# The Convolution of a Paraboloid and a Parametrized Surface

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**Abstract**

We investigate the computation of parametrizations of convolution surfaces of paraboloids and arbitrary parametrized surfaces. In particular it will turn out that the addressed problem is linear such that the convolution of a paraboloid and a rational surface admits rational parametrizations. In addition, the convolution of paraboloids and surfaces from special classes, like ruled surfaces or surfaces of rotation will be studied.

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## 1 Introduction and Definitions

In several contributions the problem has been investigated whether a one- (or two-) parameter family of quadrics possesses a rational envelope and how to construct possible parametrizations. Especially, families of spheres, defined by rational functions have been studied in [4, 8, 9, 14, 17]. In this article we will discuss the construction of the *convolution surface* (defined in (1)) of a paraboloid and a parametrized surface and we study parametrizations and geometric properties. To introduce to the subject we start with the definition of Minkowski sums of geometric objects.

Given two objects $P, Q$ in $\mathbb{R}^3$, their *Minkowski sum* $P \oplus Q$ is defined to be the set

$$P \oplus Q := \{p + q, \text{ with } p \in P, q \in Q\},$$

where $p$ and $q$ denote coordinate vectors of arbitrary points in $P$ and $Q$. Let $A = \partial P$ and $B = \partial Q$ be boundaries of $P$ and $Q$. Then, the computation of the boundary $\partial(P \oplus Q)$ is related to the computation of the *convolution surface* $A + B$ of the two boundary surfaces $A$ and $B$. We always assume in the following that $A$ and $B$ are smooth surfaces with normal vector fields $n_A$ and $n_B$, respectively. The convolution surface is defined to be

$$A + B := \{a + b, \text{ with } a \in A, b \in B, \text{ and } n_A(a) \parallel n_B(b)\}, \tag{1}$$

where $n_A(a)$ and $n_B(b)$ are parallel ($n_A(a) \parallel n_B(b)$) surface normal vectors at $a$ and $b$. In particular, if $P$ and $Q$ are convex objects, the boundary $\partial(P \oplus Q)$ of the Minkowski sum $P \oplus Q$ is exactly given by the convolution surface $A + B$. Unfortunately, for non–convex objects this property is no longer true. In general, the boundary $\partial(P \oplus Q)$ of the Minkowski sum is contained in the convolution surface $A + B$, formed by the boundaries $A = \partial P$ and $B = \partial Q$, respectively.

The computation of convolution curves/surfaces and Minkowski sums of objects occurs in various areas, like computer graphics, computational geometry and motion planning. The algorithmic problem for polynomial and polyhedral shapes as well as approximations of the convolution and Minkowski sum have been studied, see for instance [1, 3, 6, 7, 16] and the references therein.

The convolution surface is often denoted by $A \star B$. Since we are working with parametrizations only it is more convenient to denote the convolution by $A + B$ and we call it also *sum* of $A$ and $B$.

In general, the computation of the convolution surface $A + B$ of two smooth surfaces $A$ and $B$ results in the following problem. Assume that the surfaces $A$ and $B$ are parametrized by $a(u, v)$ and $b(s, t)$, respectively and that the normal vectors are denoted by $n_A(u, v)$ and $n_B(s, t)$. The convolution surface $A + B$ is formed by the sums of vectors $a, b$ whose normal vectors $n_A, n_B$ are parallel. Thus, we have to find parametrizations $a(u(s, t), v(s, t)) = a(s, t)$ and $b(s, t)$ of parts of $A$ and $B$ over a common parameter domain of the $st$-plane with the property that the normal vectors $n_A(s, t)$ and $n_B(s, t)$ at $a$ and $b$ are parallel. Let us point out that in case of an arbitrary surface $B$

there is no one-one correspondence between points $a \in A$ and $b \in B$ with $n_A(a) \parallel n_B(b)$.

The construction of the convolution $A + B$ admits a kinematic interpretation in the following way: Consider the surface $A$ together with the origin $O$ as movable system $\Sigma_t$ and let $B$ be fixed. We may denote the different positions of $A$ and $O$ by $A_t$ and $O_t$. The system $\Sigma_t$ is now moved translatory in the way that the point $O_t$ travels on $B$, and the convolution $A + B$ is obtained as envelope of $A_t$ under this translatory motion, see fig. 1, where different positions of $\Sigma$ are displayed . Another kinematic generation of $A+B$ is discussed in [11].

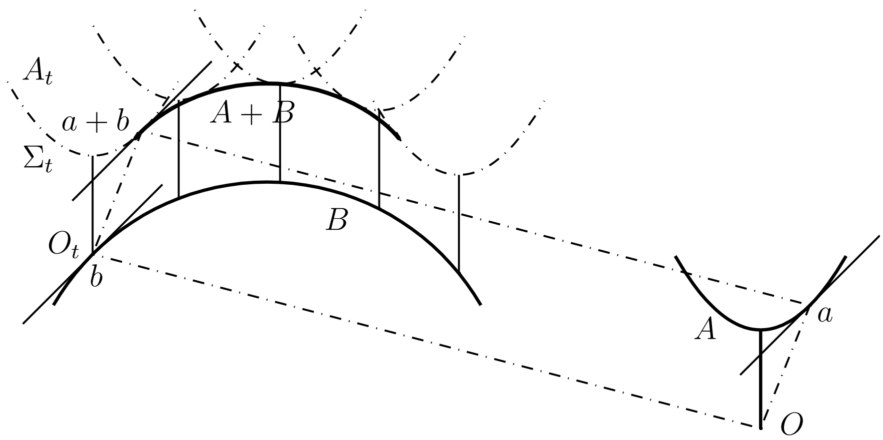


Figure 1: Kinematic generation of the convolution surface

**Remark**:

1. The construction of convolution surfaces is related to the offset computation in the following way: Let $A$ be a sphere with radius $d$, centered at $O$, then the convolution surface $A + B$ becomes the offset surface of $B$ at distance $d$.

2. The computation of convolution surfaces is affinely invariant, since the construction of corresponding points $a, b$ of $A, B$ only requires parallel surface normal vectors $n_A, n_B$. Thus, we always can work with an affine normal form of one of the surfaces we are considering. Additionally,

applying a translation with vector $c$ to one surface, say $A$, yields a translated convolution surface $(A+c)+B$, since the normal vectors (of $A$ and $B$) are not changed by a translation.

3. The parametrizations $a(s,t)$ and $b(s,t)$ of $A$ and $B$, respectively, with parallel normal vectors $n_A(s,t)$ and $n_B(s,t)$ lead to a *relative differential geometric interpretation* as follows: We assume that the tangent planes of $A$ do not contain the origin $O$ and so we can equip the surface $B$ with a *'new'* normal vector $a(s,t)$ at points $b(s,t)$. Since the normal vectors at points $a(s,t) \in A$ and $b(s,t) \in B$ are parallel, the partial derivatives $\partial a/\partial s$ and $\partial a/\partial t$ at $b(s,t)$ are tangent to $B$. Thus, $a(s,t)$ is called a *relative normalization* of the surface $B$. The surface $A$ is called *relative sphere*, playing the role the Euclidean sphere $S^2$ does as spherical image in Euclidean differential geometry. The surface $A+B$ parametrized by $(a+b)(s,t)$ is a *generalized offset* at distance 1, see [15].

   In a natural way, a *relative curvature theory* can be based on the relative *Weingarten mapping* $\omega$, expressing the partial derivatives of $a(s,t)$ by those of $b(s,t)$. In case of a locally strongly convex surface $A$, the eigenvectors of $\omega$ are always real. The corresponding *relative lines of curvature* can be characterized as in the Euclidean case: The relative normals along these curves form developable surfaces, see [19], p. 215.

The contribution of this article is the investigation of parametrizations of convolution surfaces of paraboloids and parametrized surfaces. In particular we show in section 2 that if one surface ($A$) is a paraboloid and the other surface ($B$) is parameterizable, then the sum $A+B$ possesses explicit parametrizations, too. Convolution surfaces of paraboloids and quadrics and their relation to offsets of paraboloids are discussed in section 3. Additionally, in section 4, we study the sum of a paraboloid and surfaces from special classes, like surfaces of rotation, surfaces of translation and ruled surfaces.

# 2 The sum of a paraboloid and a surface

In this section we want to investigate parametrizations of the sum $A+B$ of a paraboloid $A$ and a parametrized surface $B$. We assume that a coordinate system has been chosen in a way that the paraboloid $A$ is given by the

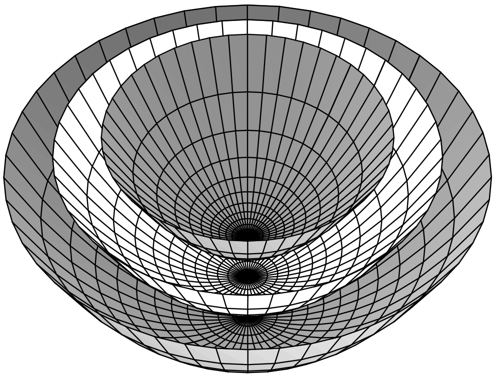

Figure 2: Elliptic paraboloid with inner and outer offset surface

equation
$$F_A = z - x^2 - cy^2 = 0, \text{ with } c \neq 0.$$
This implies that $A$ is representable by $a(u,v) = (u, v, u^2 + cv^2)$ and it is either an elliptic or a hyperbolic paraboloid depending on whether $c > 0$ or $c < 0$. The surface $B$ is assumed to admit a local parametrization $b : (s,t) \in G \subset \mathbb{R}^2 \to \mathbb{R}^3$, which is a smooth mapping.

Two points $a \in A$ and $b \in B$ are *corresponding* if the normal vectors $n_A$ and $n_B$ at $a$ and $b$, respectively, are linearly dependent,
$$n_A(a) = \lambda n_B(b), \ \lambda \neq 0. \tag{2}$$
Then, $a + b$ is a point of the convolution surface $A + B$. So, let $n_B(s,t) = (n_1, n_2, n_3)(s,t)$ be a normal vector of $B$, we rewrite condition (2) in coordinates and obtain
$$\begin{pmatrix} -2u \\ -2cv \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} n_1(s,t) \\ n_2(s,t) \\ n_3(s,t) \end{pmatrix}.$$
In case $n_3(s,t) \neq 0$ we have $\lambda = n_3(s,t)^{-1}$ and
$$u(s,t) = \frac{-n_1}{2n_3}(s,t), \ v(s,t) = \frac{-n_2}{2cn_3}(s,t). \tag{3}$$
Denoting this reparametrization by $\phi : (s,t) \to (u(s,t), v(s,t))$, the parametrization $a(\phi(s,t))$ represents in general only a part of $A$. Equation (3) is a regular reparametrization exactly if the determinant of the Jacobian $J\phi$ does not

vanish. Elaborating this, we find

$$\det(J\phi) = \frac{1}{4cn_3^3}\det(n, n_s, n_t) = \frac{1}{4cn_3^3}\Delta^2 K, \tag{4}$$

where $\Delta$ is the determinant of the *first fundamental form* (or *metric*) of $B$ and $K$ denotes $B$'s *Gaussian curvature*. This says in particular that for developable surfaces the reparametrization (3) is not invertible. The final representation of the sum $A + B$ is

$$(a+b)(s,t) = \left(\frac{-n_1}{2n_3} + b_1,\ \frac{-n_2}{2cn_3} + b_2,\ \frac{1}{4cn_3^2}(cn_1^2 + n_2^2) + b_3\right)(s,t). \tag{5}$$

**Theorem 1** *The convolution surface A+B of a paraboloid A and a parametrized surface B possesses the explicit parametrization (5). If B is a rational surface, $A + B$ is rational too.*

If $B$ is in particular the graph of a function $f(s,t)$ over the $st$-plane, the reparametrization is obtained by $u = 1/2 f_s, v = 1/(2c) f_t$, where $f_s, f_t$ denotes partial derivatives of $f$ with respect to $s$ and $t$. Finally, the sum $A+B$ possesses the parametrization

$$(a+b)(s,t) = \left(\frac{1}{2}f_s + s, \frac{1}{2c}f_t + t, \frac{1}{4}f_s^2 + \frac{1}{4c}f_t^2 + f\right)(s,t).$$

**Remarks:** A regular point of $B$ with $n_3(s,t) = 0$ has no corresponding point on the paraboloid $A$. If there is one point with this property then, in general, there exists a curve $C \in B$ with $n_3 = 0$ along $C$. $C$ is a shadow boundary of $B$ with respect to an illumination parallel to $A$'axis. In this case the convolution $A + B$ consists of non-connected parts.

Otherwise we do not need to make special assumptions on the surface $B$. In general, the correspondence, defined by (3) between points on $A$ and $B$ will not be injective. We would like to point to some geometrically special cases corresponding to a singular Jacobian (4) of the reparametrization $\phi$.

- If $B$ is a plane not parallel to $A$'s axis, the normal vector $n$ does not depend on $s, t$ but is a constant vector. Since (3) gives a single point $(u, v)$, there is a single point $a$ on $A$ which is correspondent to all points of $B$. Thus, $A + B$ is a plane translated by the fixed vector $a$.

- If $B$ is a developable surface which implies that its Gaussian curvature vanishes, $\phi$ maps $G \subset \mathbb{R}^2$ onto a curve in the $uv$-plane. Thus, there is in general only a curve $a(\tau) \in A$ which contributes to the construction of $A + B$. Section 4.3 shows that $A + B$ is a developable surface, too.

# 3 Convolution of a paraboloid and a quadric

In this section we like to discuss the convolution surface $A+B$ of a paraboloid $A$ and a quadric $B$ and we will show the relations to offset computation. So, let $F_A, F_B : \mathbb{R}^3 \to \mathbb{R}$ be two quadratic functions

$$\begin{aligned} F_A(x, y, z) &= z - x^2 - cy^2 \\ F_B(x, y, z) &= d + X^T \cdot C + X^T \cdot M \cdot X, \end{aligned}$$

where $X = (x, y, z)$, and $d \in \mathbb{R}$, $C \in \mathbb{R}^3$ and $M$ a symmetric 3x3 matrix. Then, the paraboloid $A$ is the zero set of $F_A$ and the quadric $B$ is the zero set of $F_B$.

We assume that the quadric $B$ is regular which shall mean that the extended $4 \times 4$ matrix

$$\tilde{M} = \begin{pmatrix} d & C^T \\ C & M \end{pmatrix}$$

has rank 4. The cases where $B$ is a cone or cylinder will be treated in section 4.3.

Let the paraboloid $A$ be parametrized by $a(u, v) = (u, v, u^2 + cv^2)$ and let $b(s, t)$ be a rational parametrization of $B$. Rational parametrizations of a quadric can be obtained by stereographic projection. For instance, the unit sphere $S^2 : x^2 + y^2 + z^2 - 1 = 0$ can be parametrized by the rational functions

$$x = \frac{2s}{1 + s^2 + t^2}, y = \frac{2t}{1 + s^2 + t^2}, z = \frac{1 - s^2 - t^2}{1 + s^2 + t^2}. \tag{6}$$

Expressing the reparametrization condition (2) for the quadratic polynomials $F_A, F_B$ leads to a condition for the gradient vectors

$$\nabla F_A(a) = \lambda \nabla F_B(b), \tag{7}$$

which has to be satisfied at corresponding points $a \in A$ and $b \in B$. We apply theorem 1 and obtain the following:

**Corollary 2** *The sum $A + B$ of a paraboloid $A$ and an arbitrary quadric $B$ is a rational surface.*

This applies to a variety of special cases and leads to parametrizations of the offset surfaces of paraboloids and to solutions of similar problems.

## 3.1 Offsets of parabolas

A good starting point are offsets of parabolas and we obtain the following well known result:

**Corollary 3** *The planar offset curves of parabolas are rational curves.*

This and the rationality of the offsets of paraboloids have been proved for instance by Lü, see [8, 9]. He also has proved that the offsets of ellipsoids and hyperboloids are rationally parametrizable and he has given a characterization of algebraic curves and surfaces which possess rational offsets. To

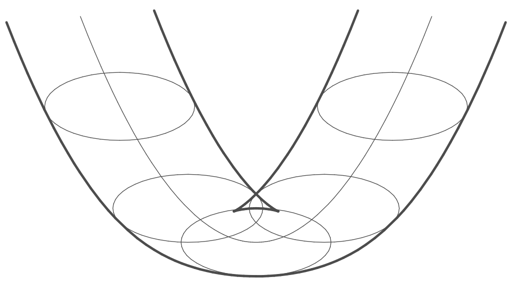

Figure 3: Convolution curve of a parabola and an ellipse

prove the result in our context we set

$$A : y - x^2 = 0, \text{ and } B : x^2 + y^2 - 1 = 0$$

and we use the parametrizations $a(t) = (t, t^2)$ and $b(u) = (\sin(u), -\cos(u))$. The condition (7) leads us to the reparametrization $t = \sin(u)/(2\cos(u))$. After substituting the trigonometric functions by rational functions we arrive at a rational parametrization of the offset curves of a parabola, which are of algebraic order 6. As a generalization of this we can note the following:

**Corollary 4** *The sum $A + B$ of a parabola $A$ and a conic $B$ is a rational curve.*

Since any conic $B$ possesses rational parametrizations, this can be proved by specializing corollary 2 to planar quadrics.

## 3.2 Offsets of paraboloids

It is known that the offset surfaces of paraboloids are rational surfaces and that their parametrizations can be calculated explicitly, see [8]. According to our approach the parametrization of the offset surfaces is a direct specialization of the convolution $A + B$ with $B$ as unit sphere.

**Corollary 5** *The offset surfaces of paraboloids are rational surfaces.*

To derive parametrizations of offsets surfaces of paraboloids we may assume that a paraboloid $A$ is given by the equation $z - x^2 - cy^2$ such that it is parametrized by $(u, v, u^2 + cv^2)$. By the use of the rational parametrization (6) of the unit sphere $S^2$ we obtain a rational representation of the offset of $A$ at distance $d$ by

$$\begin{aligned}
x(s,t) &= -\frac{s}{1-s^2-t^2} + d\frac{2s}{1+s^2+t^2} \\
y(s,t) &= -\frac{t}{c(1-s^2-t^2)} + d\frac{2t}{1+s^2+t^2} \\
z(s,t) &= \frac{s^2}{(1-s^2-t^2)^2} + \frac{t^2}{c(1-s^2-t^2)^2} + d\frac{1-s^2-t^2}{1+s^2+t^2}.
\end{aligned}$$

## 3.3 Generalized offsets of paraboloids

When defining offset surfaces one usually assumes $\mathbb{R}^3$ being a Euclidean space equipped with the canonical scalar product $\langle X_1, X_2 \rangle = X_1^T \cdot I \cdot X_2$ where $I$ denotes the identity matrix. Applying a fixed affine transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\alpha : X \mapsto X' = L \cdot X,$$

where $L$ is a regular matrix (and we have ignored the unimportant translational part), the unit sphere $S^2$ is mapped onto an ellipsoid

$$E : X'^T L^{-T} I L^{-1} X' = 0, \text{ with } L^{-T} = (L^T)^{-1}.$$

Further, all spheres are mapped to ellipsoids being similar to $E$, which means that they possess parallel principal axes and same ratios of their lengths. Since $J = L^{-T} I L^{-1}$ is a symmetric and positive definite matrix, $\mathbb{R}^3$ equipped with the scalar product

$$\langle X_1, X_2 \rangle = X_1^T \cdot J \cdot X_2 \tag{8}$$

is a Euclidean space. Angles between vectors and distances between points are now measured using the quadratic form (8). It is obvious that the offsets of paraboloids with respect to the metric defined according to (8) are rational surfaces and they are affine images of the offsets of $A$ with respect to the unit sphere $S^2$.

**Remark:** We have stated that the treatment of convolution surfaces is affinely invariant. To apply this we consider the convolution of an paraboloid $A$ and an ellipsoid $B$. We choose the $z$-axis of the coordinate system parallel to $A$'s axis and the center of $B$ is chosen to be the origin. We consider the projective extension of $\mathbb{R}^3$ and let $X, Y, Z$ be the points at infinity defined by the axes $x, y, z$. It is always possible to choose the axes $x$ and $y$ in a way that the center $O$ and the points $X, Y, Z$ are a *polar tetrahedron* of $B$ with respect to the polar system equipped with $B$. This implies that $B$ is given by a diagonal quadratic equation

$$B : -\beta_0 + \beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2 = 0.$$

By an appropriate scaling we can even obtain an affine normal form of $B$ with $\beta_i = 1, i = 0, 1, 2, 3$. Since a translation of $A$ implies a translation of the sum $A + B$ we assume that $A$ passes through the origin $O$. Additionally, let $X$ and $Y$ be conjugate with respect to $A$. Thus, $A$ can be given by the normal form $F_A = z - c_1 x^2 - c_2 y^2 = 0$, where $c_i$ may have different or equal signs depending on whether $A$ is a hyperbolic or elliptic paraboloid.

### 3.4 The Sum of a paraboloid and a hyperboloid

If the bilinear form in (8) is not positive definite but $J$ is still a regular matrix, we can obtain one of the following cases

$$J_1 = \mathrm{diag}(1, 1, -1), \text{ or } J_2 = \mathrm{diag}(-1, -1, 1),$$

when we restrict the investigation to special Euclidean normal forms. The notion of offset surfaces makes no longer sense, since the quadrics or 'unit spheres' with respect to $J_1, J_2$ are no longer convex but are hyperboloids

$$B_1 : x^2 + y^2 - z^2 - 1 = 0, \quad B_2 : -x^2 - y^2 + z^2 - 1 = 0.$$

A rational parametrization of $B_1$ may look like

$$x = \frac{2s}{1 + s^2 - t^2}, \; y = \frac{1 - s^2 + t^2}{1 + s^2 - t^2}, \; z = \frac{2t}{1 + s^2 - t^2}. \tag{9}$$

It is obtained by mapping the parametrization $(s, 0, t)$ of the plane $y = 0$ via the stereographic projection with center $(0, -1, 0)$ onto $B_1$. Analogously we get

$$x = \frac{2s}{1 - s^2 - t^2}, \; y = \frac{2t}{1 - s^2 - t^2}, \; z = \frac{1 + s^2 + t^2}{1 - s^2 - t^2}, \tag{10}$$

by mapping the parametrization $(s, t, 0)$ of the plane $z = 0$ via the stereographic projection with center $(0, 0, -1)$ onto $B_2$.

Let a paraboloid $A$ be given by $z - (x^2 + cy^2) = 0$ and parametrization $(u, v, u^2 + cv^2)$. The condition (7) leads to the reparametrization

$$u(s,t) = \frac{x}{2z}(s,t), \quad v(s,t) = \frac{y}{2cz}(s,t)$$

in both cases. Finally, the sums $A + B_1$ and $A + B_2$ are both given by

$$\left( \frac{x}{2z} + x, \frac{y}{2cz} + y, \left(\frac{x}{2z}\right)^2 + \left(\frac{y}{2cz}\right)^2 + z \right)(s,t),$$

where we have to insert parametrizations (9) and (10), respectively.

**Remark:** As in section 3.3 we may ask for affine coordinate transformations in the way that the quadratic equations of a paraboloid $A$ and a hyperboloid $B$ will be as simple as possible. Again we consider the projective extension of $\mathbb{R}^3$ and choose the $z$-axis parallel to $A$'s axis. Additionally, $O$ shall be the center of $B$. Let $B$'s curve at infinity be denoted by $C$ and let $X, Y, Z$ be the points at infinity defined by the axes $x, y, z$.

If $Z \notin C$ and $A$ is an elliptic paraboloid, then there exists a polar tetrahedron $O, X, Y, Z$ in a way that $X$ and $Y$ are conjugate points with respect to $A$ and $B$. This implies that $B$ can be determined by a diagonal quadratic form

$$B : -\beta_0 + \beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2 = 0,$$

where the signs of $\beta_i$ are according to the type of $B$. The paraboloid can be considered to be the zero set of $z - c_1 x^2 - c_2 y^2$.

We do not want to elaborate this in detail but we note that if $A$ is a hyperbolic paraboloid and $Z$ is an inner point of $C$, we obtain similar normal forms. If $Z$ is an outer point of $C$ and $A$ is a hyperbolic paraboloid, this needs not to be true. Finally, if $Z$ is on $C$, it is also not possible to find a unique normal form of $B$ but the representation essentially depends on the position of $A$ with respect to $B$.

# 4 The sum of a paraboloid and surfaces from special classes

In this section we will study the sum of a paraboloid $A$ and surfaces $B$ belonging to special classes, like surfaces of rotation, surfaces of translation, ruled surfaces and especially developable surfaces. In particular we concentrate on surfaces $B$ which are in a special position to $A$.

## 4.1 Paraboloid and surface of rotation

Let $A$ be an elliptic paraboloid which admits the representation

$$a(u,v) = (u, v, u^2 + cv^2), \text{ where } c > 0,$$

and let $B$ be affinely equivalent to a surface of rotation being generated by rotating a curve around an axis parallel to $z$. Applying an affine mapping we may assume that $B$ is generated by a Euclidean rotation, and $B$ admits a local parametrization

$$b(s,t) = (s\cos(t), s\sin(t), h(s)).$$

Here, $B$ is generated by rotating the curve $(s, h(s))$ around the $z$-axis. According to (5), a representation of the convolution $A + B$ is

$$(a+b)(s,t) = \begin{pmatrix} (\frac{1}{2}\dot{h}(s) + s)\cos(t) \\ (\frac{1}{2c}\dot{h}(s) + s)\sin(t) \\ \frac{1}{4c}\dot{h}(s)^2(c\cos^2(t) + \sin^2(t)) + h(s) \end{pmatrix}, \tag{11}$$

where the differentiation with respect to $s$ is denoted by dots. Special surfaces occur in the following cases:

1. Let $c \neq 1$. A necessary condition for $A + B$ to be affinely equivalent to a surface of rotation is

   $$\frac{1}{2}\dot{h}(s) + s = \mu(\frac{1}{2c}\dot{h}(s) + s).$$

   This restricts the function $h(s)$ to be of the form $h(s) = \alpha s^2 + d$, so that $B$ is a paraboloid of rotation. We can ignore the unimportant vertical translation corresponding to $d$ and set $d = 0$. Then $A + B$ is a paraboloid with equation

$$z = \frac{\alpha}{\alpha + 1} x^2 + \frac{\alpha c}{\alpha + c} y^2.$$

$A+B$ is an elliptic paraboloid and hence affinely equivalent to a surface of rotation exactly if $(\alpha + 1)(\alpha + c) > 0$.

2. Let $c = 1$, then $A$ is a paraboloid of rotation and the sum $A + B$ is a surface of rotation, parametrized by

$$(a + b)(s, t) = (f(s) \cos(t), f(s) \sin(t), g(s)),$$

with $f(s) = \frac{1}{2}\dot{h}(s)+s$ and $g(s) = \frac{1}{4}\dot{h}(s)^2+h$. The convolution is regular, exactly if $f^2(\dot{f}^2 + \dot{g}^2) \neq 0$ and this is equivalent to the conditions

$$\frac{1}{2}\dot{h} + s \neq 0 \text{ and } \frac{1}{4}\ddot{h} + 1 \neq 0.$$

Similarly to the just discussed case is that if $B$ is affinely equivalent to a surface parametrized by

$$b(s, t) = (s \cosh(t), s \sinh(t), h(s)).$$

These surfaces are called affine surfaces of rotation of hyperbolic type, see [19], p. 193. If we choose $A$ to be a hyperbolic paraboloid by

$$a(u, v) = (u, v, u^2 - cv^2), \text{ where } c > 0,$$

then the convolution $A + B$ has a similar parametrization to (11), but the trigonometric functions have to be substituted by the hyperbolic functions ($\cos \mapsto \cosh$, $\sin \mapsto \sinh$). The results in which cases even $A + B$ is a surface of rotation hold also for the hyperbolic type, with respect to the mentioned substitution.

## 4.2 Paraboloid and ruled surface

For the general solution of the convolution of a paraboloid $A$ with parametrization

$$a(u, v) = (u, v, u^2 + cv^2), \text{ where } c \neq 0,$$

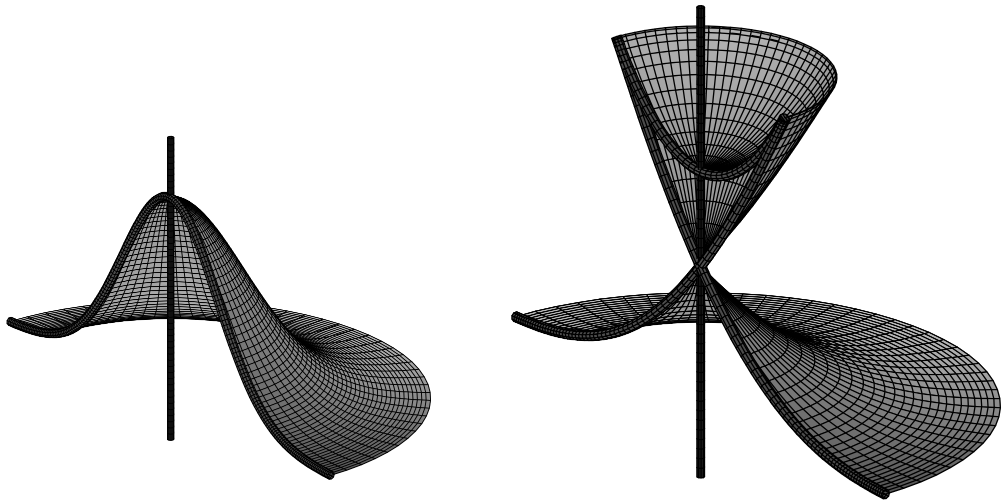

Figure 4: Left: Surface of rotation generated by a Gaussian distribution function. Right: Convolution of a paraboloid and this surface of rotation.

and a ruled surface $B$ with parametrization $b(s,t) = r(s) + tg(s)$ we can use formula (5). The parameter lines $u =$ const. are cubic curves. It is proved in [11] that the computation of the convolution of two skew ruled surfaces results in a linear problem. This implies that the convolution $A+B$ is rational for rational input surfaces $A$ and $B$.

In this section we deal with those special ruled surfaces $B$ whose generating lines $G(s)$ are horizontal (perpendicular to $A$' axis). These surfaces are called *conoidal ruled surfaces*. The orthogonal projections $G'(s)$ of the generating lines $G$ on the plane $z = 0$ form a one parameter family of lines which can be represented by the equation

$$G'(s) : x\sin(s) - y\cos(s) - p(s) = 0,$$

where $p(s)$ is the distance of $G'$ to the origin. The family $G'(s)$ envelopes the planar curve

$$\begin{pmatrix} p\sin(s) + \dot{p}\cos(s) \\ -p\cos(s) + \dot{p}\sin(s) \end{pmatrix}. \tag{12}$$

Thus, a *conoidal* ruled surface $B$ can be represented by

$$b(s,t) = \begin{pmatrix} p\sin(s) + \dot{p}\cos(s) + t\cos(s) \\ -p\cos(s) + \dot{p}\sin(s) + t\sin(s) \\ z(s) \end{pmatrix}, \tag{13}$$

with an arbitrary function $z(s)$. From (5), the resulting parametrization of $A+B$ is

$$(a+b)(s,t)=\begin{pmatrix} p\sin(s)+\dot p\cos(s)+t\cos(s)-\frac{\dot z\sin(s)}{2t} \\ -p\cos(s)+\dot p\sin(s)+t\sin(s)+\frac{\dot z\cos(s)}{2ct} \\ z(s)+\frac{\dot z^2}{4ct^2}(\cos^2(s)+c\sin^2(s)) \end{pmatrix}. \tag{14}$$

If we assume $p\equiv 0$, all generating lines $G$ of $B$ intersect the $z$-axis and if we additionally require $z(s)=ks, k\neq 0$, $B$ is a right helicoid, a helical surface which is generated by rotating $G$ around the $z$-axis and simultaneously translating $G$ in direction of $z$. The velocity of the vertical translation is proportional to the angular velocity of the rotation around $z$. A realization of $B$ is a staircase with winding stairs. Inserting the made assumptions in formula (14) gives

$$(a+b)(s,t)=\begin{pmatrix} t\cos(s)-\frac{k\sin(s)}{2t} \\ t\sin(s)+\frac{k\cos(s)}{2ct} \\ ks+\frac{k^2}{4ct^2}(\cos^2(s)+c\sin^2(s)) \end{pmatrix}. \tag{15}$$

The convolution $A+B$ is itself a helical surface exactly if $A$ is chosen to be a paraboloid of rotation, that means $c=1$. Then, its representation looks like

$$(a+b)(s,t)=\begin{pmatrix} \cos(s) & -\sin(s) & 0 \\ \sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} t \\ \frac{k}{2t} \\ (\frac{k}{2t})^2 \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ ks \end{pmatrix},$$

which shows the rotational part around the $z$-axis and the translational part in direction of $z$. The cubic curve $(t,k/(2t),(k/(2t))^2)$ generating the helical surface $A+B$ is part of the intersection of the two cylinders

$$xy=k/2,\ \text{and}\ z=y^2,$$

which, additionally, share a common line at infinity (line at infinity of planes $x=\text{const.}$). The singular points of $A+B$ are contained in the helical curves which are obtained for parameter values $t=\pm\sqrt{k/2}$, if $k>0$ or $t=\pm\sqrt{-k/2}$, if $k<0$. Figure 5 shows a right helicoid $B$ and the sum $A+B$ with a paraboloid of rotation $A$. The generating cubic curve and the singular curve on $A+B$ are plotted as tubes.

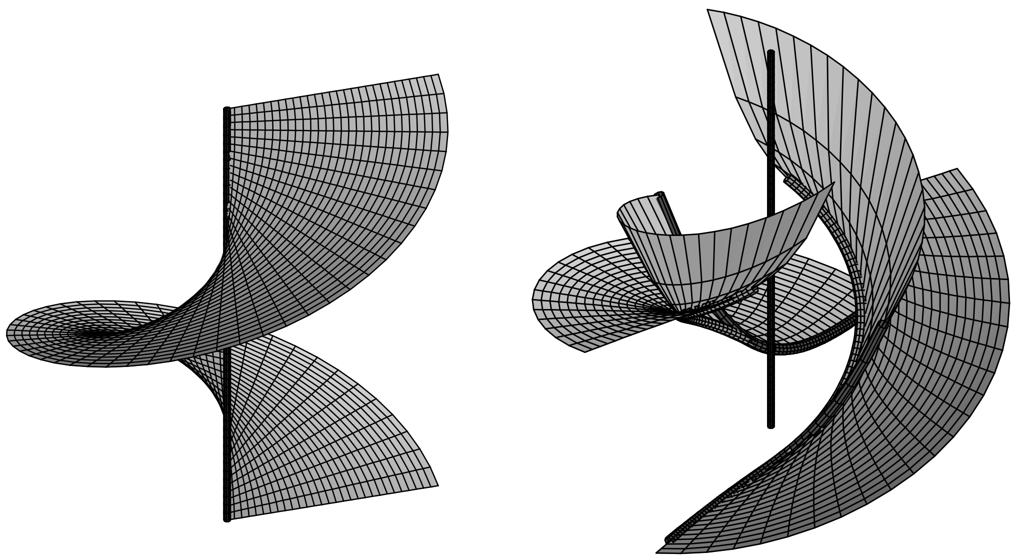

Figure 5: Left: Right helicoid. Right: Convolution of a paraboloid and a right helicoid.

### 4.3 Paraboloid and developable surface

A developable surface is a special kind of ruled surface which has the property that the tangent planes along a fixed generating line are identical. Developable surfaces possess vanishing Gaussian curvature. It is known that there are three basic types of developable surfaces, cylinders, cones and *tangent surfaces* formed by the tangent lines of twisted space curves. The last type is the general developable surface.

A tangent surface $B$ of a twisted space curve $l(s)$ admits a representation in the form

$$b(s,t) = l(s) + t\dot{l}(s), \quad l : \mathbb{R} \to \mathbb{R}^3,$$

and $B$'normal vector field is

$$n(s,t) = b_s(s,t) \times b_t(s,t) = -t\dot{l}(s) \times \ddot{l}(s),\ t \neq 0,$$

which shows that the normals in points of a fixed generating line $l(s_0)+t\dot{l}(s_0)$ are linearly dependent. In points of $l(s)$ where $t = 0$ there is no surface normal and $l(s)$ is a singular curve on $B$.

In case where $B$ is a cone with vertex in $O$, a possible parametrization is $b(s,t) = tv(s)$, where $v(s)$ denoted a vector field determining the directions of the generating lines of $B$. The normal vectors of $B$ are in this case given by $n(s,t) = -tv(s) \times \dot{v}(s)$, $(t \neq 0)$.

In case where $B$ is a cylinder, a possible representation is $b(s,t) = l(s)+tv$, where $l(s)$ is a directrix curve and $v$ is a constant vector representing the direction of $B$'s generating lines. The normal vectors of $B$ are $n = \dot{l} \times v$.

We recognize in all three cases that the quotients $n_1/n_3$ and $n_2/n_3$ are functions of $s$ only and independent of $t$. Forming the convolution $A+B$ according to (5) leads to

$$(a+b)(s,t) = \begin{pmatrix} -n_1/(2n_3) \\ -n_2/(2cn_3) \\ (cn_1^2 + n_2^2)/(4cn_3^2) \end{pmatrix}(s) + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}(s,t).$$

For fixed $s = s_0$ the convolution $A+B$ carries a straight line $b(s_0,t) + a(s_0)$, which is only a translated version of $b(s,t_0)$ by the vector $a(s_0)$. Since the tangent plane of $A+B$ at $(a+b)(s,t)$ is parallel to the tangent planes of $B$ at $b(s,t)$, it follows that $A+B$ is developable too.

**Remark:** If $B$ is a cylinder whose generating lines are not parallel to $A$'s axis, the unit normal vectors of $B$ are contained in a great circle $C_g$ in the unit sphere $S^2$. Since the tangent planes of the sum $A+B$ are all parallel to those of $B$, their unit normal vectors lie in $C_g$, too, which proves that $A+B$ is a cylinder.

An analogous result for cones $B$ does not hold. In general, $A+B$ is a developable surface sharing the same curve at infinity with $B$. If $B$ is especially a cone of revolution, the following holds. The unit normal vectors of $B$ parametrize a small circle $C_s$ in $S^2$. Constructing $A+B$, its unit normals parametrize at least a part of $C_s$, but in general the sum $A+B$ is a *developable surface of constant slope* with respect to the direction which is perpendicular to the carrier plane of $C_s$. If $B$ is a cone of revolution whose axis is parallel to $A$'s axis and if additionally $A$ is a paraboloid of revolution, then $A+B$ is a cone of revolution again.

**Corollary 6** *The sum $A+B$ of a paraboloid $A$ and a developable surface $B$ is itself a developable surface. In case where $B$ is a cylinder, $A+B$ is a cylinder too. The sum $A+B$ of a paraboloid and a cone of rotation $B$ is a developable surface of constant slope with respect to the carrier plane of $B$'s spherical image.*

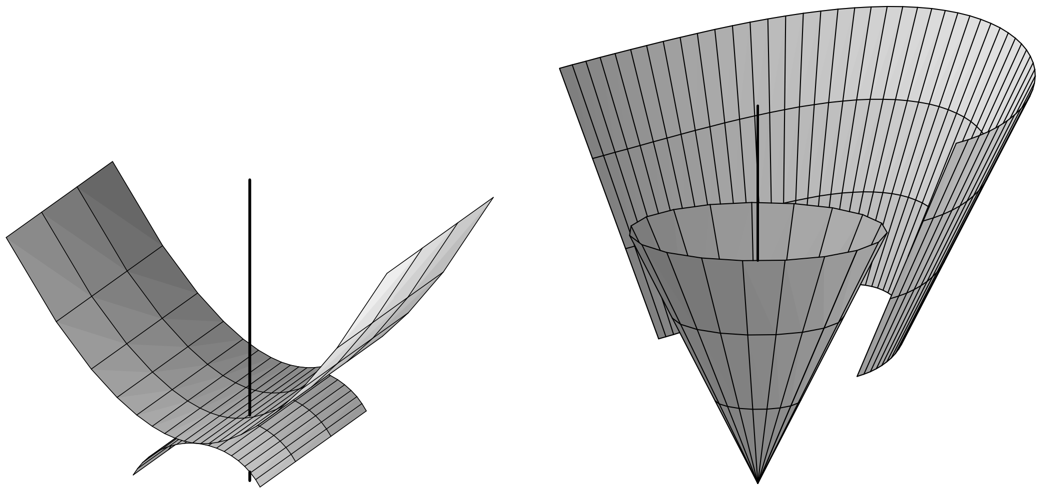

Figure 6: Left: Part of a parabolic cylinder as sum of a part of a cylinder of rotation and a paraboloid. Right: Developable of constant slope as sum of a paraboloid and a cone of revolution.

### 4.4 Paraboloid and translational surface

A surface $B$ which is itself the sum of two curves $G$ and $H$ is called *translational surface*, since it is generated by translating $H$ along $G$ (or $G$ along $H$). A parametrization of $B$ is obtained by adding representations of $G = G(s)$ and $H = H(t)$.

Here, we study the sum of a paraboloid $A$ and special translational surfaces which are generated by planar curves $G$ and $H$, whose carrier planes are vertical (parallel to $A$'s axis). Additionally, we assume that the curves $G$ and $H$ are given by the following parametrizations

$$G : (s, 0, g(s)) \text{ and } H : (0, t, h(t)).$$

This implies that the *translational surface* $B$ is represented by

$$b(s,t) = (s, t, g(s) + h(t)).$$

We show that the sum $A+B$ of $B$ and the paraboloid $A$ with parametrization $a(u,v) = (u, v, u^2 + cv^2)$, is a translational surface again. It is a straight forward calculation and according to (5) we obtain

$$(a+b)(s,t) = \begin{pmatrix} \frac{1}{2}\dot{g}(s) + s \\ 0 \\ \frac{1}{4}\dot{g}^2(s) + g(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2c}\dot{h}(t) + t \\ \frac{1}{4c}\dot{h}^2(t) + h(t) \end{pmatrix}, \tag{16}$$

where, for simplification, 'dot' denotes both differentiations with respect to $s$ and $t$. But since $g$ and $h$ are univariate functions, it is always clear what 'dot' means.

**Corollary 7** *Given a translational surface $B$ and a paraboloid $A$ in the way that the carrier planes of the parameter curves of $B$ are parallel to $A$'s axis. Then, $A+B$ is again a translational surface with the property that the carrier planes of the parameter curves of $A+B$ are parallel to $A$'s axis.*

We want to note that we have started with a surface $B$ which is a graph surface over the $st$-plane and whose parameter curves are also planar graph curves in planes $x =$ const. and $y =$ const., respectively. The convolution $A+B$ is a translational surface with curves in planes $x =$ const. and $y =$ const., but in general $A+B$ is not a graph surface over the $st$-plane.

Figure 7 shows on the left hand side a translational surface generated by translating a parabola along a cosine curve, plotted as tube. Both curves are graphs over $s$ and $t$, respectively. The right hand side figure shows the convolution surface. One generating curve is again a parabola, the second curve (plotted as tube) is parametrized by trigonometric functions and is in general not a graph curve.

# 5 Generalization in n–space

It is a straight forward generalization that the results we have obtained for the sum $A+B$ of a paraboloid $A$ and a parametrized surface $B$ hold analogously in $\mathbb{R}^n$. So, let $A$ be the zero set of the equation

$$F_A = x_n - (c_1x_1^2 + c_2x_2^2 + \ldots + c_{n-1}x_{n-1}^2), (c_1, \ldots, c_{n-1}) \neq (0, \ldots, 0). \tag{17}$$

Using parameters $(u_1, \ldots, u_{n-1}) = U$ in $\mathbb{R}^{n-1}$, the paraboloid $A$ in $\mathbb{R}^n$ is parametrized by

$$a(U) = \left(u_1, \ldots, u_{n-1}, \sum_{i=1}^{n-1} c_i u_i^2\right).$$

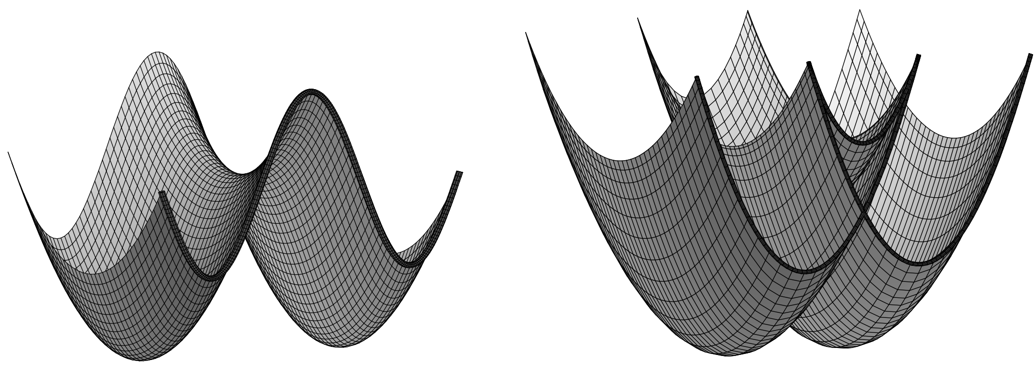

Figure 7: Left: Translational surface generated by translating a parabola along a cosine curve . Right: Convolution of a paraboloid and the translational surface.

Let $B$ be a regular surface, represented by the parametrization

$$B := b(V) = (b_1(V), \dots, b_n(V)), \text{ with } V = (v_1, \dots, v_{n-1}).$$

Analogously to the treatment in $\mathbb{R}^3$, the computation of $A + B$ requires the determination of the correspondence between points $a \in A$ and $b \in B$ in a way that normal vectors $n_A$ and $n_B$ at $a$ and $b$ are linearly dependent. This leads to

$$\begin{pmatrix} -2c_1u_1 \\ \vdots \\ -2c_{n-1}u_{n-1} \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} N_1(V) \\ \vdots \\ N_{n-1}(V) \\ N_n(V) \end{pmatrix},$$

where $N(V)$ is a normal vector field of $B$. It is again obvious that by letting $\lambda = 1/N_n(V)$ one obtains a reparametrization

$$u_i(V) = \frac{-N_i}{2c_iN_n}(V), i = 1, \dots, n-1, \; N_n(V) \neq 0.$$

A parametrization of the sum $A + B$ is thus found by

$$(a + b)(V) = (\dots, b_i - \frac{N_i}{2c_iN_n}(V), \dots, b_n(V) + \frac{1}{4N_n^2}(\sum(\frac{N_i^2}{c_i}(V))).$$

Assuming that $b_i(V)$ are rational functions, the resulting representation $(a + b)(V)$ is a rational parametrization of the sum $A + B$.

## 6 Conclusion

We have shown that the sum or convolution surface $A+B$ of a paraboloid $A$ and a quadric $B$ or an arbitrary parametrizeable surface $B$ possesses explicit parametrizations. Offset surfaces of paraboloids can be treated within this concept. Further we have investigated convolution surfaces of paraboloids $A$ and surfaces $B$ from special families, like surfaces of rotation, skew and developable ruled surfaces and translational surfaces. We have found special geometric properties of the convolution, if the surface $B$ is in a special position to the paraboloid $A$ (parameter curves are situated in planes parallel or perpendicular to the axis of $A$).

The reason for the special behavior of a paraboloid $A$ with respect to the construction of the sum $A+B$ with another surface $B$ is the following: *For any given plane $E$ in space which is not parallel to $A$'s axis, there exists exactly one point $a$ at $A$ in the way that the tangent plane at $a$ is parallel to $E$.*

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