G^1-Hermite Interpolation of Ruled Surfaces

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Abstract. This article discusses two methods for G^1-Hermite interpolation of ruled surfaces with low degree rational ruled surfaces. We will interpret ruled surfaces as one-parameter families of straight lines. Given two generating lines G_1, G_2 and tangent planes at points of these lines, we want to construct polynomial or rational ruled surfaces of low degree interpolating these boundary data.

§1. Introduction and Fundamentals

Ruled surfaces are among the simplest surfaces used for modeling and design, since one family of parameter curves are straight lines. Applications of ruled surfaces include surface modeling, motion design and wire-cut EDM. In the last years several articles deal with design of ruled surfaces, see for instance [2,5,7]. Different viewpoints and techniques can be chosen to study ruled surfaces. Here, they shall be treated as (differentiable) one-parameter families of lines.

A ruled surface Φ in Euclidean space $\mathbb{R}^3$ possesses a parametric representation

$$\mathbf{x}(u,v) = \mathbf{a}(u) + v\mathbf{r}(u), \qquad u \in I, v \in \mathbb{R}, \tag{1}$$

where $\mathbf{a}(u)$ denotes a directrix curve and $\mathbf{r}(u) \neq (0,0,0)$ denotes a vector field. In the following, we assume sufficient differentiability and regularity of the functions involved. The generating lines $G(u)$ of Φ are obtained by inserting a constant u_0 into (1). If $\mathbf{r}(u) = \mathbf{c}$ is a constant vector, (1) parametrizes a general cylinder.

The tangent plane at a regular surface point is spanned by the partial derivative vectors $\mathbf{x}_u = \dot{\mathbf{a}} + v\dot{\mathbf{r}}$ and $\mathbf{x}_v = \mathbf{r}$. Thus, the surface normal at $\mathbf{x}$ is

$$\mathbf{n}(u,v) = \mathbf{x}_u \times \mathbf{x}_v = \dot{\mathbf{a}}(u) \times \mathbf{r}(u) + v(\dot{\mathbf{r}}(u) \times \mathbf{r}(u)) = \mathbf{n}_1(u) + v\mathbf{n}_2(u).$$

A generating line $G(u_0)$ is called non–torsal iff $\det(\dot{\mathbf{a}}(u_0), \mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) \neq 0$, which expresses linear independence of $\mathbf{n}_1 = \dot{\mathbf{a}} \times \mathbf{r}$ and $\mathbf{n}_2 = \dot{\mathbf{r}} \times \mathbf{r}$.

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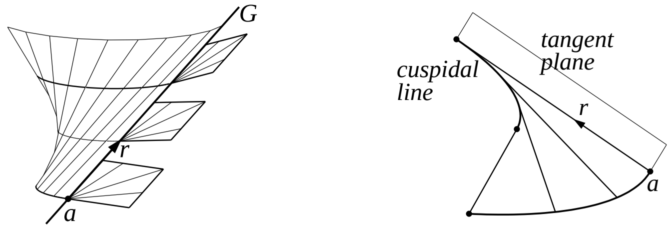

Fig. 1. Non-torsal ruled surface and torsal ruled surface with cuspidal line.

The surface normals along the generator $G(u_0)$ are parametrized by

$$\mathbf{y}(v,w) = \mathbf{a}(u_0) + v\mathbf{r}(u_0) + w(\mathbf{n}_1(u_0) + v\mathbf{n}_2(u_0)).$$

Because of bilinearity in the parameters v and w, this parametrization represents a hyperbolic paraboloid. The points $\mathbf{a} + v\mathbf{r}$ on $G(u_0)$ are in bijective correspondence to the normals $\mathbf{n}_1 + v\mathbf{n}_2$ (or tangent planes) by

$$\mathbf{a}(u_0) + v\mathbf{r}(u_0) \mapsto \mathbf{n}_1(u_0) + v\mathbf{n}_2(u_0). \tag{2}$$

We can extend formula (2) for the parameter value $v = \infty$. This implies that the point at infinity of $G(u_0)$ possesses a tangent plane with normal vector $\mathbf{n}_2(u_0)$. The mapping (2) is called contact projectivity along a non–torsal generator. This leads to the following Lemma.

Lemma 1. *Two ruled surfaces $\mathbf{x}_1, \mathbf{x}_2$ are tangent at all points of a common generator G, iff $\mathbf{x}_1, \mathbf{x}_2$ possess same tangent planes at three points of G.*

A generator $G(u_0)$ is called torsal if all regular points of $G(u_0)$ possess the same tangent plane. Analytically, we have

$$\det(\dot{\mathbf{a}}(u_0), \mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) = 0. \tag{3}$$

There are two cases to be distinguished. First, if $rank(\mathbf{r}(u_0), \dot{\mathbf{r}}(u_0)) = 1$, all points on $G(u_0)$ are regular and the common surface normal along $G(u_0)$ is $\dot{\mathbf{a}} \times \mathbf{r}$. If Φ is a cylinder surface, all rulings are of this type and parallel to each other. Thus, $G(u_0)$ are called cylindrical. Secondly, if $G(u_0)$ is not cylindrical, there exists exactly one singular point on $G(u_0)$, whose parameter value is

$$v_c = -\frac{(\dot{\mathbf{a}} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \mathbf{r})}{(\dot{\mathbf{r}} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \mathbf{r})}. \tag{4}$$

It is called cuspidal point. The surface normal in all other points of $G(u_0)$ is $\dot{\mathbf{a}} \times \mathbf{r}$. A ruled surface Φ is called torsal, if it is developable, which expresses that Φ can be represented as envelope of its one-parameter family of tangent planes

$$(\mathbf{x} - \mathbf{a}) \cdot (\dot{\mathbf{a}} \times \mathbf{r}) = 0.$$

The singular curve $\mathbf{x}(u, v_c(u)) = \mathbf{c}(u)$ on Φ is called line of regression or cuspidal line. If the curve $\mathbf{c}(u)$ consists of one point only, Φ is called cone.

If $G(u_0)$ is a non-torsal generator, the parameter value $v_s = v_c$ in (4) parametrizes the striction curve $\mathbf{a}(u) + v_s\mathbf{r}(u)$. The distribution parameter

$$\delta(u) = -\frac{\det(\dot{\mathbf{a}}, \mathbf{r}, \dot{\mathbf{r}})}{(\mathbf{r} \times \dot{\mathbf{r}})^2}(u)$$

is a Euclidean differential invariant of first order. It measures how fast the tangent plane turns around $G(u_0)$ if the point $\mathbf{x}(u_0, v)$ travels along $G(u_0)$. It is a signed invariant, and is zero for torsal generators (except for some cylindrical generators of higher order). For more details on Euclidean line geometry see for instance [1,3].

§3. An Elementary Method for G^1-Hermite Interpolation

Let $R : \mathbf{x}(u, v) = \mathbf{y}(u) + v\mathbf{r}(u)$ be a given ruled surface in $\mathbb{R}^3$, $u \in I$ and $v \in \mathbb{R}$. We assume that there is a cartesian coordinate system such that all generating lines $G(u) = \mathbf{y}(u) + v\mathbf{r}(u)$ intersect two parallel planes $E_1 : z = 0$ and $E_2 : z = 1$. Let $\mathbf{c}_1(u)$, $\mathbf{c}_2(u)$ be the intersection curves of R with E_1, E_2. We pick a sequence of generating lines G_i, $i = 1, \ldots, N$, and compute tangent planes along them. Our task is to determine a ruled surface S which interpolates two adjacent generators $G, H \subset \{G_i\}$ such that S and R are tangent at all points of G and H. Let

$$\mathbf{a} = G \cap E_1, \quad \mathbf{b} = H \cap E_1, \quad \mathbf{p} = G \cap E_2, \quad \mathbf{q} = H \cap E_2.$$

The intersection points of the tangent planes at $\mathbf{a}, \mathbf{b}$ and $\mathbf{p}, \mathbf{q}$ along G, H with E_1, E_2 are the inner points $\mathbf{c}, \mathbf{r}$, see Fig. 2.

We construct a low degree rational (or polynomial) ruled surface S as tensor product surface of degrees $(d, 1)$,

$$S : \mathbf{x}(u, v) = (1 - v)\mathbf{k}_1(u) + v\mathbf{k}_2(u),$$

such that the planar intersection curves $\mathbf{k}_1$ and $\mathbf{k}_2$ interpolate points $\mathbf{a}, \mathbf{b}$ and $\mathbf{p}, \mathbf{q}$ plus the given tangents determined by $\mathbf{c}$ and $\mathbf{r}$. This yields that S and R possess common generating lines G, H and same tangent planes at the points $\mathbf{a}, \mathbf{p}$ and $\mathbf{b}, \mathbf{q}$.

Applying Lemma 1, R and S are tangent at all points of G (or H), if they have the same tangent plane at a *third* point, different from $\mathbf{a}, \mathbf{p}$ (or $\mathbf{b}$, $\mathbf{q}$). For simplicity we choose this third point as midpoint of $\mathbf{a}, \mathbf{p}$ (or $\mathbf{b}, \mathbf{q}$), see Fig. 2.

There is a one parameter family of conics

$$\mathbf{k}_1(u) = \frac{(1-u)^2\mathbf{a} + 2tu(1-u)\mathbf{c} + u^2\mathbf{b}}{(1-u)^2 + 2tu(1-u) + u^2}$$

satisfying the G^1-requirements in the plane E_1. Since we set the weights at $\mathbf{a}, \mathbf{b}$ to 1, we use a special parametrization here. Additionally there is a one parameter family of conics

$$\mathbf{k}_2(u) = \frac{(1-u)^2\mathbf{p}w_1 + 2u(1-u)\mathbf{r} + u^2\mathbf{q}w_2}{w_1(1-u)^2 + 2u(1-u) + w_2u^2}$$

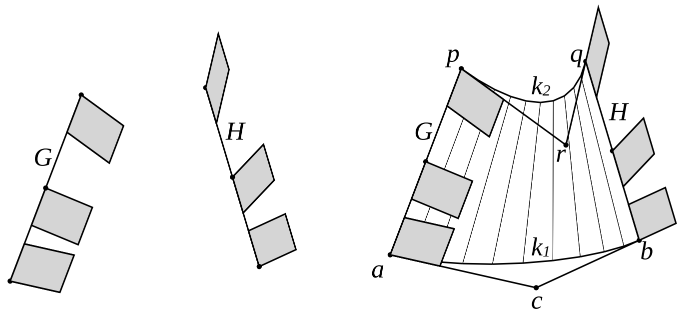

Fig. 2. G^1 Hermite boundary data and solution of the interpolation problem.

satisfying the G^1-requirements in the plane E_2. We put weights w_1, w_2 to points $\mathbf{p}, \mathbf{q}$ to be flexible with the parametrization. Further, let $\mathbf{n}$ and $\mathbf{m}$ be the given surface normals at the midpoints $1/2(\mathbf{a}+\mathbf{p})$ and $1/2(\mathbf{b}+\mathbf{q})$.

Inserting the G^1 condition for the midpoints, we find that the surface S is tangent to R at all points of G and H iff the following conditions on the parameters t, w_1, w_2 hold:

$$tw_1 = -\frac{(\bar{\mathbf{r}}-\bar{\mathbf{p}})\cdot\bar{\mathbf{n}}}{(\bar{\mathbf{c}}-\bar{\mathbf{a}})\cdot\bar{\mathbf{n}}} = \tau_1, \quad tw_2 = -\frac{(\bar{\mathbf{q}}-\bar{\mathbf{r}})\cdot\bar{\mathbf{m}}}{(\bar{\mathbf{b}}-\bar{\mathbf{c}})\cdot\bar{\mathbf{m}}} = \tau_2. \tag{5}$$

Here, $\bar{\mathbf{x}} = (x_1, x_2)$ denotes the vector built by the first two coordinates of the vector $\mathbf{x} \in \mathbb{R}^3$.

Expressing w_1, w_2 in terms of t yields a one parameter family of ruled surfaces $S(t)$ solving the G^1-Hermite interpolation problem.

Useful solutions

The above discussed algorithm results in useful solutions if τ_1, τ_2 possess the same sign. If this is not the case, one of the conics possesses points at infinity. This is caused by a too large difference of the distribution parameters at G and H. To avoid this, we can choose generators G, H 'closer'; or, alternatively, we let E_2 be the plane $z = 0.5$, for instance, which reduces the turning angle of the tangent planes. If the segments on G, H are sufficiently small and G and H are close enough, τ_1, τ_2 will have same sign. For a useful distance measure between lines, see [6].

If G is a torsal generator, $(\bar{\mathbf{c}}-\bar{\mathbf{a}})\cdot\mathbf{n} = 0$ and $(\bar{\mathbf{r}}-\bar{\mathbf{p}})\cdot\bar{\mathbf{n}}$ vanishes too. Thus, the G^1-condition along a torsal generator is already satisfied, and the weight w_1 can be chosen arbitrarily.

If both generators are torsal, all weights can be chosen arbitrarily, for instance $= 1$. The solution S is in general a non–torsal polynomial ruled surface of degree 4 with two torsal generators.

Theorem 2. *Given G^1-Hermite boundary data of a ruled surface, there is a one parameter family $S(t)$ of rational ruled surfaces of degrees (2,1) and order 4 which interpolate the given data. The surfaces $S(t)$ carry a one parameter family of conics, and intersect two given planes E_1, E_2 in conics.*

The presented method is general and works for arbitrary input data and the planes E_1, E_2 need not be parallel.

In Section 4 we will compute a lowest degree solution of the G^1-Hermite interpolation problem, which consists of a smoothly joined pair of ruled quadrics. But, since quadrics never possess torsal generators, this method is restricted to ruled surfaces without torsal generators. To derive this method we have to introduce some theory about lines in space.

A modeling scheme using cubic ruled surfaces is difficult because it can be proved that cubic surfaces do not fit all possible data. A combined method consisting of quadric pairs and cubic surface segments is discussed in [5].

§3. Local Coordinates of Lines

A local parametrization of the set of lines in $\mathbb{R}^3$, or at least in a domain of interest shall be constructed. With some restrictions, a line G can be mapped onto a vector $\mathbf{G} = (g_1, \ldots, g_4) \in \mathbb{R}^4$. This implies that a ruled surface Φ is mapped onto a curve and a two parametric family of lines is mapped onto a surface in $\mathbb{R}^4$. Thus, the G^1-Hermite interpolation of ruled surfaces in $\mathbb{R}^3$ will be translated to G^1-Hermite interpolation with curves in $\mathbb{R}^4$.

For practical purposes, it is often sufficient to consider patches of ruled surfaces, bounded by two planes which enclose the domain of interest in $\mathbb{R}^3$. In the following we will assume that these two planes are parallel and are chosen to be $E_1 : z = 0$ and $E_2 : z = 1$, perpendicular to the z-axis of the coordinate system. The intersection points $\mathbf{g}_1 = (g_1, g_2, 0)$ and $\mathbf{g}_2 = (g_3, g_4, 1)$ of a line G and the planes E_1, E_2 define a parametrization of all *non–horizontal lines* $\mathcal{L}$ by

$$\begin{aligned} &\mu : \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{L} \\ &\mathbf{G} = (g_1, g_2, g_3, g_4) \mapsto \mu(\mathbf{G}) = G. \end{aligned} \tag{6}$$

Parametrization μ is a local mapping, and depends essentially on the coordinate system. In applications, the z-axis of the coordinate system can be computed as solution of a regression problem, see [6].

Some Linear Subsets of $\mathbb{R}^4$ and their μ-Images

Given two non–intersecting lines G, H in $\mathbb{R}^3$, there is a unique bilinear tensor product surface (hyperbolic paraboloid)

$$\mathbf{x}(u, v) = (1 - v)\left((1 - u)\mathbf{g}_1 + u\mathbf{g}_2\right) + v\left((1 - u)\mathbf{h}_1 + u\mathbf{h}_2\right).$$

The μ^{-1}-image curve of $\mathbf{x}$ is the straight line segment in $\mathbb{R}^4$ connecting $\mathbf{G} = \mu^{-1}(G)$ and $\mathbf{H} = \mu^{-1}(H)$, see Fig. 3.

If G and H are intersecting, $\mathbf{h}_1 - \mathbf{g}_1$ and $\mathbf{h}_2 - \mathbf{g}_2$ are linearly dependent. Analytically, this says that

$$(h_1 - g_1)(h_4 - g_4) - (h_2 - g_2)(h_3 - g_3) = 0.$$

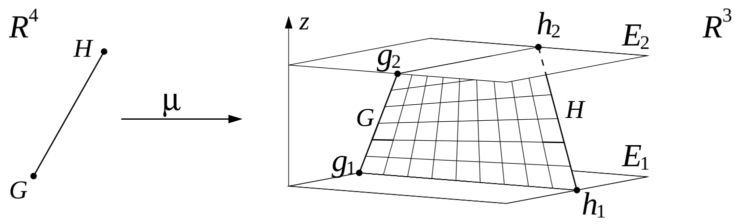

Fig. 3. Hyperbolic paraboloid in $\mathbb{R}^3$ as μ-image of a straight line $\mathbb{R}^4$.

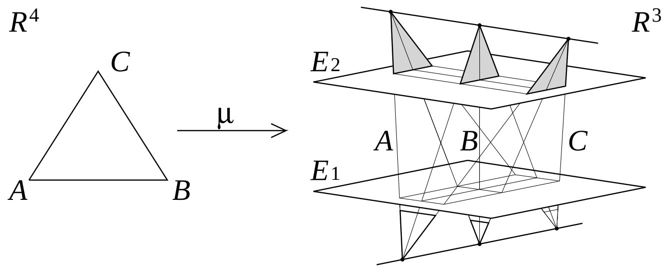

Fig. 4. Hyperbolic net in $\mathbb{R}^3$ as μ-image of a plane in $\mathbb{R}^4$.

The direction vector $\mathbf{H} - \mathbf{G} = (h_1 - g_1, \ldots, h_4 - g_4)$ satisfies the quadratic equation of the indefinite quadratic form

$$I : \langle \mathbf{X}, \mathbf{X} \rangle_i = x_1 x_4 - x_2 x_3 = 0, \mathbf{X} \in \mathbb{R}^4. \quad (7)$$

Vectors satisfying (7) shall be called isotropic. The μ-image of an isotropic line $(1-u)\mathbf{G} + u\mathbf{H}$ (line with isotropic direction vector) is a pencil of lines, spanned by intersecting lines G, H. We summarize.

Corollary 3. *The μ-image of a line G in $\mathbb{R}^4$ is a pencil of lines or a hyperbolic paraboloid, depending on whether G is an isotropic line or not.*

Let E be a plane in $\mathbb{R}^4$ which is parametrized by $E : \mathbf{A} + \sigma(\mathbf{B} - \mathbf{A}) + \tau(\mathbf{C} - \mathbf{A})$. What does the corresponding set of lines in $\mathbb{R}^3$ look like? We compute the isotropic directions in E and obtain the following quadratic equation in the homogeneous parameter $(\sigma : \tau)$:

$$\langle \sigma(\mathbf{B} - \mathbf{A}) + \tau(\mathbf{C} - \mathbf{A}), \sigma(\mathbf{B} - \mathbf{A}) + \tau(\mathbf{C} - \mathbf{A}) \rangle_i = 0. \quad (8)$$

If (8) vanishes identically, the plane E is called isotropic since it contains only isotropic directions. The family of lines $\mu(E)$ in $\mathbb{R}^3$ is a bundle (all lines through a fixed point) or a field (lines in a fixed plane).

Otherwise, the equation (8) has two, one or zero solutions and the planes are of hyperbolic, parabolic or elliptic type. The family of lines $\mu(E)$ in $\mathbb{R}^3$ is called hyperbolic, parabolic or elliptic net.

A plane E of hyperbolic type carries two 1-parameter families of isotropic lines, parallel to the isotropic directions. The corresponding hyperbolic net $\mu(E)$ consists of two 1-parameter families of pencils of lines. The vertices of these pencils form the two horizontal focal lines or axes of the net, see Figure 4.

A plane E of elliptic type in $\mathbb{R}^4$ contains no real isotropic directions such that the focal lines of the elliptic net $\mu(E)$ are conjugate imaginary. There are no pencils contained in that net $\mu(E)$ and pairwise distinct lines G, H of $\mu(E)$ are skew.

A plane E of parabolic type carries a 1-parameter family of isotropic lines such that the parabolic net $\mu(E)$ consists of a 1-parameter family of pencils of lines. The vertices of these pencils lie on one horizontal focal line and the planes containing the pencils pass through this focal line. The vertices and the planes are in a projective correspondence.

Corollary 4. *The μ-image of a plane E in $\mathbb{R}^4$ is a bundle or field of lines in case when E contains only isotropic directions and otherwise it is an elliptic, parabolic or hyperbolic net of lines.*

Ruled Surfaces as μ-images of curves $\subset \mathbb{R}^4$

Consider a smooth curve $\mathbf{C}(t)$ in $\mathbb{R}^4$, different from a straight line. The ruled surface $\mu(\mathbf{C}(t))$ is parametrizable in the form

$$\mathbf{x}(t, v) = (1 - v)\mathbf{c}_1(t) + v\mathbf{c}_2(t), \tag{9}$$

where $\mathbf{c}_1, \mathbf{c}_2$ are the intersection curves of $\mu(\mathbf{C})$ with the planes E_1, E_2.

The tangent line $\mathbf{C}(t_0) + \lambda\dot{\mathbf{C}}(t_0)$ determines the first order properties of the ruled surface $\mu(\mathbf{C})$ at the generating line $\mu(\mathbf{C}(t_0))$. If $\dot{\mathbf{C}}(t_0)$ is isotropic, $\mu(\mathbf{C}(t_0))$ is a torsal generator and the cuspidal point $\mathbf{v}(t_0)$ is the vertex of the pencil of lines $\mu(\mathbf{C} + \lambda\dot{\mathbf{C}})(t_0)$.

If $\dot{\mathbf{C}}(t_0)$ is not isotropic, the hyperbolic paraboloid determined by the tangent line $\mathbf{C}(t_0) + \lambda\dot{\mathbf{C}}(t_0)$ touches the ruled surface $\mu(\mathbf{C})$ in all points of the generator $\mu(\mathbf{C}(t_0))$.

Let $\mathbf{C}(t)$ be a planar curve in a non–isotropic plane E. The ruled surface $\mu(\mathbf{C}(t))$ is contained in the net of lines $\mu(E)$. If E is isotropic, $\mu(\mathbf{C}(t))$ is a cone or the set of tangent lines of a planar curve, depending on whether $\mu(E)$ is a bundle or field of lines.

In particular, if $\mathbf{C}$ is a conic in a non–isotropic plane, $\mu(\mathbf{C})$ is a rational ruled surface of order ≤ 4. The parametrization (9) is a (2,1) tensor product representation and $\mathbf{c}_1, \mathbf{c}_2$ are conics in the planes E_1, E_2. The degree is less than 4 if $\mathbf{c}_1$ and $\mathbf{c}_2$ possess common points for common parameter values. These parameter values as well as the common points need not to be real. Additionally, common points could lie at infinity.

In general, let $\mathbf{C}(t)$ be a rational curve of degree d in $\mathbb{R}^4$. A $(d, 1)$ rational tensor product point representation of $\mu(\mathbf{C})$ is given by (9). The order of the ruled surface is at most $2d$, since the planar curves $\mathbf{c}_1, \mathbf{c}_2$ are in general of degree d.

§4. Interpolation by Pairs of Quadrics

Given two generating lines A, B of a ruled surface R and tangent planes (or surface normals) at points of A and B, we want to construct a ruled surface,

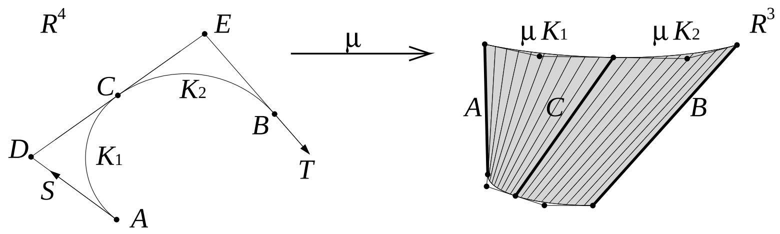

Fig. 5. Isotropic biarc in $\mathbb{R}^4$ and pair of quadrics in $\mathbb{R}^3$.

consisting of a pair of ruled quadrics, which interpolates the given boundary data. Since ruled quadrics do not possess torsal generators, we have to restrict ourselves to ruled surfaces R without torsal generators. This implies that the distribution parameter δ has no sign changes in the interval determined by A and B.

The local parameterization μ maps generating lines A, B with associated tangent planes onto points $\mathbf{A}$, $\mathbf{B}$ with associated tangent lines with direction vectors $\mathbf{S}$ and $\mathbf{T}$, respectively. In general, the data $\mathbf{A}, \mathbf{B}, \mathbf{T}, \mathbf{S}$ span a 3–space in $\mathbb{R}^4$.

In Section 3 we introduced an indefinite quadratic form I in formula (7). Similar to Euclidean space, one can define isotropic circles with respect to I. A curve $\mathbf{K}$ is an isotropic circle if it is a planar intersection of an isotropic sphere $\Sigma : \langle \mathbf{X}, \mathbf{X} \rangle_i = r$. Additionally, we require that the plane carrying $\mathbf{K}$ is not tangent to Σ. This excludes degeneracies where $\mu(\mathbf{K})$ is contained in a bundle or a field or is the union of two pencils.

An isotropic circle (i-circle) in a plane of hyperbolic or elliptic type is a conic $\mathbf{K}$, whose points $\mathbf{X}$ possess constant isotropic distance r from a given center $\mathbf{M}$,

$$\|\mathbf{M} - \mathbf{X}\|_i = \sqrt{\langle \mathbf{M} - \mathbf{X}, \mathbf{M} - \mathbf{X} \rangle_i} = r = const.$$

An isotropic circle in a plane of parabolic type is a parabola $\mathbf{K}$ with isotropic axis. The following theorem holds.

Theorem 5. *The μ-images of isotropic circles $\mathbf{K}$ are ruled quadrics.*

This theorem allows us to apply an isotropic biarc construction in $\mathbb{R}^4$. We will construct two isotropic circles $\mathbf{K}_1(t), \mathbf{K}_2(t)$, which interpolate the given data $\mathbf{A}, \mathbf{T}$ and $\mathbf{B}, \mathbf{S}$ and join smoothly at a point $\mathbf{C}$. Taking into account that no torsal generators occur in the segment under consideration, we can assume that

$$sign(\langle \mathbf{S}, \mathbf{S} \rangle_i) = sign(\langle \mathbf{T}, \mathbf{T} \rangle_i).$$

This allows the normalization $\|\mathbf{S}\|_i = \|\mathbf{T}\|_i = 1$. Further, by letting $\mathbf{Y} = \mathbf{B} - \mathbf{A}$ we have to guarantee that $sign(\langle \mathbf{Y}, \mathbf{Y} \rangle_i) = sign(\langle \mathbf{T}, \mathbf{T} \rangle_i)$, which can always be achieved by an appropriate choice of the data lines A, B.

The biarc $\mathbf{K}_1, \mathbf{K}_2$ possesses inner Bézier points $\mathbf{D}, \mathbf{E}$ and junction point

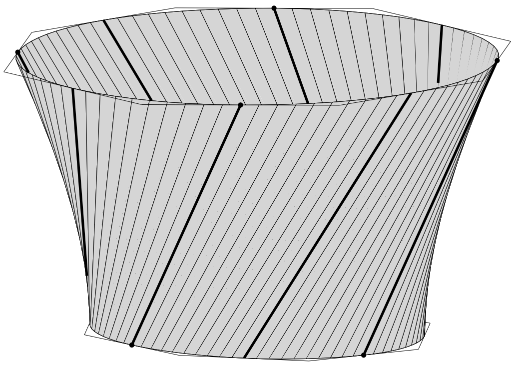

Fig. 6. Sequence of pairs of ruled quadrics interpolating G^1-Hermite data.

C, see Fig. 5, with

$$\mathbf{D} = \mathbf{A} + \lambda\mathbf{S},\ \mathbf{E} = \mathbf{B} - \mu\mathbf{T},\ \mathbf{C} = \frac{\mu}{\lambda+\mu}\mathbf{D} + \frac{\lambda}{\lambda+\mu}\mathbf{E}.$$

The weights w_1, w_2 of the inner control points $\mathbf{D}, \mathbf{E}$ are computed by

$$w_1 = \frac{\langle \mathbf{C}-\mathbf{A}, \mathbf{D}-\mathbf{A}\rangle_i}{\|\mathbf{C}-\mathbf{A}\|_i \|\mathbf{D}-\mathbf{A}\|_i}, \quad w_2 = \frac{\langle \mathbf{B}-\mathbf{C}, \mathbf{E}-\mathbf{C}\rangle_i}{\|\mathbf{B}-\mathbf{C}\|_i \|\mathbf{E}-\mathbf{C}\|_i}.$$

The points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ and weights w_1, w_2 determine a biarc if and only if

$$\langle \mathbf{E}-\mathbf{D}, \mathbf{E}-\mathbf{D}\rangle_i = (\lambda+\mu)^2.$$

Elaborating this gives a bilinear equation in λ and μ,

$$\langle \mathbf{Y}, \mathbf{Y}\rangle_i - 2\lambda\langle \mathbf{Y}, \mathbf{S}\rangle_i - 2\mu\langle \mathbf{Y}, \mathbf{T}\rangle_i + 2\lambda\mu\left[\langle \mathbf{S}, \mathbf{T}\rangle_i - 1\right] = 0.$$

We summarize the result:

Theorem 6. *Given G^1-Hermite boundary data satisfying above restrictions, there exists a one-parameter family of isotropic biarcs in $\mathbb{R}^4$, interpolating the given data. Finally, there is a one-parameter family of quadric pairs solving the G^1-Hermite interpolation problem in $\mathbb{R}^3$.*

To obtain a unique solution we might let $\lambda = \mu$ or $\lambda + \mu \to \min$, see [4]. More generally, optimization in curve design is often done by minimizing a functional involving second derivatives. Thus, we can require that

$$F = \int_{t_0}^{t_1} \langle \ddot{\mathbf{C}}, \ddot{\mathbf{C}}\rangle_e dt \to \min, \tag{10}$$

where $\mu(\mathbf{C}(t))$ is a ruled surface. Here, $\langle \mathbf{X}, \mathbf{X}\rangle_e = x_1^2+x_2^2+x_3^2+x_4^2+x_1x_3+x_2x_4$ denotes a Euclidean scalar product in $\mathbb{R}^4$ which induces a distance measure between lines, see [6]. The minimization of (10) is equivalent to minimizing the functional

$$3\int_{t_0}^{t_1}\int_0^1 [(1-u)\ddot{\mathbf{c}}_1 + u\ddot{\mathbf{c}}_2]^2 du dt, \tag{11}$$

where $(1-u)\mathbf{c}_1 + u\mathbf{c}_2$ are level curves of the ruled surface $\mu(\mathbf{C})$. Thus, we minimize the linearized curvatures of all these level curves.

Theorem 7. *Minimizing the functional (11) over all level curves* $(1-u)\mathbf{c}_1 + u\mathbf{c}_2$ *of the ruled surface* $S = \mu(\mathbf{C}(t))$ *equals minimizing the linearized bending energy of the image curve* $\mathbf{C}(t)$ *in* $\mathbb{R}^4$ *with respect to the Euclidean scalar product* $\langle , \rangle_e$.

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