Sphere-geometric aspects of bisector surfaces

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The bisector surface B of two smooth input objects P and Q is the locus of centers of spheres which are tangent to P and Q, respectively. This definition already indicates that methods from sphere geometry, in particular Lie-sphere geometry apply nicely to the construction of these surfaces. The computation of bisector surfaces of general input surfaces results in the solution of a system of nonlinear equations. We show that if both surfaces are canal surfaces or if one surface is a Lie-sphere, the construction is elementary.

1 Introduction

Given two geometric objects P and Q in Euclidean 3–space $\mathbb{R}^3$, their *bisector* B is defined as locus of equidistant points from P and Q. Since distances are measured orthogonal to both objects, the bisector B is the set of centers of spheres touching both P and Q. We do not require that the distance from B to P and Q is minimal and discuss only the *untrimmed bisector*, see [8]. The objects P and Q shall be points, smooth curves and surfaces.

A possible method to study bisectors in $\mathbb{R}^3$ from a sphere-geometric point of view uses concepts from *Laguerre geometry*. A detailed description of the planar case is given in [11], general monographs on sphere geometry are for instance [2, 4] and a brief introduction and some details are given in [13]. Within Laguerre geometry one still distinguishes between oriented spheres and oriented planes. *Lie-sphere geometry* provides a unifying concept to deal with all these elements and this will be applied in the following.

A *Lie-sphere* is defined to be either an oriented sphere or an oriented plane or a point in $\mathbb{R}^3$. We will use a quadric model $\mathcal{L}$ where Lie-spheres are represented as points and oriented contact between two elements is determined by the *conjugacy relation* with respect to the quadric $\mathcal{L}$. Envelopes of one-parameter family of Lie-spheres are called *Lie-canal surfaces*. This class of surfaces consists of canal surfaces, together with curves and developable surfaces, as one parameter families of points and oriented planes. All other smooth surfaces are denoted by *general surfaces*, from the sphere geometric point of view.

The computation of the bisector surface of two input surfaces is difficult in general and results in a system of nonlinear equations. We show that if both input surfaces are Lie-canal surfaces or one input surface is a Lie-sphere, the construction of the bisector is either linear or quadratic and we call this an *elementary construction*. Several results about geometric properties of bisector surfaces can be found in [5, 6, 8, 12].

2 The quadric model of Lie-sphere geometry

Let $\mathbb{R}^3$ be the real Euclidean 3–space. We identify points in $\mathbb{R}^3$ by their coordinate vectors $\mathbf{x} = (x_1, x_2, x_3)^\top$ with respect to a Cartesian coordinate system. The canonical Euclidean dot product is denoted by $\mathbf{x}^\top \cdot \mathbf{y}$ and for the squared norm $\|\mathbf{x}\|^2$ of vectors we use also $\mathbf{x}^2$.

An oriented sphere $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ in $\mathbb{R}^3$ is uniquely determined by its center $\mathbf{m} = (m_1, m_2, m_3)^\top$ and its signed radius r. Thus there is a bijective correspondence between points $M = (m_1, m_2, m_3, r)^\top$ in $\mathbb{R}^4$ and oriented spheres in $\mathbb{R}^3$. We make the arrangement that positive radii shall represent spheres with normal vectors pointing outwards. Points $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ are considered as spheres of zero radius.

Two spheres S_1, S_2 with centers $\mathbf{m}_1, \mathbf{m}_2$ and radii r_1, r_2 respectively, are in *oriented contact* if

$$(\mathbf{m}_1 - \mathbf{m}_2)^2 - (r_1 - r_2)^2 = 0. \qquad (1)$$

An oriented plane E in $\mathbb{R}^3$ is determined by an equation $e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 = 0$. We also use the notation $e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0$ and assume the normalization $\mathbf{e}^\top \cdot \mathbf{e} = 1$, which makes the description unique. An oriented plane $E : e_0 + \mathbf{e}^\top \cdot \mathbf{x} = 0$ is in *oriented contact* to an oriented sphere $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ exactly if

$$e_0 + \mathbf{e}^\top \cdot \mathbf{m} + r = 0, \text{ with } \mathbf{e}^\top \cdot \mathbf{e} = 1. \qquad (2)$$

Points and spheres or points and planes are said to be in oriented contact in case of incidence.

A bijective mapping is called a *Lie-transformation* if it maps Lie-spheres to Lie-spheres and preserves oriented contact. These transformations are not necessarily point-preserving, but they can map points to oriented spheres. While $\mathbb{R}^4$ can be considered as model of Laguerre geometry, where points represent oriented spheres, a point model of Lie-sphere geometry is given by a quadric $\mathcal{L}$ of index 1 in $\mathbb{R}^5$, where the index denotes the largest dimension of subspaces contained in the quadric.

Let P^5 be the projective extension of $\mathbb{R}^5$. Points in P^5 are denoted by capital bold face letters $\mathbf{X}$ and are identified with their homogeneous coordinate vectors $\mathbf{X} = (X_0, \ldots, X_5)\mathbb{R}$. The *Lie quadric* $\mathcal{L}$ is determined by the quadratic equation

$$\mathcal{L} : 2X_0X_5 + X_1^2 + X_2^2 + X_3^2 - X_4^2 = 0. \tag{3}$$

The correspondence between Lie-spheres X in $\mathbb{R}^3$ and points $\mathbf{X} \in \mathcal{L} \subset P^5$ is given by the mapping λ,

$$\begin{array}{ll} \text{sphere } (\mathbf{m}, r) & \mapsto \mathbf{M} = (1, \mathbf{m}, r, -\frac{1}{2}(\mathbf{m}^2 - r^2))\mathbb{R}, \\ \text{point } (\mathbf{p}) & \mapsto \mathbf{P} = (1, \mathbf{p}, 0, -\frac{1}{2}\mathbf{p}^2)\mathbb{R}, \\ \text{plane } (e_0, \mathbf{e}) & \mapsto \mathbf{E} = (0, \mathbf{e}, -1, e_0)\mathbb{R}, \text{ with } \|\mathbf{e}\| = 1. \end{array} \tag{4}$$

Points $\mathbf{x} \in \mathbb{R}^3$ have λ-images $\mathbf{X}$ in $X_4 = 0$, while oriented planes E are mapped to points $\mathbf{E} \in X_0 = 0$. The point $\mathbf{Z} = (0, 0, 0, 0, 0, 1)\mathbb{R}$ does not occur as λ-image of spheres, planes or points of $\mathbb{R}^3$ but is considered as λ-image of the 'point' ∞, which is used to compactify $\mathbb{R}^3$ in the sense of Möbius geometry (one-point compactification). The quadratic cone $X_0 = 0, X_1^2 + X_2^2 + X_3^2 - X_4^2 = 0$ consists of the λ-images of oriented planes and is often referred to as *Blaschke cone (cylinder)*. The quadric $\mathcal{L} \cap X_4 = 0$ is projectively equivalent to S^3 and is a point model of the *Möbius geometry* in $\mathbb{R}^3$.

The group of Lie-transformations is represented by the bijective projective transformations $\kappa : P^5 \to P^5$ which map $\mathcal{L}$ onto itself. It is a 15-parametric group and contains the groups of Laguerre transformations and Möbius transformations as subgroups. Laguerre transformations preserve oriented planes and Möbius transformations preserve points.

The mapping λ is quadratic. The projection

$$\pi : \mathcal{L} \to P^3 : X_4 = X_5 = 0, \tag{5}$$

with 'center' $\mathbf{Z} \vee (0, 0, 0, 0, 1, 0)\mathbb{R}$ onto P^3 maps a finite point $\mathbf{X}$ with $X_0 \neq 0$ to the center $(X_1/X_0, X_2/X_0, X_3/X_0)$ of the corresponding Lie-sphere X. This fact shall motivate the special choice of the coordinate system, although the classical literature [1] often uses a different normal form of $\mathcal{L}$.

The stereographic projection from $\mathbf{Z}$ onto $X_5 = 0$ leads to a point model of Laguerre geometry not considering the images of oriented planes. We note that $\mathbb{R}^4$ is considered as Lorentz space with $(+++-)$ as the signature of the corresponding scalar product.

The oriented contact of Lie spheres is determined by the bilinear form

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X_0Y_5 + X_5Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3 - X_4Y_4, \tag{6}$$

with respect to the Lie-quadric $\mathcal{L}$ (3). It is not difficult to show that two Lie-spheres X, Y are in oriented contact exactly if $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$. We note that any oriented plane is in oriented contact with ∞ and oriented spheres or points are never in contact with ∞.

3 Bisector Constructions

Let P and Q be two general parametrized oriented surfaces in $\mathbb{R}^3$ whose λ-image surfaces are parametrized by $\mathbf{P}(u, v)$ and $\mathbf{Q}(s, t)$. A Lie-sphere X is tangent to P and Q if its image point $\mathbf{X}$ satisfies the linear conjugacy relations

$$\begin{array}{lll} \langle \mathbf{P}, \mathbf{X} \rangle = 0, & \langle \mathbf{P}_u, \mathbf{X} \rangle = 0, & \langle \mathbf{P}_v, \mathbf{X} \rangle = 0, \\ \langle \mathbf{Q}, \mathbf{X} \rangle = 0, & \langle \mathbf{Q}_s, \mathbf{X} \rangle = 0, & \langle \mathbf{Q}_t, \mathbf{X} \rangle = 0, \end{array} \tag{7}$$

where partial derivatives $\partial F/\partial t$ of a function F are denoted by F_t. Additionally $\langle \mathbf{X}, \mathbf{X} \rangle = 0$ holds, since $\mathbf{X}$ is λ-image of a Lie-sphere. If $X_0 \neq 0$ then the center of the Lie-sphere X is contained in the bisector B of $P$and Q.

In general the computation of B amounts to the problem of eliminating the parameters u, v and s, t from the equations (7). This leads to a system of two equations $F(\mathbf{X}) = 0, G(\mathbf{X}) = 0$. Together with $\langle \mathbf{X}, \mathbf{X} \rangle = 0$ this defines a two-dimensional manifold $\mathbf{B}$ in $\mathcal{L}$. The projection $B = \pi(\mathbf{B})$ of $\mathbf{B}, B_0 \neq 0$ onto $\mathbb{R}^3$ is the bisector surface of the two input surfaces P and Q.

Consider rational input surfaces P and Q, the elimination leads to rather complicated equations $F(\mathbf{X}) = G(\mathbf{X}) = 0$ even for simple input surfaces. Moreover it is difficult to understand geometric properties of the solution.

3.1 Offset-invariance of the bisector surface

Let P and Q be two surfaces in $\mathbb{R}^3$ and let B be their bisector surface. We consider offset surfaces P_d and Q_d which are the envelopes of oriented spheres with radii d centered at P and Q, respectively.

B is the locus of centers of spheres $S(u, v)$ which are tangent to P and Q. By reducing the radius function of $S(u, v)$ by $-d$ and keeping their centers fixed one obtains a family of spheres tangent to the offset surfaces P_d and Q_d. Thus B is also the bisector surface of P_d and Q_d.

The offset surfaces P_d and Q_d are images of P and Q under a Lie-transformation (Laguerre transformation) δ which increases the radius of oriented spheres by d.

3.2 Bisector surfaces of Lie spheres

Let P and Q be two Lie-spheres $\neq \infty$ in $\mathbb{R}^3$ which are not in oriented contact. The λ-image $\mathbf{B}$ of their bisector B is the intersection of $\mathcal{L}$ with the 3-space $\langle \mathbf{P}, \mathbf{X} \rangle = \langle \mathbf{Q}, \mathbf{X} \rangle = 0$. In general B is a quadric of index 0, but for certain configurations B is a plane considered as set of points. This happens if P, Q are points or spheres with equal radii or if P, Q are oriented planes.

We consider three Lie-spheres P, Q and R which are not touching each other and compute the set of all Lie-spheres touching them. The image points $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ define a plane $E \in P^5$ which intersects $\mathcal{L}$ in a conic $\mathbf{C}$. The family of spheres C corresponding to $\mathbf{C}$ envelop a Dupin cyclide Φ in general. The conjugacy conditions

$$\langle \mathbf{P}, \mathbf{X} \rangle = \langle \mathbf{Q}, \mathbf{X} \rangle = \langle \mathbf{R}, \mathbf{X} \rangle = 0$$

define a plane F, the polar plane of E with respect to $\mathcal{L}$. The conic $\mathbf{D} = F \cap \mathcal{L}$ corresponds to a family of spheres touching the same Dupin cyclide, since Φ admits two generations as canal surface. If P, Q and R are oriented planes in general position, the oriented planes C envelope a cone of revolution Φ and D is the family of spheres touching Φ with centers at Φ's axis.

4 Lie canal surfaces

One-parameter families of Lie-spheres $C(u)$ correspond to curves $\mathbf{C}(u)$ on $\mathcal{L}$. We assume sufficient smoothness.

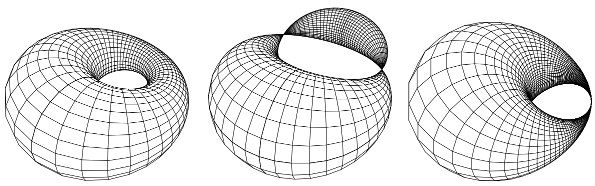

Figure 1: Dupin cyclides

The tangent lines of $\mathbf{C}$ are $\mathbf{T}(u_0) = \lambda\mathbf{C}(u_0) + \mu\dot{\mathbf{C}}(u_0)$, where $\dot{\mathbf{C}} = \mathrm{d}\mathbf{C}/\mathrm{d}u$.

If $\langle \dot{\mathbf{C}}, \dot{\mathbf{C}} \rangle \geq 0$ holds, these Lie-spheres $C(u)$ envelope a Lie canal surface. This is either a canal surface, a curve in case of $\mathbf{C}_4(u) = 0$ or a developable surface in case of $\mathbf{C}_0(u) = 0$. If $\langle \dot{\mathbf{C}}, \dot{\mathbf{C}} \rangle \leq 0$ holds, no real envelope exists.

If $\langle \dot{\mathbf{C}}, \dot{\mathbf{C}} \rangle(u) = 0$ holds identically in an interval, $\mathbf{C}$ is an *asymptotic curve* in $\mathcal{L}$. The curve $\mathbf{C}(u) \in \mathcal{L}$ define a one-parameter family of planes $E(u)$ determined by $\mathbf{T} \cap X_0 = 0$ and an incident curve, determined by $\mathbf{T} \cap X_4 = 0$. This is called a *surface strip*.

Lie spheres tangent to $\mathbf{C}(u)$ are computed as solutions of

$$\langle \mathbf{C}(u), \mathbf{X} \rangle = 0, \langle \dot{\mathbf{C}}(u), \mathbf{X} \rangle = 0, \langle \mathbf{X}, \mathbf{X} \rangle = 0. \quad (8)$$

4.1 Bisector surfaces of two Lie-canal-surfaces

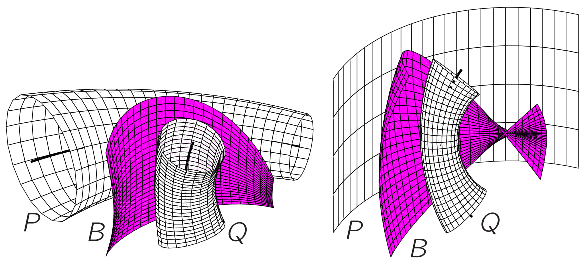

Figure 2: Bisector surfaces of Lie canal surfaces

We are given two Lie-canal-surfaces P and Q with λ-images $\mathbf{P}(u)$ and $\mathbf{Q}(v)$ in $\mathcal{L}$. The bisector surface B consists of all centers of oriented spheres X tangent to P and Q. Any solution $\mathbf{B} \in \mathcal{L}$ of

$$\begin{array}{ll} \langle \mathbf{P}, \mathbf{X} \rangle = 0, & \langle \mathrm{d}\mathbf{P}/\mathrm{d}u, \mathbf{X} \rangle = 0, \\ \langle \mathbf{Q}, \mathbf{X} \rangle = 0, & \langle \mathrm{d}\mathbf{Q}/\mathrm{d}v, \mathbf{X} \rangle = 0, \end{array} \quad (9)$$

satisfying $\langle \mathbf{X}, \mathbf{X} \rangle = 0$ and $X_0 \neq 0$ projects onto a point $\pi(\mathbf{B})$ of the bisector surface B. For any (u, v) the solution of (9) is a line $\mathbf{G}(u, v)$ in P^5. Thus we obtain $\mathbf{B}(u, v)$ as intersection $\mathbf{G}(u, v) \cap \mathcal{L}$ which proves

Theorem 1 *The bisector surface of two Lie canal surfaces $P, Q \in \mathbb{R}^3$ can be constructed in an elementary way. The parametrization of the bisector B of two Lie canal surfaces is given by square roots. The construction is linear if P and Q are both curves or developable surfaces.*

We note that the bisector of two rational developable surfaces or two rational curves is a rational surface since $\mathbf{G}(u, v) \cap \mathcal{L}$ is linear in u, v.

5 Two parametric families of Lie spheres

Two parametric families of Lie-spheres correspond to surfaces $\mathbf{P}(u, v)$ in $\mathcal{L}$. The tangent space T^2 is spanned by $\mathbf{P}$ and the derivative points $\mathbf{P}_u, \mathbf{P}_v$. The real intersection $\mathcal{L} \cap T^2$ can consist of one point, or one or two lines of $\mathcal{L}$. Of particular interest are the two-dimensional submanifolds $\mathbf{P}(u, v) = (1, \mathbf{p}(u, v), 0, -\frac{1}{2}\mathbf{p}^2(u, v))\mathbb{R}$ of the Möbius-quadric $M = \mathcal{L} \cap y_4 = 0$. Since M is of index 0, T^2 intersects $\mathcal{L}$ in a single point. This also follows from intersecting $s\mathbf{P}_u + t\mathbf{P}_v$ with $\mathcal{L}$. Since $\langle \mathbf{P}_u, \mathbf{P}_u \rangle = \mathbf{p}_u^2$, $\langle \mathbf{P}_v, \mathbf{P}_v \rangle = \mathbf{p}_v^2$ and $\langle \mathbf{P}_u, \mathbf{P}_v \rangle = \mathbf{p}_v^\top \cdot \mathbf{p}_v$, we obtain

$$s^2\mathbf{p}_u^2 + 2st\mathbf{p}_u^\top \cdot \mathbf{p}_v + t^2\mathbf{p}_v^2 > 0.$$

5.1 Lie-sphere and a parametrized surface

We consider a general parametrized surface $P = \mathbf{p}(u, v)$ with λ-image $\mathbf{P}(u, v)$ in $\mathcal{L}$ and a Lie-sphere $\mathbf{Q}$. The bisector surface B of P and Q is the projection $\pi(\mathbf{B})$ of the surface $\mathbf{B} \in \mathcal{L}$ which is a solution of

$$\begin{aligned} &\langle \mathbf{P}, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_u, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_v, \mathbf{X} \rangle = 0, \\ &\langle \mathbf{Q}, \mathbf{X} \rangle = 0. \end{aligned} \tag{10}$$

Since (10) determines lines $\mathbf{G}(u, v)$ in P^5, we obtain $\mathbf{B}(u, v) = \mathbf{G}(u, v) \cap \mathcal{L}$. We can state

Theorem 2 *The bisector surface of a parametrized surface $P \in \mathbb{R}^3$ and a Lie-sphere Q can be constructed in an elementary way and the parametrization of B involves square roots. If Q is a point or an oriented plane the construction is linear. If P is a rational offset surface, B is rational too.*

The bisector surface B of P and an oriented plane E is an anticaustic of P with respect to parallel illumination perpendicular to the plane E.

5.2 Lie canal surface and parametrized surface

Let Q be a Lie canal surface with λ-image $\mathbf{L}(t)$ and let P be a general surface whose λ-image has the parametrization $\mathbf{P}(u, v)$. We consider a fixed oriented sphere $\mathbf{L}(t_0)$. The bisector B_{t_0} of Q and $\mathbf{L}(t_0)$ is obtained as projection of the solution $\mathbf{B}_{t_0} \in \mathcal{L}$ of

$$\begin{aligned} &\langle \mathbf{P}, \mathbf{X} \rangle = 0, \qquad \langle \mathbf{P}_u, \mathbf{X} \rangle = 0, \quad \langle \mathbf{P}_v, \mathbf{X} \rangle = 0, \\ &\langle \mathbf{L}(t_0), \mathbf{X} \rangle = 0. \end{aligned} \tag{11}$$

Only those points $\mathbf{X} \in \mathbf{B}_{t_0}$ contribute to B which satisfy the linear relation

$$R^4(t_0) : \langle \mathbf{L}_t(t_0), \mathbf{X} \rangle = 0.$$

Performing this for all t leads to a representation of B.

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