian sphere.



Blaschke model for oriented planes in 3-space

- All oriented planes $E{:}\,\mathbf{nx}+d=0$ which are tangent to a given sphere with center $\mathbf{p}$ and signed radius r satisfy
  $$\mathbf{n}\cdot\mathbf{p} + d = r$$
- Their Blaschke image points E^b lie in the hyperplane H:
  $p_1u_1 + p_2u_2 + p_3u_3 + u_4 - r = 0$
- The hyperplanar cut of B with H is an ellipsoid on B



Surface recognition with PCA on the Blaschke image

- Given data points $\mathbf{p}_i$, $i=1,\dots,k$ in R^3 with estimated tangent planes
  $E_i{:}\,d_i+a_ix+b_iy+c_iz=0$ with $a_i^2+b_i^2+c_i^2=1$.
- Blaschke image points $\mathbf{b}_i = (a_i,b_i,c_i,d_i)$ in R^4.
- Compute best fitting hyperplane
  $H: h_0 + h_1u_1+\dots+h_4u_4 = 0$,
  to image points $\mathbf{b}_i$ in R^4
- H passes through barycenter $\mathbf{c} = (\Sigma\mathbf{b}_i)/k$
  ⇒ Compute new coordinate vectors $\mathbf{q}_i = \mathbf{b}_i\text{-}\mathbf{c}$
- H has vanishing $h_0=0$

Surface recognition with PCA on the Blaschke image of estimated tangent planes

- Minimize the homogeneous quadratic function

$$F(h_1,\dots,h_4) = F(h) = h^T \cdot C \cdot h, \text{ with } C = \frac{1}{k}\sum_{i=1}^{k} q_i \cdot q_i^T,$$

under the constraint $\boldsymbol{h}^2=1$. This is an ordinary eigenvalue problem.

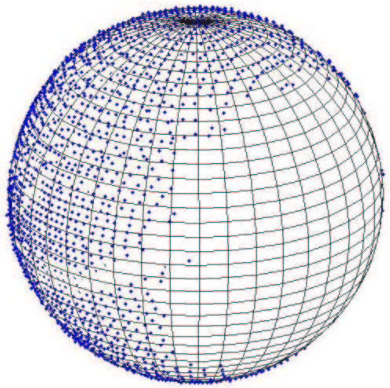
- Best fitting hyperplane H_1 belongs to the smallest eigenvalue λ_1 of C.

Surface recognition with PCA on the Blaschke image

- Distribution of eigenvalues (EV) $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ gives important information about the type of the surface S:
  - *One small EV* and *surface-like* Blaschke image: *S = sphere*
  - *One small EV* and *curve-like* Blaschke image: *S = general cone or cylinder, special developable surface*
  - *Two small EV* and *curve-like* Blaschke image: *S = cone or cylinder of rotation*
  - *Three small EV*: not possible
  - *Four small EV*: *S = plane*

Example: Sphere

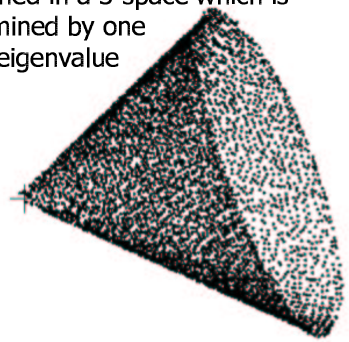
Blaschke image is surface-like and is contained in a 3-space which is determined by one small eigenvalue



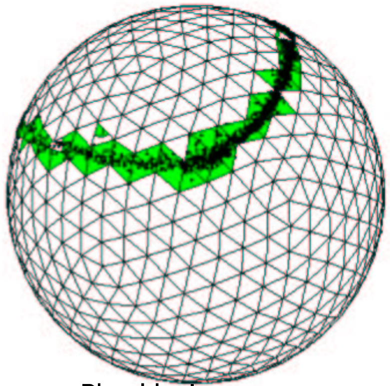
$$\lambda_1 = 0.00004, \lambda_2 = 0.18820, \lambda_3 = 0.35731, \lambda_4 = 0.37100$$

Example: General cone

Blaschke image is curve-like and is contained in a 3-space which is determined by one small eigenvalue



Data points of general quadratic cone

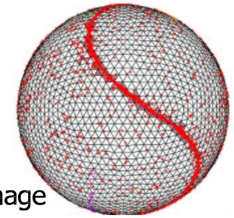


Blaschke image orthogonally projected onto S^2

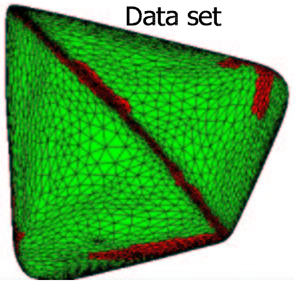
$$\lambda_1 = 0.00433, \lambda_2 = 0.01480, \lambda_3 = 0.17442, \lambda_4 = 0.57563$$

Example: Developable surface

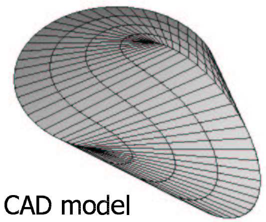
- The Blaschke image of developable surfaces are curves on B
- Use accumulator array to detect curve like arrangement in the set of estimated tangent planes



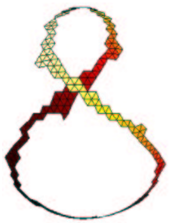
Gaussian image



Data set

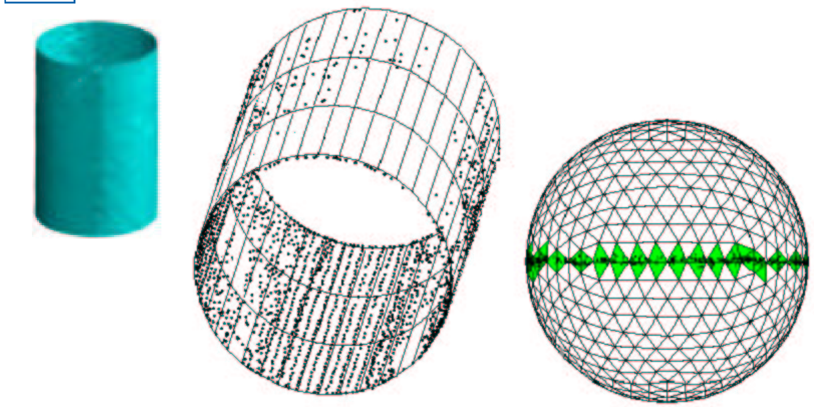


CAD model



Blaschke image

Example: Cylinder of revolution



Blaschke image is curve-like and is contained in a plane which is determined by two small eigenvalues

$$\lambda_1 = 0.00013,\ \lambda_2 = 0.00023,\ \lambda_3 = 0.48055,\ \lambda_4 = 0.51496$$

Shape recognition in point clouds

- Detected edges are represented by black triangles and lead to a pre-segmentation of the data points
- Clustering of the pre-segmented data with help of the Blaschke model
- PCA for the recognized clusters in the Blaschke image
- Four small eigenvalues indicate that the Blaschke image of a region is a point-like cluster
  $\Rightarrow$ planar region

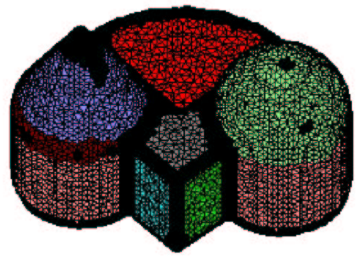
R_1 : 0.00001, 0.00018, 0.00175, 0.00315

R_2 : 0.00001, 0.00046, 0.00112, 0.00201

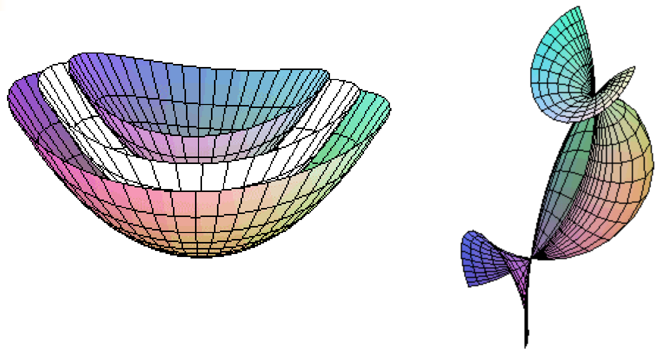
Shape recognition in point clouds

- Pre-segmentation with help of computed edges of the object
- Clustering of the pre-segmented data with help of the Blaschke model
- Computation of hyperplanes of regression to detected clusters gives the following eigenvalues:

Planar vertical region left:
eigenvalues: 0.00000, 0.00006, 0.00098, 0.00200.
Cylinder of rotation, front:
eigenvalues: 0.00002, 0.00069, 0.10908, 0.62807.
Cone of rotation:
eigenvalues: 0.00001, 0.00079, 0.36814, 0.55629.
Spherical part:
eigenvalues: 0.00000, 0.07370, 0.33614, 0.51365.



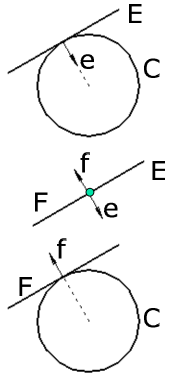
Laguerre sphere geometry Part2



Contents

2D Laguerre geometry

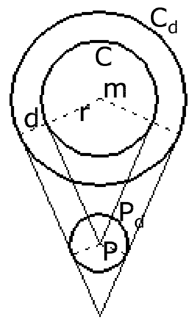
- An oriented (or.) circle C in 2D (R^2) is given by
  C: $(\mathbf{x}-\mathbf{m})^2-r^2=0$,
  and the orientation is determined by or. normals. Points are considered as circles of radius 0.
- An oriented line E in 2D is given by
  E: $e_0+e_1x_1+e_2x_2=e_0+\mathbf{ex}=0$. We always assume that $\mathbf{e}^2=1$.



- E and C are said to be in oriented contact iff
  $e_0+e_1m_1+e_2m_2+r=e_0+\mathbf{em}+r=0$, $\mathbf{e}^2=1$.

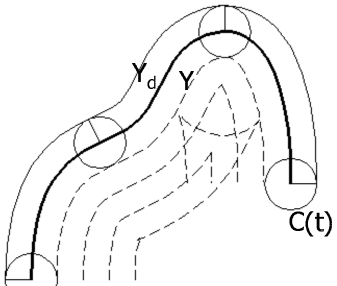
2D Laguerre transformations

- A Laguerre trafo T consists of two mappings
  T_C:$\mathbf{C} \rightarrow \mathbf{C}$, T_E:$\mathbf{E} \rightarrow \mathbf{E}$,
  which are bijective on the sets of circles $\mathbf{C}$ and lines $\mathbf{E}$, respectively, and preserve or. contact of circles and lines.
- *Motions* and *similarities* in 2D are examples for point-preserving Laguerre trafos.
- A *dilation* D maps the circle C:$(\mathbf{x}-\mathbf{m})^2-r^2=0$ onto the circle C_d:$(\mathbf{x}-\mathbf{m})^2-(r+d)^2=0$. D is not point- preserving but maps points to circles of radius d.



Offsets of 2D curves

A dilation D maps a curve
Y: $\mathbf{y}(t) = (y_1,y_2)(t)$
onto the offset Y_d which is constructed as envelope of the 1-par. family of circles
$C(t)$: $(\mathbf{x}-\mathbf{y}(t))^2-d^2=0$,
which are centered at Y.



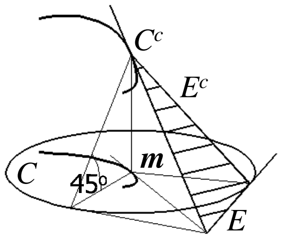
Remark: An oriented curve possesses oriented one-sided offset curves. The offsets of not oriented curves consist locally of two not connected components, the inner part and the outer part.

Cyclographic model – 2D case

- An or. circle C: $(\mathbf{x}-\mathbf{m})^2-r^2=0$ is mapped to the point $C^c=(m_1,m_2,r)$ in 3-space A^3. R^2 is embedded in A^3 as plane $x_3=0$.
- An or. line E: $e_0+\mathbf{ex}=0$, $\mathbf{e}^2=1$, is mapped to the plane E^c: $e_0+e_1x_1+e_2x_2+x_3=0$, $e_1^2+e_2^2=1$.
- Oriented contact of C and E is realized by incidence of C^c and E^c.
- If A^3 is equipped with the 'scalar product'
  $\langle x, y\rangle_c = x^t I_c y$, with $I_c=diag(1,1,-1)$,
  A^3 is the Lorenz space $R^3{}_1$.
- Laguerre trafos T are transformed to affine mappings in $R^3{}_1$, which satisfy
  $\langle x, y\rangle_c = \langle Tx, Ty\rangle_c$.

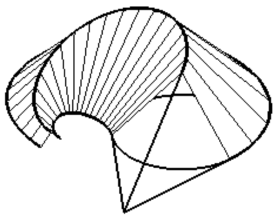
Cyclographic model – 2D case

A one-par. family of circles
$C(t)$: $(\mathbf{x}-\mathbf{m}(t))^2-r(t)^2=0$
is identified with the curve
$C^c(t)=(m_1,m_2,r)(t)$ in $R^3{}_1$.



The circles $C(t)$ possess a real envelope, exactly if
$m_t^2 - r_t^2 = \langle C_t^c, C_t^c\rangle_c \geq 0$.
Notation: $x_t := dx/dt$.

The envelope of $C(t)$ is traced out by the developable of constant slope 45°, passing through $C(t)^c$.



3D Laguerre geometry

- An oriented (or.) sphere S in 3D is given by
  S: $(\mathbf{x}-\mathbf{m})^2-r^2=0$,
  and the orientation is determined by or. normals. Points are considered as spheres of radius 0.
- An oriented plane E in 3D is given by
  E: $e_0+e_1x_1+e_2x_2+e_3x_3 = e_0+\mathbf{ex}=0$, $\mathbf{e}^2=1$,
- E and S are said to be in oriented contact iff
  $e_0+e_1m_1+e_2m_2+e_3m_3+r=e_0+\mathbf{em}+r=0$, $\mathbf{e}^2=1$.
  $\mathbf{e}$ denotes the unit normal vector of E.

3D Laguerre transformations

- A Laguerre trafo T consists of two mappings T_S:$\boldsymbol{S} \rightarrow \boldsymbol{S}$, T_E:$\boldsymbol{E} \rightarrow \boldsymbol{E}$, which are bijective on the sets of spheres $\boldsymbol{S}$ and planes $\boldsymbol{E}$, respectively, and preserve or. contact of spheres and planes.
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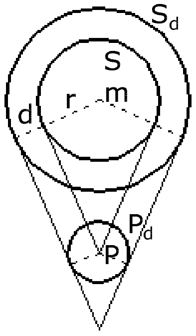
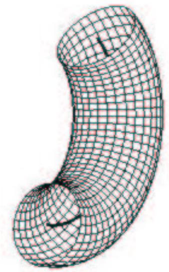
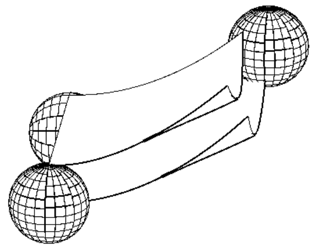


Offsets of curves and surfaces

- A dilation D maps a curve Y: $\mathbf{y}(t) = (y_1,y_2,y_3)(t)$ onto the *offset* Y_d which is the envelope of the spheres $S(t)$: $(\mathbf{x}-\mathbf{y}(t))^2-d^2=0$, centered at Y. Y_d is called *pipe surface*.



- A dilation D maps a surface Y: $\mathbf{y}(u,v) = (y_1,y_2,y_3)(u,v)$ onto the *offset* Y_d which is the envelope of the spheres $S(u,v)$: $(\mathbf{x}-\mathbf{y}(u,v))^2-d^2=0$, centered at Y.

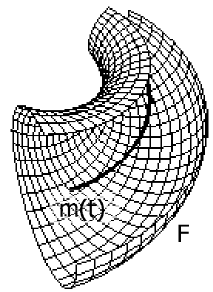


Cyclographic model – 3D case,1

- An or. sphere S: $(\mathbf{x}-\mathbf{m})^2-r^2=0$ is mapped to the point $S^c=(m_1,m_2, m_3,r)$ in 4-space A^4. R^3 is embedded in A^4 as hyperplane $x_4=0$.
- An or. plane E: $e_0+\mathbf{ex}=0$, $\mathbf{e}^2=1$, is mapped to the 3-space E^c: $e_0+e_1x_1+e_2x_2+e_3x_3+x_4=0$, $e_1^2+e_2^2+ e_3^2=1$.
- Oriented contact of S and E is realized by incidence of S^c and E^c.
- If A^4 is equipped with the 'scalar product' $\langle x, y\rangle_c = x^t I_c y$, with $I_c=diag(1,1,1,-1)$, A^4 is the Lorenz space $R^4{}_1$.
- Laguerre trafos T appear as affine mappings in $R^4{}_1$, which satisfy $\langle x, y\rangle_c = \langle Tx, Ty\rangle_c$.

Cyclographic model – 3D case, 2

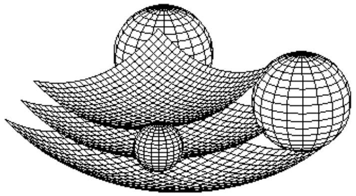
A one-par. family of spheres $S(t)$: $(\mathbf{x}-\mathbf{m}(t))^2-r(t)^2=0$ is mapped to the curve $S^c(t)=(m_1,m_2, m_3, r)(t)$. The spheres $S(t)$ possess a real envelope F, exactly if $m_t^2 - r_t^2 = \langle S_t^c, S_t^c\rangle_c \geq 0$.



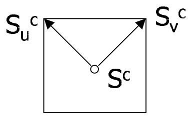
The envelope of a one-par. family of spheres $S(t)$ is called *canal surface*.

Cyclographic model – 3D case, 3

- A two-par. family of spheres $S(u,v): (x-\mathbf{m}(u,v))^2-r(u,v)^2=0$ is mapped to the surface $S^c(u,v)=(m_1,m_2,m_3,r)(u,v)$.



- The two-par. family of spheres $S(u,v)$ possesses a real envelope, exactly if



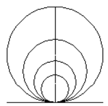
$(\lambda m_u + \mu m_v)^2 - (\lambda r_u + \mu r_v)^2 = \langle \lambda S^c_u + \mu S^c_v, \lambda S^c_u + \mu S^c_v \rangle_c \ge 0$, for all (λ,μ).

Cyclographic mapping and image

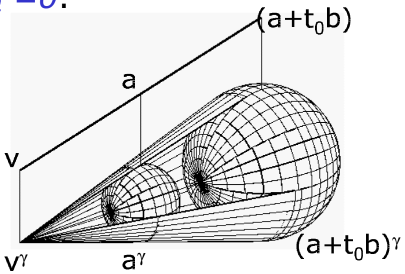
- Let $\gamma: R^4_1 \rightarrow \mathbf{S}$ be the mapping, which maps points $P=(p_1,p_2,p_3,p_4)\in R^4_1$ onto spheres P^γ in R^3, with center $P'=(p_1,p_2,p_3)$ and radius p_4.
- γ is called *cyclographic mapping*.
- Let F be a curve or surface in R^4_1. By F^γ we will always denote the *envelope* of the one- or two-par. family of spheres corresponding to F.
- F^γ is called *cyclographic image* of F.
- A line $L{:}a+tb$ in R^4_1 corresponds to a one-par. family of spheres $S(t)$. The family $S(t)$ has a real envelope $F=L^\gamma$, if $\langle b, b\rangle_c \ge 0$.

Cones and cylinders of rotation

- $F=L^\gamma$ is a cylinder of rot. with radius d, if $b_4=0$ and $a_4=d$.
- F is a cone of rot. in the general case.
- F is a parabolic pencil of spheres, if $\langle b, b\rangle_c = 0$.
- F is a line if $b_4=0$ and $a_4=0$.



A cone of rot. is always determined by the common tangent planes of two or. spheres.



Families of cones of rotation

- Let $G{:}a(u)+vb(u)$ be a ruled surface in R^4_1, consisting of a one-par. family of lines $L(u)$.
- Since any line $L(u)$ corresponds to a cone of revolution $L(u)^\gamma$, the cyclographic image G^γ in R^3 is envelope of a one-par. family of cones of revolution $L(u)^\gamma$.
- The axes of the cones $L(u)^\gamma$ lie on the ruled surface G', the orthogonal projection of G onto $R^3{:}x_4=0$.

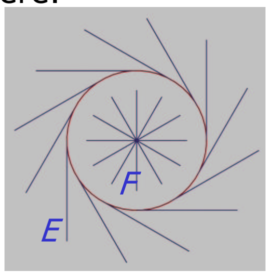
Other Models of Laguerre Geometry

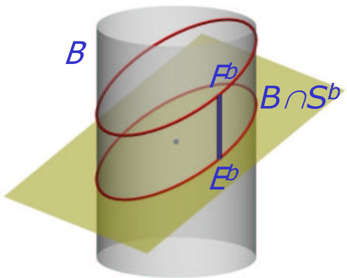
Blaschke model – 3D case,1

- A point $S^c=(m_1,m_2, m_3,r)$ is mapped to the 3-space $S^b:r +m_1x_1+\dots+m_3x_3+x_4=0$ in 4-space $R^4{}_1$.
- A 3-space E^c: $e_0+e_1x_1+e_2x_2+e_3x_3+x_4=0$ is mapped to the point $E^b = (e_1,e_2,e_3,e_0)$, with $e_1{}^2+e_2{}^2+ e_3{}^2=1$.
- Image points E^b of planes E are contained in the Blaschke cylinder $B:x_1{}^2+x_2{}^2+x_3{}^2=1$.
- Parallel planes $E:e_0+\mathbf{nx}=0,F: f_0+\mathbf{nx}=0$ have image points E^b,F^b in a generating line of B.
- Laguerre trafos T appear as automorphic linear (projective) transformations of B.

Blaschke model – 3D case, 2

- The cross sections of $B:x_1{}^2+x_2{}^2+x_3{}^2=1$ with hyperplanes $x_4=const$. are copies of the unit sphere.





- All planes E tangent to a sphere S correspond to the points E^b, lying in the intersection $B\cap S^b$.

Blaschke model – 3D case, 3

- A non-developable surface F considered as envelope of its 2-par. family of tangent planes
  $E(u,v):e_0(u,v)+\mathbf{e}(u,v)\mathbf{x}=0,$
  corresponds to a surface
  $E^b(u,v): (e_1,e_2,e_3,e_0)(u,v),\ \mathbf{e}(u,v)^2=1$
  in the Blaschke cylinder B.
- A developable surface F considered as envelope of its 1-par. family of tangent planes
  $E(t):e_0(t)+\mathbf{e}(t)\mathbf{x}=0,$
  corresponds to a curve
  $E^b(t): (e_1,e_2,e_3,e_0)(t),\ \mathbf{e}(t)^2=1,$
  in the Blaschke cylinder B.

Isotropic model – 3D case

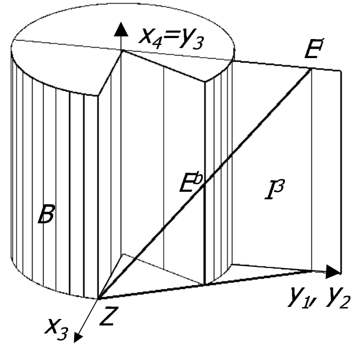
Applying a *stereographic projection* to B with center $Z=(0,0,1,0)$ and image space $x_3=0$ yields the isotropic model I^3. The 3-space I^3 is spanned by the coordinate vectors $x_1=y_1, x_2=y_2, x_4=y_3$.

A point $E^b=(e_1,e_2,e_3,e_0)$ maps to the point

$$E^i = 1/(1-e_3)(e_1,e_2,e_0)$$

in I^3. E^i is called *isotropic image* of E.

The stereographic projection and its inverse are rational transformations.



Models of Laguerre geometry

R^3	Sphere $S:(x-m)^2-r^2=0$	Plane $E: e_0+ e_1x_1+\ldots+e_3x_3=0, e^2=1$
CM	Point $S^c=(m_1,m_2,m_3,r)$	Hyperplane $E^c: e_0+e_1x_1+\ldots+e_3x_3+x_4=0, e^2=1$
BM	Hyperplane $S^b: r +m_1x_1+\ldots+m_3x_3+x_4=0$	Point $E^b = (e_1,e_2,e_3,e_0), e^2=1$
IM	Paraboloid in I^3 $2y_3+(y_1^2+y_2^2)(r+m_3)+2y_1m_1+2y_2m_2+r-m_3=0$	Point $E^i = 1/(1-e_3)(e_1,e_2,e_0)$

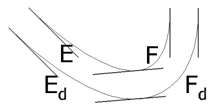
Parallel planes E,F have image points E^i,F^i in I^3 which lie on y^3-parallel lines.

Rational Offset Surfaces

Rational offset surfaces, 1

- A surface F is called *rational offset surface* if it possesses a parametrization $\mathbf{f}(u,v)$ and unit normal vectors $\mathbf{e}(u,v)$ such that its offset surfaces F_d admit rational parametrizations $\mathbf{f}(u,v)+d\mathbf{e}(u,v)$.
- Let $E(u,v):e_0(u,v)+\mathbf{e}(u,v)\mathbf{x}=0$ be F's tangent planes. Then $\mathbf{e}(u,v)$ is a rational parametrization of F's spherical (Gaussian) image. The offset surfaces F_d of F are envelopes of the translated planes

$$E_d(u,v):(e_0(u,v)+d)+\mathbf{e}(u,v)\mathbf{x}=0.$$



Rational offset surfaces, 2

- Let $E(u,v)$, $E_d(u,v)$ be tangent planes of F and F_d. Their Blaschke images E^b and E_d^b are
  $E^b(u,v)$: $(e_1,e_2,e_3,e_0)(u,v)$, $e(u,v)^2=1$,
  $E_d^b(u,v)$: $(e_1,e_2,e_3,e_0+d)(u,v)$, $e(u,v)^2=1$.
- E^b and E_d^b are rational surfaces and E_d^b is a translated version of E^b in x_4-direction.
- The isotropic images of F and F_d are
  $E^i(u,v) = 1/(1-e_3)(e_1,e_2,e_0)(u,v)$,
  $E_d^i(u,v) = 1/(1-e_3)(e_1,e_2,e_0+d)(u,v)$.
  These are rational surfaces in I^3.

Offsets of paraboloids, 1

From an arbitrary rational parametrization $\mathbf{f}(u,v)$ it is not always clear if F is a rational offset surface. Reparametrizations are often necessary.

Example: $F: z-x^2-cy^2=0$ is a paraboloid with parametrization $\mathbf{f}(u,v) = (u,v,\ u^2+cv^2)$ and normal vectors $\mathbf{n}(u,v)=(-2u,-2cv,1)$.
The unit normals of F have to be a parametrization of part of the unit sphere S^2.
Thus, $\mathbf{n}$ has to be a multiple of

$$\mathbf{e}(s,t) = (\cos(s)\cos(t),\ \sin(s)\cos(t),\ \sin(t)).$$

We obtain the condition

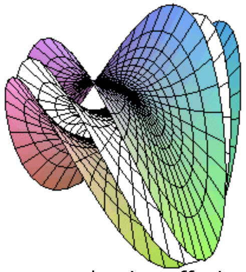
$$\mathbf{n}(u,v)=\lambda \mathbf{e}(s,t).$$

Offsets of paraboloids, 2

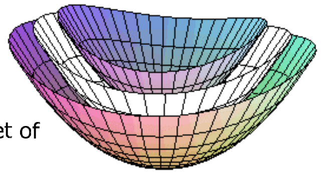
With the reparametrization
$u = -\cos(s)\cos(t)/(2\sin(t))$,
$v = -\sin(s)\cos(t)/(2c\sin(t))$,
we obtain the representation

$$\mathbf{p}_d(s,t) = \mathbf{f}(s,t) + d\mathbf{e}(s,t)$$

This parametrization can be converted into a rational representation.



inner and outer offset of a hyperbolic paraboloid



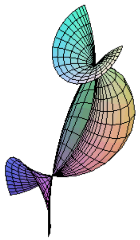
inner and outer offset of an elliptic paraboloid

Rational offset surfaces, general concept

- Let $\mathbf{e} = (e_1,e_2,e_3)$ be a rational parametrization of the unit sphere S^2, with polynomials a,b,c in u and v:
  $e_1=2ac/N$, $e_2 = 2bc/N$, $e_3=(a^2+b^2-c^2)/N$,
  with $N=(a^2+b^2+c^2)$. Let $h(u,v)$ be an arbitrary rational function.
- The envelope F of the two-par. family of planes
  $E(u,v)$: $h(u,v)+ e_1x_1+ e_2x_2+ e_3x_3 = 0$ is a *rational offset surface*.
- The offsets F_d of F are envelopes of the planes
  $E_d(u,v)$: $(h+d)+ e_1x_1+ e_2x_2+ e_3x_3 = 0$.
- If a,b,c are polynomials in t and $h(t)$ is a rational function, F and F_d are developable.

Parabolic Dupin cyclide

Example: Let $a=u$, $b=v$, $c=1$. Then
$e_1=2u/N$, $e_2 = 2v/N$, $e_3=(u^2+v^2-1)/N$,
with $N=(u^2+v^2+1)$.
We choose $h(u,v)=q(u,v)/N$ where $q(u,v)$ is an arbitrary quadratic polynomial and we obtain
$E(u,v)$: $q(u,v) + 2ux+2vy+(u^2+v^2-1)z = 0$.
The isotropic image of the planes $E(u,v)$ is the paraboloid $E^i(u,v) = (u,v,q(u,v))$.



The envelope F of planes $E(u,v)$ is a *parabolic Dupin cyclide* (alg.order 3) and all its offset surfaces F_d are of the same type.

Cones and cylinders of rotation

- Let $a(t)$, $b(t)$, $c(t)$ be linear polynomials in t. The spherical image $\mathbf{e}=(e_1, e_2, e_3)$ with $e_1=2ac/N$, $e_2 = 2bc/N$, $e_3=(a^2+b^2-c^2)/N$ and $N=(a^2+b^2+c^2)$ is a circle.
- We choose $h(t)=q(t)/N$ where $q(t)$ is an arbitrary quadratic polynomial and we obtain
  $E(t)$: $q(t) + 2acx+2bcy+(a^2+b^2-c^2)z = 0$.
  The isotropic image $E^i(t)$ of $E(t)$ is a (special) conic, a planar section of a paraboloid of rot.
- The envelope F of planes $E(t)$ and all offset surfaces F_d are *cones or cylinders of rotation*. The direction of the generators is $\mathbf{e} \times \mathbf{e}_t$, the vertex of the cone is $E \cap E_t \cap E_{tt}$.

Rational offset surfaces and the isotropic model

Considering tangent planes
$E(u,v)$: $e_0(u,v)+\mathbf{e}(u,v)\mathbf{x}=0$
of a surface F. Their isotropic images are
$E^i(u,v) = 1/(1-e_3)(e_1,e_2,e_0)(u,v)$.

Theorem:

Let $Y(u,v)=(y_1,y_2,y_3)(u,v)$ be an arbitrary rational surface in I^3. The corresponding family of planes in R^3 is
$E(u,v)$: $e_0+ e_1x_1+ e_2x_2 + e_3x_3 =0$, with
$(e_0,e_1,e_2,e_3)=(2y_3, 2y_1, 2y_2, y_1^2+y_2^2-1)/N$,
and $N = (y_1^2+y_2^2+1)$.
Its envelope is a *rational offset surface*.

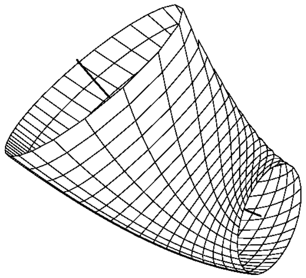
Developable rational offset surfaces and the isotropic model

Theorem:

Let $Y(t)=(y_1,y_2,y_3)(t)$ be an arbitrary rational curve in I^3. The corresponding family of planes in R^3 is
$E(t)$: $e_0+ e_1x_1+ e_2x_2 + e_3x_3 =0$, with
$(e_0,e_1,e_2,e_3)=(2y_3, 2y_1, 2y_2, y_1^2+y_2^2-1)/N$,
and $N = (y_1^2+y_2^2+1)$.
Its envelope is a *developable rational offset surface*.

Rational canal surfaces

- The family $S(t):(\mathbf{x}-\mathbf{m}(t))^2-r(t)^2=0$ is called *rational*, if $\mathbf{m}(t)$ is a rational curve and $r(t)^2$ is rational function.
- If $S(t)$ possesses a real envelope F, it is proved that any real component of F admits a *rational parametrization*.
- If additionally $r(t)$ is rational, $S^c(t)=(m_1,m_2,m_3,r)(t)$ is a rational curve in R^4 and F is a *rational offset surface*.
- F is (part of the) envelope of a rational family of cones of revolution.



Envelopes of quadratic cones

- Let $S_i(t):(\mathbf{x}-\mathbf{m}_i(t))^2-r_i(t)^2=0,\ i=1,2,$ be two rational families of or. spheres.
- The common tangent planes of S_1 and S_2 envelope a cone of rot. D. We call $D(t)$ a *rational 1-par. family of cones of rotation*.
- The isotropic images of the tangent planes of $S_1(t)$ and $S_2(t)$ are two paraboloids of rotation
  $\Phi_1: 2y_3+(y_1^2+y_2^2)(r_1+m_{13})+2y_1m_{11}+2y_2m_{12}+r_1-m_{13}=0,$
  $\Phi_2: 2y_3+(y_1^2+y_2^2)(r_2+m_{23})+2y_1m_{21}+2y_2m_{22}+r_2-m_{23}=0.$
- The intersection $d(t)$ of $\Phi_1(t)$ and $\Phi_2(t)$ is the isotropic image of the tangent planes of $D(t)$.
- The curves $d(t)$ are planar sections of paraboloids of rot (Φ_1, Φ_2) = *isotropic circles*.

Envelopes of quadratic cones

- The family of curves $d(t)$ is the isotropic image of the envelope F of the cones $D(t)$.
- A parametrization of $d(t)$ is a dual parametrization of F (as set of tangent planes).
- The orthogonal projection $d'(t)$ of $d(t)$ onto the y_1y_2-plane is a family of circles

$$d'(t): (y_1^2+y_2^2)(R+M_3)+2y_1M_1+2y_2M_2+R-M_3=0,$$

where $M=m_2-m_1$, and $R=r_2-r_1$. The circles $d'(t)$ have rational centers

$$n(t)=1/(R+M_3)\ (M_1,M_2)$$

and rational squared radii

$$s(t) = (M_1^2+M_2^2+M_3^2-R^2)/(R+M_3)^2.$$

Rational parametrizations of envelopes of quadratic cones, 1

- We show that any real component of the envelope F of a rational family of cones of rot. $D(t)$ admits real rational parametrizations.
- The envelope is real if $s(t)$ is not negative.
- Then we compute a decomposition

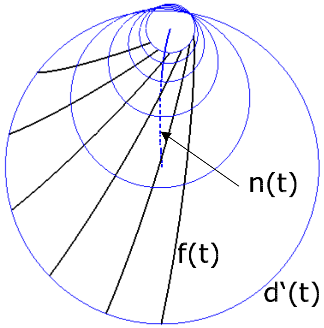
$$s(t) = s_1(t)^2+s_2(t)^2,$$

  with rational functions $s_1(t)$ and $s_2(t)$.
- Since $s(t)=s_d(t)/(s_n(t)^2)$, where $s_d(t)$ is a polynomial, we only have to decompose $s_d(t)$.
- This leads to

$$s_d(t)=\prod_{i=1}^{n} s_0(t-z_i)(t-\overline{z}_i)=(g_1+ig_2)(g_1-ig_2)$$

Rational parametrizations of envelopes of quadratic cones, 2

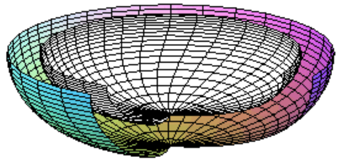
- With $s_1=g_1/s_n$, $s_2=g_2/s_n$, we obtain at first a solution $f(t)=(n_1+s_1,n_2+s_2)$ such that $f(t)$ satisfies $d'(t)$ for all t.
- Then a global parametrization $f(t,u)$ which satisfies $d'(t)$ identically for all t and u is computed.



- With $-2y_3=(y_1^2+y_2^2)(r_1+m_{13})+2y_1m_{11}+2y_2m_{12}+r_1-m_{13}$, we compute a parametrization of the surface $d(t)$ in I^3. This is a dual parametrization of the envelope of $D(t)$.

Corollaries

- The envelope F of a rational one-par. family of cones of rotation $D(t)$ is *a* rational offset surface.
- The offsets F_d of rational non-developable ruled surfaces F are rational and all its Laguerre transforms $T(F_d)$.
- The offsets F_d of rational canal surfaces F admit real rational parametrizations.
- The offsets of non-developable quadrics (ellipsoids, hyperboloids, paraboloids) and its Laguerre transforms admit real rational parametrizations.



Ellipsoid of rotation and outer offset surface

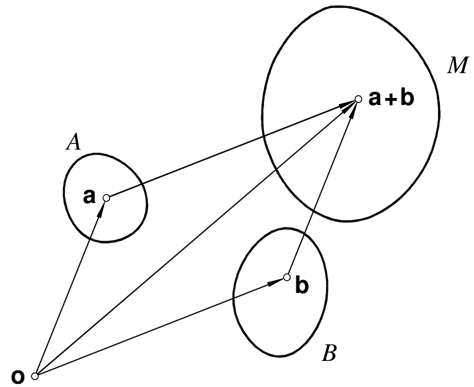
Classical geometric methods for the computation of Minkowski sum boundary surfaces

Contents

Definition and properties of the Minkowski Sum

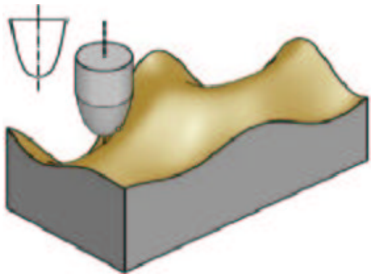
The Minkowski sum

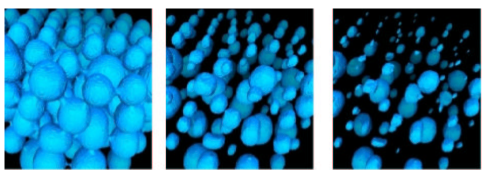
- We are given two objects *A* and *B* in R^3
- Their *Minkowski sum* is defined as the set
  $M = A \oplus B := \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}$,
  a and **b** ... coordinate vectors of arbitrary points in *A* and *B*
- Result is independent of the choice of the origin (up to a translation)



Applications

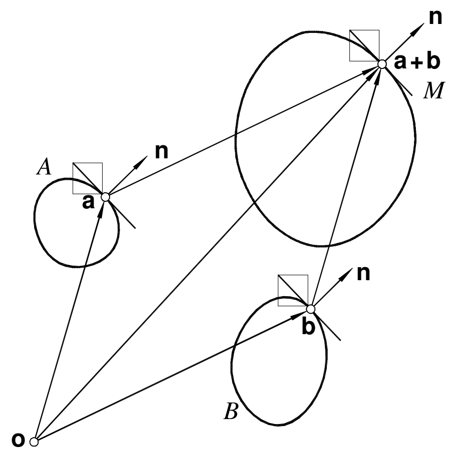
- NC tool path generation
- Robot motion planning
- Mathematical morphology





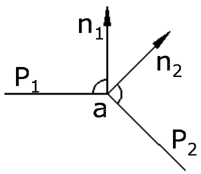
Minkowski sum of convex objects, 1

- Minkowski sum of two *convex bodies* is again a convex body
- Consider *outward unit normal vectors* to the points on the boundary surfaces of *A* and *B*
- Compute the sum **a**+**b** of those pairs (**a**,**b**), where the unit normal vectors are the same



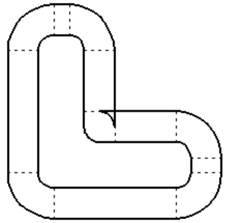
Minkowski sum of convex objects, 2

- Smooth case: this yields all boundary points of the Minkowski sum $M = A \oplus B$
- Non-smooth case: a vector n is called a normal at a point $\boldsymbol{a}$ if it is normal to a support plane P

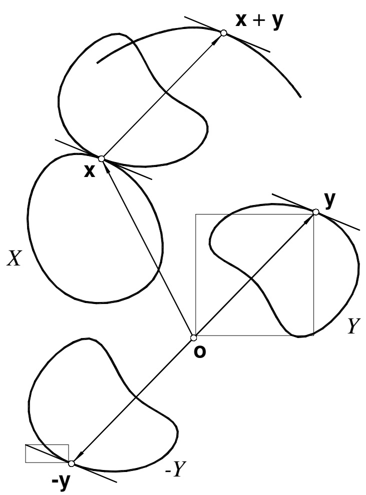


Minkowski sum of non-convex objects

- Again: search for point pairs ($\boldsymbol{a},\boldsymbol{b}$) with parallel outward unit normals and compute the vector sum $\boldsymbol{a}+\boldsymbol{b}$.
- In general, this gives a superset of the boundary of M.
- $\Rightarrow$ *Convolution surface*
- Trim away certain parts, which do not lie on the boundary of M.



Kinematic interpretation



- Let X be fixed, reflect c at the origin and let $-Y$ be movable.
- Convolution surface $X+Y$ is a *point trajectory of a translatory motion* of $-Y$ with respect to X, where the two surfaces remain in *point contact*.
- Equivalent: convolution surface is a part of the envelope of Y undergoing a translatory motion such that a reference point $\boldsymbol{p}$ in the moving system runs on X

Connection to offsets

- General offsets are the convolution surfaces of an arbitrary surface and a convex surface and appear in 3-axis sculptured surface machining
- Classical offsets are obtained, if the latter surface is a sphere.

Convolution surfaces

- Given two surfaces *A* and *B* in implicit or parametric form with normal vectors n_A, n_B.
  Points *a* (in *A*) and *b* (in *B*) are said to be corresponding, exactly if $n_A(a)$ is parallel to $n_B(b)$.
- Construction of the convolution *A+B*:
  Find parametrizations *a(u,v)* and *b(u,v)* over a common parameter domain in a way that $n_A(a) = \lambda\, n_B(b)$.

Convolution of paraboloid and parametrized surface, 1

- Let *A* be a paraboloid with $F_A=z-x^2-cy^2=0$, which admits the parametrization $a=(u,v,u^2+cv^2)$. Its normals are $n_A=(-2u,-2cv,1)$.
- Let *B* be parametrized by *b(s,t)* and let $n(s,t) = (n_1, n_2, n_3)(s,t)$ be a normal vector field of *B*.
- The condition $n_A(a) = \lambda\, n_B(b)$ gives
  $(-2u,-2cv,1)=\lambda\,(n_1, n_2, n_3)(s,t)$
  and leads to the reparametrization

$$\Phi : (s,t) \rightarrow (u(s,t), v(s,t))$$

$$u(s,t) = \frac{-n_1}{2n_3},\ v(s,t) = \frac{-n_2}{2cn_3}.$$

Convolution of paraboloid and parametrized surface, 2

- The determinant of the Jacobian of Φ is

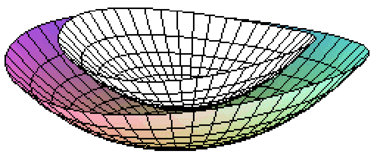
$$\det(J\Phi) = \frac{\det(n, n_s, n_t)}{4cn_3^{\,3}} = \frac{\Delta^2 K}{4cn_3^{\,2}}$$

  where *K* is the Gauss curvature and Δ is the determinant of the first fundamental form of B.
- The convolution *A+B* of *A* and *B* is given by

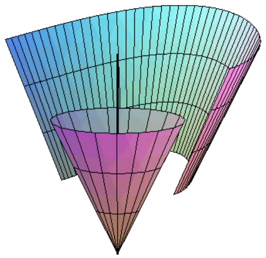
$$(a+b)(s,t) = \left[\frac{-n_1}{2n_3}+b_1,\ \frac{-n_2}{2cn_3}+b_2,\ \frac{cn_1^{\,2}+n_2^{\,2}}{4cn_3^{\,2}}+b_3\right]$$

Convolution of paraboloid and parametrized surface, 3

Theorem: The convolution *A+B* of a paraboloid *A* and a rational surface *B* is rational. If *B* is developable, *A+B* is developable, too.



Paraboloid (inside) and sum of paraboloid and ellipsoid



Cone of revolution and sum of paraboloid and cone

Explicit parameterization of the convolution surface of two ruled surfaces

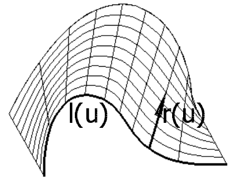
Ruled surfaces

- Parameterization:

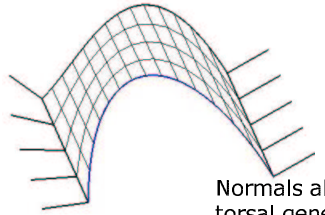
  $\mathbf{x}(u,\lambda) = \mathbf{l}(u) + \lambda \mathbf{r}(u)$

  $\mathbf{l}(u)$ ... directrix curve

  $\mathbf{r}(u)$ ... direction vector of the generator or ruling



- Non-torsal generator: Normal vectors turn around.
- Torsal generator: all its points have the same tangent plane and same unit normal
- Ruled surface with only torsal rulings: developable surface, otherwise skew ruled surface



Normals along non-torsal generator

Normals along torsal generator are parallel

Convolution surface of two skew ruled surfaces, 1

- We are given two ruled surfaces

  X...$\mathbf{x}(u,\lambda) = \mathbf{l}(u) + \lambda \mathbf{r}(u)$

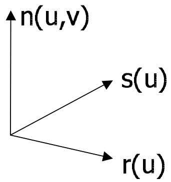
  Y...$\mathbf{y}(v,\mu) = \mathbf{m}(v) + \mu \mathbf{s}(v)$

- Problem: Find reparametrizations

  $\lambda = \lambda(u,v)$

  $\mu = \mu(u,v)$

  with the property that the normals in points $x(u,\lambda(u,v))$ and $y(v,\mu(u,v))$ are parallel.



- Solution: Unit normal of the convolution has to be orthogonal to $\mathbf{r}(u)$, $\mathbf{s}(v)$:

  $n(u,v) = \alpha\, r(u) \times s(u)$

Convolution surface of two skew ruled surfaces, 2

- The normals n_x and n_y at points x and y are $n_x=(l_u+\lambda r_u)\times r$ and $n_y=(m_v+\mu s_v)\times s$.
- We obtain the conditions $(l_u+\lambda r_u)\,(r\times s)=0$ and $(m_v+\mu s_v)(r\times s)=0$.
- From that the reparametrizations are

$$\lambda(u,v) = -\frac{\det(l_u, r, s)}{\det(r_u, r, s)},\quad \mu(u,v) = -\frac{\det(m_v, r, s)}{\det(s_v, r, s)}.$$

Explicit parameterization

- This gives the following explicit parameterization *(x+y)(u,v)* of the convolution surface *X+Y*:

$$(x+y)(u,v) = l(u,v) - \frac{\det(l_u, r, s)}{\det(r_u, r, s)} \mathrm{r(u)} + \mathrm{m(v)} - \frac{\det(m_v, r, s)}{\det(s_v, r, s)} \mathrm{s(v)}.$$

Results

- The convolution surface of two ruled surfaces can be *explicitly parameterized.*
- *Parallel rulings* yield *straight lines* on the convolution surface.
- If the two surfaces possess the *same curve at infinity*, the convolution surface *contains a ruled surface.*
- For *torsal rulings*, only the *singular point* (cuspidal point) contributes to the Minkowski sum.
- If the two given ruled surfaces *X* and *Y* are *rational*, *(x+y)(u,v)* is a *rational parameterization* of the convolution surface *X+Y*.

Examples

Convolution surface of two hyperbolic paraboloids

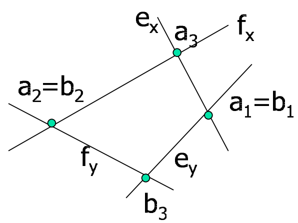
- We are given two hyperbolic paraboloids

  X … $\mathbf{l}(u) = \mathbf{a}_0 + \mathbf{a}_1 u,\ \mathbf{r}(u) = \mathbf{a}_2 + \mathbf{a}_3 u$

  Y … $\mathbf{m}(v) = \mathbf{b}_0 + \mathbf{b}_1 v,\ \mathbf{s}(v) = \mathbf{b}_2 + \mathbf{b}_3 v$

- *X* and *Y* intersect the ideal plane in lines $e_x = a_3 a_1$, $f_x = a_3 a_2$, and $e_y = b_3 b_1$, $f_y = b_3 b_2$.

- By a reparametrisation we may assume that $a_1 = b_1$ and $a_2 = b_2$.



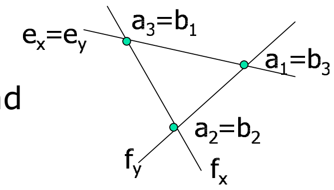
General case

- Since a translation of X(or Y) only results in a translated convolution $X+Y$, we let $a_0=b_0=(0,0,0)$.
- Further we may choose $a_1=b_1\ (0,1,0)$, $a_2=b_2(1,1,1)$, $a_3=(0,0,a)$ and $b_3=(b,0,0)$.
- We obtain the following parametrization of the convolution $X+Y$:

$$(x+y)(u,v)=\frac{1}{abuv}\begin{bmatrix}(au+bv+abuv)(u+v(1+bv))\\(au+bv+2abuv)(u+v)\\(au+bv+abuv)(u(1+au)+v)\end{bmatrix}$$

Special cases of the convolution of hyperbolic paraboloids

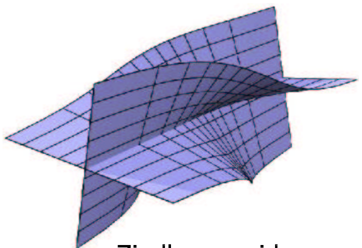
- The surfaces X and Y share a line at infinity ($e_x=e_y$) but have different axes ($a_3 \neq b_3$).
- We may choose $a_1=(0,0,1)$, $b_1=(1,0,0)$, $a_2=b_2=(0,1,0)$, $a_3=(a,0,0)$ and $b_3=(0,0,b)$, and obtain the following parametrization for $X+Y$:

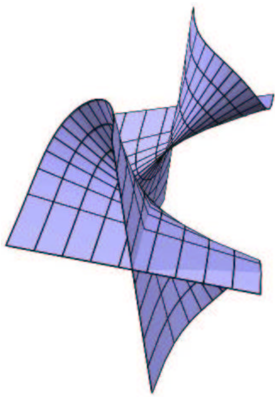


$$(x+y)(u,v)=\frac{1}{abuv}\begin{bmatrix}(au)(bv^2-au^2)\\-(au^2+bv^2)\\bv(au^2-bv^2)\end{bmatrix}$$

Common line at infinity

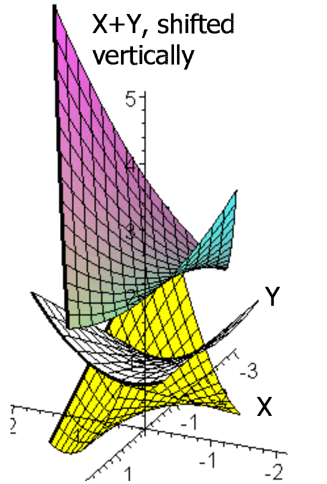
- Depending on whether $ab>0$ or $ab<0$, the convolution surface $X+Y$ of two hyperbolic paraboloids with a common line at infinity is projectively equivalent to the Zindler conoid or to the Plücker conoid.


Zindler conoid


Plücker conoid

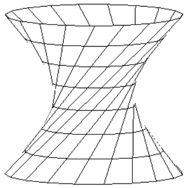
Parallel axes and common lines at infinity

- Parallel axes ($a_3 = b_3$): Convolution surface is a paraboloid with parallel axis, can be elliptic or hyperbolic, see figure.
- Parallel axis and common line(s) at infinity: Convolution surface is a *hyperbolic paraboloid* with the same line(s) at infinity.

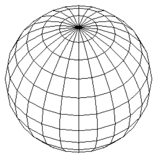


Pseudoeuclidean offsets of ruled surfaces

- Let $X: x^2+y^2-z^2-1=0$ be a hyperboloid of revolution. X may be considered as sphere in *pseudoeuclidean 3-space* $R^3{}_1$, (Minkowski space, Lorenz space).
- *Pseudoeuclidean offset surfaces* of skew rational ruled surfaces possess real rational parameterizations.
  - Euclidean counterpart: Pottmann, Lü, Ravani, 1996 more complicated



$x^2+y^2-z^2-1=0$



$x^2+y^2+z^2-1=0$

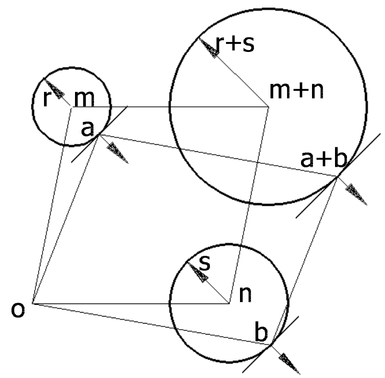
Euclidean offsets of ruled surfaces

- Using the complex extension C^3 of Euclidean space R^3, every quadric – especially a sphere – is a ruled surface.
- Computation of complex rational parameterizations of the Euclidean offsets of a rational ruled surface.
- The *convolution surface* of any *quadric* and a *rational ruled surface* admits *complex rational parametrizations*. It can be proved that even *real rational parametrizations* exist.
- It can be proved that the *convolution of two quadrics* admit *real (improper) rational parametrizations*.

Parameterizing the convolution surface of two canal surfaces

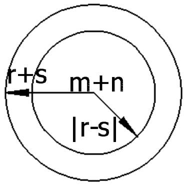
Minkowski sum of spheres

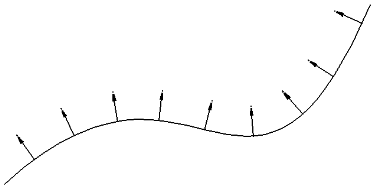
- Consider two balls (solid spheres) in R^3 with centers $\boldsymbol{m},\boldsymbol{n}$ and radii r,s
- Add pairs of points with parallel outward unit normal vector
- Minkowski sum is again a ball with center $\boldsymbol{m}+\boldsymbol{n}$ and radius $r+s$



Minkowski sum of spheres

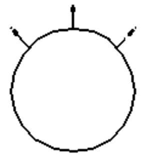
- We only need the boundary spheres S,T of the balls A,B to compute the Minkowski sum $A \oplus B$
- But: the Minkowski sum $S \oplus T$ is a different set, bounded by two concentric spheres with center $m+n$ and radii $r+s$ and $|r-s|$
- Consider oriented surfaces (surfaces plus normal vector field)

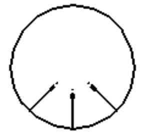




Oriented spheres

- The orientation of the spheres will be represented by the sign of the radius r:
  - $r>0$: surface normal pointing to the *exterior*
  - $r<0$: surface normal pointing to the *interior*
  - $r=0$: *point*





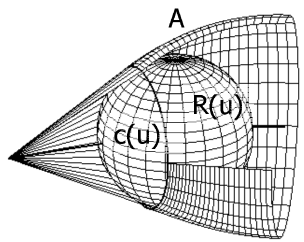
- $S \oplus T$ is a *sphere* with *center* $\mathbf{m}+\mathbf{n}$ and *radius* $r+s$

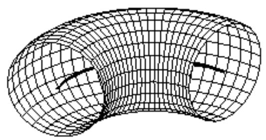
Canal surfaces

- A *canal surface* A is the envelope of a one-parameter family of spheres
  $R(u) :(x-m(u))^2 - r(u)^2=0$
  with centers $m(u)$ and radii $r(u)$.
- The envelope A consists of the characteristic circles $c(u)=R(u) \cap R_u(u)$, where $R_u(u)$ denotes the derivative of $R(u)$ w.r.t. u,
  $R_u(u):(x-m(u))m_u + r(u)r_u(u)=0$.

Canal surfaces

- Exactly if $m_u^2-r_u^2 \geq 0$ holds, the intersection $R(u) \cap R_u(u)$ is a real circle $c(u)$. The canal surface A is tangent to the sphere $R(u)$ in points of $c(u)$.
- The orientation of the inscribed spheres $R(u)$ induces an orientation of the canal surface A.
- Special case: $r(u)=const.$:
  A is called *pipe surface*.

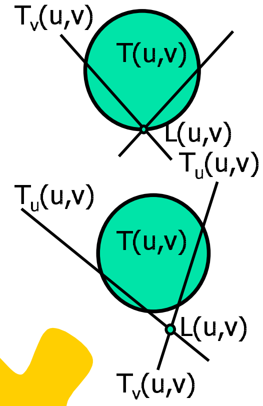




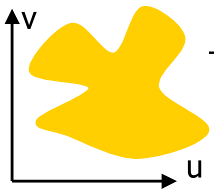
Convolution of canal surfaces, 1

- Consider two canal surfaces
  A... $R(u): (x-m(u))^2 - r(u)^2 = 0$
  B... $S(u): (x-n(v))^2 - s(v)^2 = 0$
- The convolution surface of each pair of spheres $R(u)$,$S(v)$ is a sphere
  $T(u,v): (x-(m(u)+n(v)))^2 - (r(u)+s(v))^2 = 0$.
- The envelope of the two-parameter family of spheres $T(u,v)$ is obtained as solution of the system of equations
  $T(u,v) : (x-(m+n))^2 - (r+s)^2 = 0,$
  $T_u(u,v): (x-(m+n))m_u + (r+s)r_u = 0,$
  $T_v(u,v): (x-(m+n))n_v + (r+s)s_v = 0.$

Convolution of canal surfaces, 2



- For fixed (u_0,v_0) the equations of T_u,T_v are linear and represent planes and $L(u,v)=T_u \cap T_v$ is a straight line.
- The intersection $I=L\cap T$ is contained in the envelope of the family $T(u,v)$.
- For a fixed (u_0,v_0) the intersection $I(u_0,v_0)$ can consist of 2,1 or 0 real points of the envelope. The figure shows a possible pre-image of the real part of the envelope of $T(u,v)$.

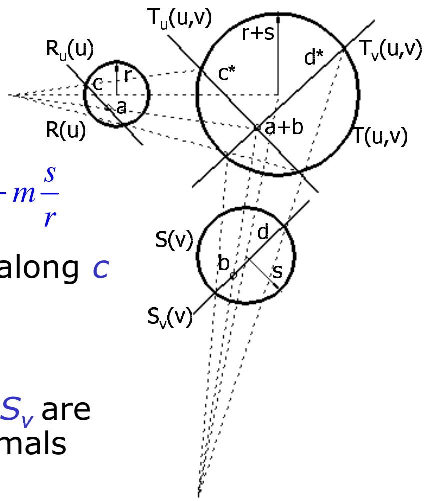


Convolution of canal surfaces, 3

Apply a similarity ρ which maps R to T:

- ρ maps R_u to T_u and $c=R \cap R_u$ to a circle $c^*=T\cap T_u$.

$$\rho: x' = x\frac{r+s}{r} + n - m\frac{s}{r}$$



- The surface normals of R along c are parallel to the surface normals of T along c^*.
- Analogously, the surface normals of S along $d= S\cap S_v$ are parallel to the surface normals of T along $d^*=T\cap T_v$.

Convolution of canal surfaces, 4

- The unit normal vector of the envelope surface at each point $\boldsymbol{x}$ of $I(u,v)$ agrees with the unit normal vector of the canal surface A at a point $\boldsymbol{a}$ of c and with the unit normal vector of the canal surface B at a point $\boldsymbol{b}$ of d.
- $\boldsymbol{a}$ and $\boldsymbol{b}$ are corresponding point pairs for the generation of the convolution surface.
- $\boldsymbol{a}+\boldsymbol{b}$ lies on the sum-sphere T, thus $\boldsymbol{x}=\boldsymbol{a}+\boldsymbol{b}$.

Result

- The convolution surface *A+B* of two canal surfaces *A* and *B*, which are the envelopes of the one-parameter families of spheres *R(u)* and *S(v)*, respectively, is the *envelope* of the two-parameter family of spheres *T(u,v).*

  Thus: *A+B* can be considered as cyclographic image.

- The solution of the system

  $$\{T(u,v)=0,\ T_u(u,v)=0,\ T_v(u,v)=0\}$$

  yields an *explicit parameterization* $\mathbf{x}(u,v)$ of the convolution surface *A+B*.

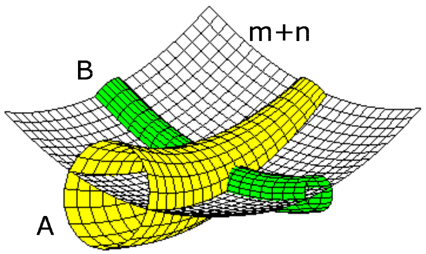
Convolution of canal surfaces and the cyclographic mapping

- Canal surface *A* enveloped by spheres *R(u)* is the cyclographic image of the curve $R^c(u) \subset R^4$. ($A = (R^c(u))^\gamma$)

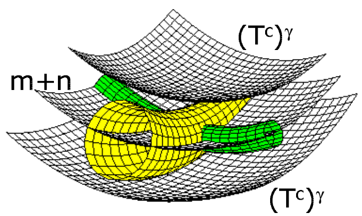
- Likewise, *B* is the cyclographic image of the curve $S^c(v) \subset R^4$. ($B = (S^c(v))^\gamma$)

- This implies that the convolution *A+B* is the cyclographic image of the translational surface $T^c(u,v) = R^c(u)+S^c(v) \subset R^4$.
  ($A+B = (T^c(u,v))^\gamma = (R^c(u)+S^c(v))^\gamma$).

Two canal surfaces and their convolution surface

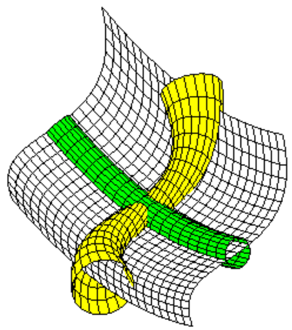


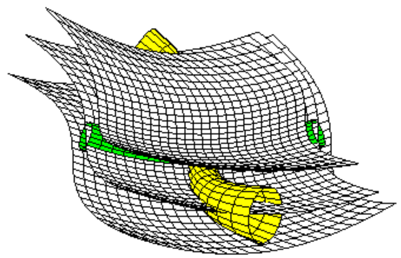
Canal surfaces *A* and *B*, enveloped by spheres *R(u)* and *S(v)* whose centers are on curves *m(u)* and *n(v)*. Translational surface *m(u)+n(v)*.



Convolution surface *A+B* enveloped by spheres *T(u,v)* whose centers are on *m(u)+n(v)*.

Two canal surfaces and their convolution surface, 2





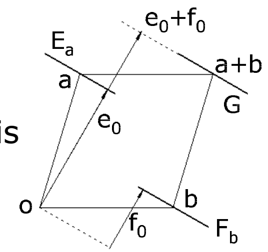
Convolution Surfaces and the Isotropic Model

Isotropic image of the Convolution in I^3, 1

- Let A: $a(u,v)$ and B: $b(s,t)$ be two parametrized surfaces. We study the isotropic images A^i, B^i and $(A+B)^i$ of the surfaces A, B and $A+B$.
- We consider A and B to be families of tangent planes $E(u,v): e_0+ex=0$, $F(s,t): f_0+fx=0$, where $e^2=1$ and $f^2=1$ shall hold.
- Surfaces A and B possess the isotropic images
  $A^i: E^i=1/(1-e_3)(e_1,e_2,e_0)$, and
  $B^i: F^i=1/(1-f_3)(f_1,f_2,f_0)$.

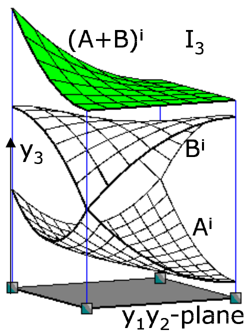
Isotropic image of the Convolution in I^3, 2

- Points a and b are corresponding, exactly if the tangent planes E_a, F_b are parallel. This implies that $e_a=f_b$, ($e_a^2=f_b^2=1$).
- The convolution $C=A+B$ is formed by $a+b$, with respect to corresponding points a, b. The tangent plane at $a+b$ to $A+B$ is
  G: $e_0+f_0+gx = 0$,
  with $g=e=f$ and $g^2= e^2=f^2=1$.
- The isotropic image of $C=A+B$ is
  $G^i=1/(1-g_3)(g_1,g_2,e_0+f_0)$.
- Thus, E^i,F^i and $G^i=E^i+F^i$ have identical first and second coordinate.



Correspondence in I^3

- The correspondence between points a and b of A and B with respect to parallel tangent planes is realized in I^3 by the correspondence between A^i and B^i by vertical lines (y^3-parallel).
- The problem of finding correspondences between points a and b of A and B is translated to the problem of finding simultaneous parametrizations of A^i and B^i over the y_1y_2-plane.



Results

- Let $F^i=(u,v,f(u,v))$ and $G^i=(u,v,g(u,v))$ be rational surfaces in I^3, parametrized over the y_1y_2-plane. F and G are isotropic images of *rational offset surfaces*. Their *convolution* $H=F+G$ is a *rational offset surface* and a parametrization of H's isotropic image is given by $H^i=(u,v,f(u,v)+g(u,v))$.
- In particular, if $f(u,v)$ is a (general) quadratic polynomial, F is a parabolic Dupin cyclide.
- The convolution $H=F+G$ of a parabolic Dupin cylide and a rational offset surface G, whose isotropic image G^i is graph of a rational function, is a *rational offset surface*.

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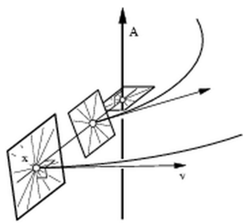
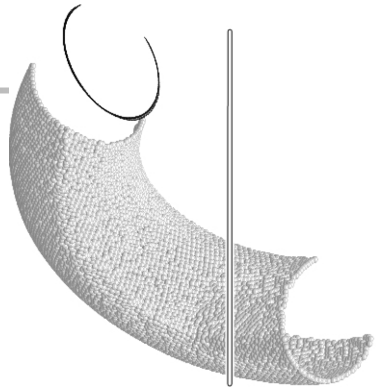
Line Geometry for 3D Shape Understanding and Reconstruction

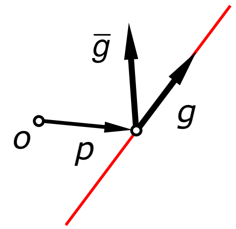
M. Hofer, B. Odehnal, J. Wallner





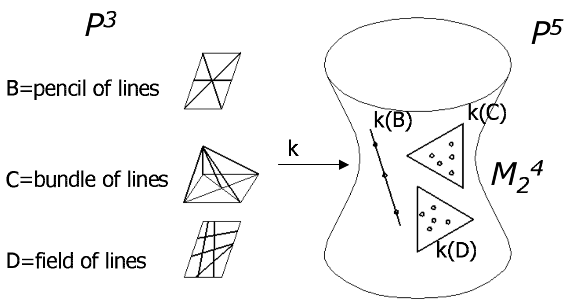
Line geometry, 1

- Describe lines by their *Plücker coordinates*
  $$(g, \bar{g}) = (g, p \times g)$$
  where p is a point of the line G parallel to the vector g
- *Plücker relation*
  $$g \cdot \bar{g} = 0$$
  $$g = (g_1, g_2, g_3), \bar{g} = (g_4, g_5, g_6).$$



Line geometry, 2

- Interpret Plücker coordinates $(g_1,\ldots,g_6)$ of a line G as homogeneous coordinates of a point in projective 5-space
- Yields Klein mapping k from lines in P^3 to points of a quadric M_2^4 in P^5



A point model related to the Klein model

- Klein model not well suited for the solution of approximation problems
- Alternative: Introduce the normalization $\|l\| = 1$ and interpret $(l, \bar{l})$ as point in R^6
- Results in a mapping from *lines* in Euclidean 3-space to *points* of a *four-dimensional manifold* M^4 in R^6
- M^4 is of algebraic order 4: Intersection of the cylinder S^5 and the cone Γ^5
  $$S^5 : x^2 = 1, \Gamma^5 = x \cdot \bar{x} = 0$$

Metric

- The canonical Euclidean metric in R^6 is used to define the distance between lines G and H by

$$d(G,H) = (g-h)^2 + (\bar{g}-\bar{h})^2$$

- It is a useful distance measure for lines whose distance to the origin is not too large
- Region of interest is necessary anyway for a practical distance measure between lines

Linear line complex

- A *linear complex* $C = (c_1,\ldots,c_6)$ is a 3-parameter set of lines whose Plücker coordinates $(g_1,\ldots,g_6)$ satisfy a linear homogeneous equation

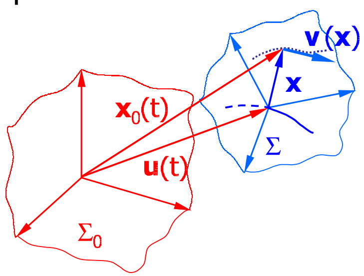
$$\bar{c}\cdot g + c\cdot\bar{g} = c_4 g_1 + c_5 g_2 + c_6 g_3 + c_1 g_4 + c_2 g_5 + c_3 g_6 = 0$$

- Image in R^6 is a *hyperplanar cut* of M^4
- Intersections of M^4 with lower dimensional linear subspaces also possess a simple geometric meaning (linear congruences, in particular spread; bundle, field, regulus)

Kinematics

- One-parameter motion in Euclidean 3-space
- At each time instant the velocity vector field is linear in x and computed with

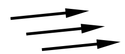
$$v(x) = \bar{c} + c \times x$$



One-parameter motions with constant velocity vector field, 1

1. uniform translation

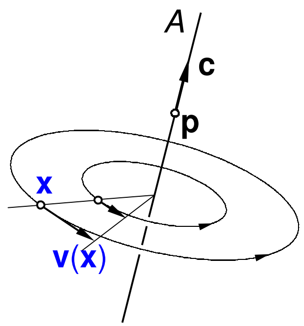
$$c = 0 \Rightarrow v(x) = \bar{c}.$$



2. uniform rotation

$$v(x) = \bar{c} + c \times x \text{ with } c\cdot\bar{c} = 0.$$

Rotation axis A has direction vector c and passes through points p with $\bar{c} = c \times p$.
($\bar{c}$ … moment vector of the axis A)



One-parameter motions with constant velocity vector field, 2

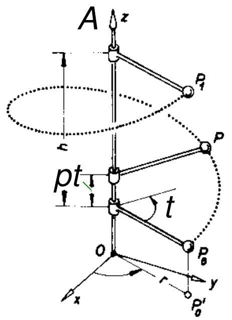
3. *uniform helical motion*

$v(x) = \bar{c} + c \times x$ with $c \cdot \bar{c} \neq 0$.

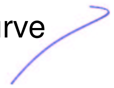
Helical motion is the composition of
- a *rotation* about an axis A and
- a *proportional translation* parallel to A

$$x(t) = x_0 \cos(t) - y_0 \sin(t)$$
$$y(t) = x_0 \sin(t) + y_0 \cos(t)$$
$$z(t) = z_0 + pt$$

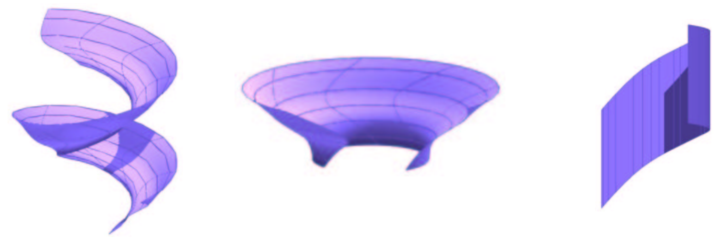
p … pitch



Uniform helical motion

A curve  generates for pitch p these types of surfaces:

$p \neq 0$ (helical motion)	$p = 0$ (pure rotation)	‘$p = \infty$’ (pure translation)



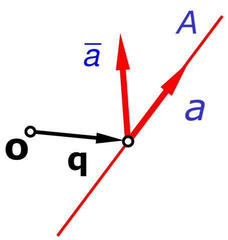
helical surface | rotational surface | cylindrical surface

Axis and pitch of a helical motion

- From the vector $C = (c, \bar{c}) \in R^6$ the *axis* $A = (a, \bar{a})$ and the *pitch p* of the underlying helical motion are calculated by:

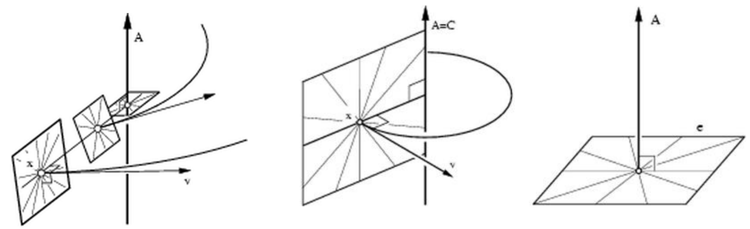
$$a = \frac{c}{\|c\|}, \quad \bar{a} = \frac{\bar{c} - pc}{\|c\|}, \quad p = \frac{c \cdot \bar{c}}{c^2}$$

- a … direction vector of axis A
- $\bar{a}$ … moment vector of axis A



Linear line complex and surface normals, 1

- *Path normals* of a helical motion, rotation, translation

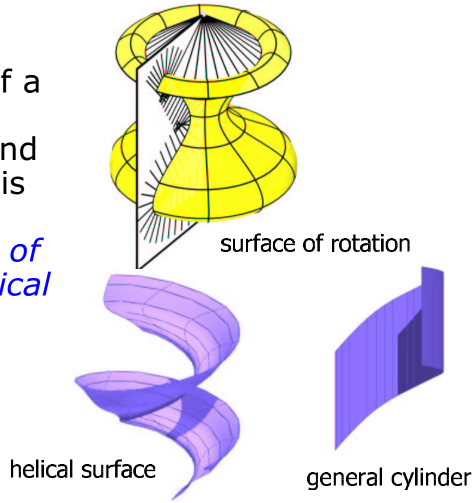


- *Path normals* of a helical motion lie in a linear line complex, i.e. a 3-par. family of lines whose Plücker coordinates satisfy the linear homogeneous equation

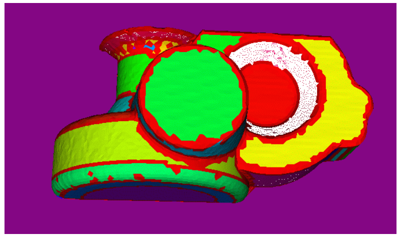
$$\bar{c} \cdot g + c \cdot \bar{g} = 0$$

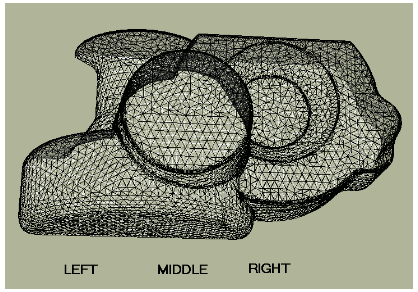
Linear line complex and surface normals, 2

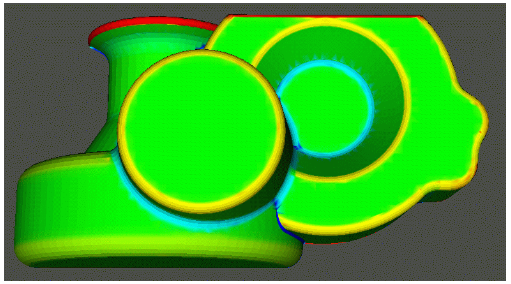
- The *normal lines* of a C^1 surface lie in a *linear complex* if and only if the surface is contained in a *cylinder*, a *surface of revolution* or a *helical surface*.



Recognition and Reconstruction of Rotational and Helical Surfaces



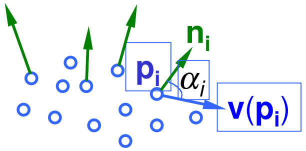




Estimation of the generating motion of a kinematic surface

Given: Point cloud $p_1, p_2, \ldots, p_N$ representing a surface

1. step: estimation of surface normal vectors $n_1, n_2, \ldots, n_N$



2. step: calculate an appropriate motion with the velocity vector field $v(x) = \bar{c} + c \times x$.

ideally: $\alpha_i = \pi/2 \Leftrightarrow$ normals lie in linear line complex

$\Rightarrow$ *approximation problem in line space*

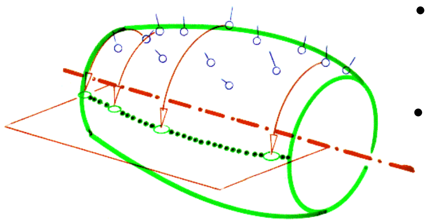
Fitting a linear line complex to surface normals

- Estimated surface normals determine a cloud C of points on the manifold M^4 in R^6
- Fitting a 5-dimensional subspace to the point cloud C yields an approximating linear complex
- Equivalent to *principal component analysis* on C

Shape info from the spectrum

- The distribution of eigenvalues gives important information on special shapes:
  - *Three* small eigenvalues (in relation to extension of data point cloud in R^3): surface is part of a *sphere* or a *plane*
  - *Two* small eigenvalues: surface is part of a *right circular cylinder*
  - *One* small eigenvalue: surface is part of a general *cylinder, rotational surface or helical surface* (decision based on the axis of the complex)

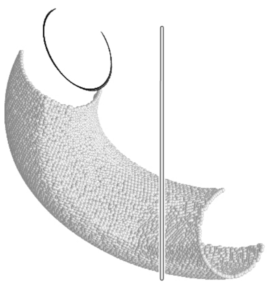
Reconstruction process



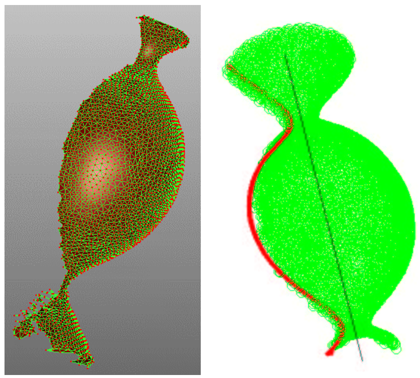
- Estimation of *surface normals* in the *data points*.
- Computation of the *one parameter motion* whose path normals fit the estimated normals best.
- Reconstruction of the *profile curve* by moving the data points into an appropriate plane.
- Reconstruction of the *surface* by applying the one parameter motion to the profile curve.

Examples

- Reconstructed *axis* and *profile curve* of a *helical surface*



- Reconstructed *axis* and *profile* (meridian) *curve* of a *surface of rotation*



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