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Parallel Projections in Multidimensional Space

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Summary

Two aspects of parallel projections in the Euclidean n-space $\mathbf{E^n}$ will be considered. On the one hand this is a review on results concerning the relations between m-dimensional axonometric views of $\mathbf{E^n}$ and parallel projections with $(n-m)$-dimensional projectors (e.g. POHLKE's theorem). This requires a discussion of singular affine transformations. In view of applications in computer graphics the matrix representation is emphasized instead of the a more elegant coordinate-free treatment usually used in linear algebra.

On the other hand an applications of parallel projections in CAGD is presented: All Bézier curves of the same order are affine transforms of a particular curve in $\mathbf{E^n}$. This idea gives raise to a type of closed algebraic interpolation curves that are invariant under cyclic permutations of the base points. If the base points form a regular n-gon, then this interpolating curve coincides with the circumcircle.

The affine n-space $\mathbf{A}^n$

Let $\Re^n$ be a *n-dimensional real vectorspace*, e.g.

$$\Re^n = \{\mathbf{x} = (x_1, \dots, x_n)\} \mid x_i \text{ real}, i = 1, .., n\}, \quad \text{where}$$

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n), \rho\mathbf{x} = (\rho x_1, \dots, \rho x_n) \text{ for all } \mathbf{x}, \mathbf{y} \in \Re^n, \rho \in \Re.$$

$\mathbf{S^d}$ is called *d-dimensional (linear) subspace*, if there is a d-dimensional subvectorspace $\mathbf{U} \subset \Re^n$ and

$$\mathbf{S^d} = \{\mathbf{x} = \mathbf{p} + \mathbf{u} \mid \mathbf{u} \in \mathbf{U}\} =: \mathbf{p} + \mathbf{U}.$$

Then for all $\mathbf{x}, \mathbf{y} \in \mathbf{S^d}$ we have $(\mathbf{x} - \mathbf{y}) \in \mathbf{U}$. The set of all subspaces $\mathbf{S^d} \subset \Re^n$, $d \in \{0, ..., n\}$ is called the *real affine n-space* $\mathbf{A^n}$. This space contains especially for

$d = 0$: *points* $\mathbf{P} = \{\mathbf{p}\}$ (for short: point $\mathbf{p}$),
$d = 1$: *lines* $\mathbf{L} = \mathbf{p} + \Re\mathbf{u} = \{\mathbf{p}+t\mathbf{u} \mid t \in \Re\}$, $\mathbf{u} \neq \mathbf{o}$ (=zero-vector),
$d = n-1$: *hyperplanes* $\mathbf{H} = \mathbf{p} + \Re\mathbf{u}_1 + ... + \Re\mathbf{u}_{n-1}$, linear independent $\{\mathbf{u}_1, ..., \mathbf{u}_{n-1}\}$.

The points $\mathbf{p}_0, ..., \mathbf{p}_k$ are called *independent*, if linear independent $\{\mathbf{p}_1 - \mathbf{p}_0, ..., \mathbf{p}_k - \mathbf{p}_0\}$. Each $d+1$ independent points $\mathbf{e}_0, ..., \mathbf{e}_d$ in $\mathbf{S^d}$ define an *affine coordinate system* $S(\mathbf{e}_0; \mathbf{e}_1, ..., \mathbf{e}_d)$ in $\mathbf{S^d}$ with the *origin* $\mathbf{e}_0$ and with the *unit points* $\mathbf{e}_1, ..., \mathbf{e}_d$ by the rule

$$\mathbf{x} \mapsto (\xi_1, ..., \xi_d)_S, \quad \text{if} \quad \mathbf{x} = \mathbf{e}_0 + \sum_{i=1}^{d} \xi_i(\mathbf{e}_i - \mathbf{e}_0). \tag{1}$$

In particular we obtain

$$e_0 \mapsto (0, ..., 0)_S,\ e_1 \mapsto (1, 0, ..., 0)_S,\ e_d \mapsto (0, ..., 0, 1)_S.$$

In the representation $\mathbf{S^d} = \mathbf{p} + \mathbf{U}$ point $\mathbf{p}$ is an arbitrary point in $\mathbf{S^d}$. $\mathbf{U}$ indicates the *direction* of $\mathbf{S^d}$ in the sense that two spaces $\mathbf{S}_i = \mathbf{p}_i + \mathbf{U}_i$, $i \in \{1, 2\}$ are called *(completely) parallel*, if $\mathbf{U}_1 \subset \mathbf{U}_2$ or $\mathbf{U}_1 \supset \mathbf{U}_2$.[1]

The *intersection* $\mathbf{S}_1 \cap \mathbf{S}_2$ of two spaces $\mathbf{S}_i = \mathbf{p}_i + \mathbf{U}_i$, $i \in \{1, 2\}$ is either empty or

$$\mathbf{S}_1 \cap \mathbf{S}_2 = \mathbf{p} + (\mathbf{U}_1 \cap \mathbf{U}_2), \text{ provided } \mathbf{p} \in (\mathbf{S}_1 \cap \mathbf{S}_2).$$

The smallest linear space that contains $\mathbf{S}_1$ and $\mathbf{S}_2$, is the *span*

$$[\mathbf{S}_1\mathbf{S}_2] = \mathbf{p}_1 + \Re(\mathbf{p}_2 - \mathbf{p}_1) + (\mathbf{U}_1 + \mathbf{U}_2).$$

Here $(\mathbf{U}_1 + \mathbf{U}_2) := \{\mathbf{u}_1 + \mathbf{u}_2 \mid \mathbf{u}_1 \in \mathbf{U}_1, \mathbf{u}_2 \in \mathbf{U}_2\}$ determines a space $\mathbf{P}$ parallel to $\mathbf{S}_1$ and $\mathbf{S}_2$; we sweep out $[\mathbf{S}_1\mathbf{S}_2]$ by moving $\mathbf{P}$ along the line $[\mathbf{p}_1\mathbf{p}_2] = \mathbf{p}_1 + \Re(\mathbf{p}_2 - \mathbf{p}_1)$. Any $(k+1)$ independent points $\mathbf{p}_0, ..., \mathbf{p}_k \in \mathbf{A^n}$ span a k-dimensional space

$$[\mathbf{p}_0 ... \mathbf{p}_k] := \{\mathbf{x} = t_0\mathbf{p}_0 + ... + t_k\mathbf{p}_k \mid t_0 + ... + t_n = 1\}.$$

For any two spaces with nonempty intersection the *dimension theorem* says

$$\dim \mathbf{S}_1 + \dim \mathbf{S}_2 = \dim(\mathbf{S}_1 \cap \mathbf{S}_2) + \dim[\mathbf{S}_1\mathbf{S}_2].$$

Affine transformations

Let $\mathbf{S}$, $\mathbf{S}'$ be two linear spaces in $\mathbf{A^n}$. A global map

$$\alpha\colon \mathbf{S} \to \mathbf{S}', \quad \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x}\alpha$$

is called *affine transformation*, if for all $\mathbf{x}, \mathbf{y} \in \mathbf{S}$ and $\xi \in \Re$

$$\mathbf{z} = \mathbf{x} + \xi(\mathbf{y} - \mathbf{x}) \quad \Longrightarrow \quad \mathbf{z}\alpha = \mathbf{x}\alpha + \xi(\mathbf{y}\alpha - \mathbf{x}\alpha).$$

This means that the line $[\mathbf{xy}] \subset \mathbf{S}$ is mapped either on the point $\mathbf{x}\alpha = \mathbf{y}\alpha$ or on the line $[\mathbf{x}\alpha\mathbf{y}\alpha] \subset \mathbf{S}'$ preserving the ratio for each three points of $[\mathbf{xy}]$.

According to a standard result of linear algebra for any affine transformation α: $\mathbf{S} = \mathbf{p} + \mathbf{U} \to \mathbf{S}' = \mathbf{p}' + \mathbf{U}'$ there is a linear map

$$l_\alpha\colon \mathbf{U} \to \mathbf{U}' \quad \text{with} \quad l_\alpha(\mathbf{u} + \mathbf{v}) = l_\alpha\mathbf{u} + l_\alpha\mathbf{v} \quad \text{and} \quad l_\alpha(\rho\mathbf{u}) = \rho l_\alpha\mathbf{u}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{U}$, and $\rho \in \Re$, and with

$$\mathbf{y}\alpha - \mathbf{x}\alpha = l_\alpha(\mathbf{y} - \mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbf{S}.$$

This implies that the image set $\mathbf{S}\alpha = \{\mathbf{x}\alpha \mid \mathbf{x} \in \mathbf{S}\} \subset \mathbf{S}'$ is a linear space. The restriction of an affine transformation α: $\mathbf{S} \to \mathbf{S}'$ on any subspace $\mathbf{T} \subset \mathbf{S}$ is again affine.

The *kernel* of l_α, $\ker l_\alpha := \{\mathbf{u} \mid l_\alpha(\mathbf{u}) = \mathbf{o}\}$ is a subvectorspace of $\mathbf{U}$. The affine transformation α is called *regular*, if $\ker l_\alpha = \{\mathbf{o}\}$, otherwise *singular*. The set $\mathbf{F}_{\mathbf{p}\alpha} := \{\mathbf{x} \mid \mathbf{x}\alpha = \mathbf{p}\alpha\}$ is called *fibre* of α. We deduce $\mathbf{F}_{\mathbf{p}\alpha} = \mathbf{p} + \ker l_\alpha$. For all $\mathbf{p} \in \mathbf{S}$ the fibres $\mathbf{F}_{\mathbf{p}\alpha}$ form a bundle of parallel linear spaces. The *defect* of α, $d_\alpha := \dim(\ker l_\alpha)$ indicates the decrease in dimension by applying α.

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[1] More general: $\mathbf{S}_1$ and $\mathbf{S}_2$ are called *m-parallel*, if $m = \frac{\dim(\mathbf{U}_1 \cap \mathbf{U}_2)}{\min\{\dim \mathbf{U}_1, \dim \mathbf{U}_2\}}$.

Any singular affine transformation α: $\mathbf{S} \to \mathbf{S}'$ is the composition of a parallel projection ι: $\mathbf{S} \to \mathbf{S}_0$ and a regular affine transformation β: $\mathbf{S}_0 \to \mathbf{S}'$. The fibres $\mathbf{F}_{\mathbf{p}\alpha}$ are the *projectors* of ι: $\mathbf{p} \mapsto \mathbf{p}\iota = (\mathbf{F}_{\mathbf{p}\alpha} \cap \mathbf{S}_0)$. Any parallel projection is a singular affine transformation.[2]

Let $d = \dim \mathbf{S}$, $d' = \dim \mathbf{S}'$. Then for any affine coordinate system $S(\mathbf{e}_0; \mathbf{e}_1, ..., \mathbf{e}_d)$ in $\mathbf{S}$ and S' in $\mathbf{S}'$ and for any affine transformation

$$\alpha\colon \mathbf{S} \to \mathbf{S}', \quad \mathbf{x} = (\xi_1, ..., \xi_d)_S \mapsto \mathbf{x}\alpha = (\xi'_1, ..., \xi'_{d'})_{S'}$$

there is a unique d' by d matrix $A_{d'd} = (a_{ij})$ and a vector $\mathbf{a} = (a_1, ..., a_{d'})$ such that

$$\begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_{d'} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{d'} \end{pmatrix} + \begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & & \vdots \\ a_{d'1} & \dots & a_{d'd} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix}. \tag{2}$$

The column vectors of $A_{d'd}$ are coordinates of $(\mathbf{e}_i\alpha - \mathbf{e}_0\alpha)$ with respect to S'; vector $\mathbf{a}$ consists of the S'-coordinates of $\mathbf{e}_0\alpha$.

For any independent points $\mathbf{a}_0, ..., \mathbf{a}_d \in \mathbf{S}$ and any points $\mathbf{a}'_0, ..., \mathbf{a}'_d \in \mathbf{S}'$ there is a unique affine transformation α: $\mathbf{S} \to \mathbf{S}'$, where $\mathbf{a}_i\alpha = \mathbf{a}'_i$ for $i = 0, ..., d$. Any point $\mathbf{x} \in \mathbf{S}$ is mapped according to

$$\alpha: \quad \mathbf{x} = \sum_{i=0}^{d} \xi_i \mathbf{a}_i \quad \mapsto \quad \mathbf{x}\alpha = \sum_{i=0}^{d} \xi_i \mathbf{a}'_i, \quad \text{provided, } \xi_0 + ... + \xi_d = 1. \tag{3}$$

Here $\xi_1, ..., \xi_d$ are the coordinates of $\mathbf{x}$ with respect to $S(\mathbf{a}_0; \mathbf{a}_1, ..., \mathbf{a}_d)$ and ξ_0 is defined as $\xi_0 = 1 - \xi_1 - ... - \xi_d$.

Affine transformations in the Euclidean space $\mathbf{E}^n$

Using the *dot-product* $\mathbf{u}.\mathbf{v} := u_1v_1 + ... + u_nv_n$ and the *norm* $\|\mathbf{v}\| := \sqrt{\mathbf{u}.\mathbf{u}}$ in $\Re^n$ we are able to define the *distance* $d(\mathbf{p}, \mathbf{q}) := \|\mathbf{p} - \mathbf{q}\|$ for each two points $\mathbf{p}$, $\mathbf{q}$ and to define orthogonality: The spaces $\mathbf{S}_i = \mathbf{p}_i + \mathbf{U}_i$, $i = 1, 2$ are called *(completely) orthogonal*, if $\mathbf{u}.\mathbf{v} = 0$ for all $\mathbf{u}_1 \in \mathbf{U}_1$ and $\mathbf{u}_2 \in \mathbf{U}_2$. The affine space $\mathbf{A}^n$ together with these definitions is called *Euclidean n-space* $\mathbf{E}^n$.

In $\mathbf{E}^n$ for any given space $\mathbf{S}^d = \mathbf{a} + \mathbf{U}$ there is a space $\mathbf{N}^{n-d} = \mathbf{p} + \mathbf{U}^\perp$ at any given point $\mathbf{p}$ that is completely orthogonal to $\mathbf{S}^d$. The mapping $\mathbf{U} \mapsto \mathbf{U}^\perp$ is involutory and matches the rule $(\mathbf{U}_1 + \mathbf{U}_2)^\perp = \mathbf{U}_1^\perp \cap \mathbf{U}_2^\perp$.[3]

Any $(d+1)$ points $\mathbf{e}_0, ..., \mathbf{e}_d$ form a *Cartesian base* in $\mathbf{S}^d = [\mathbf{e}_0 ... \mathbf{e}_d]$, if $\|\mathbf{e}_i - \mathbf{e}_0\| = 1$ and $[\mathbf{e}_i\mathbf{e}_0]$ and $[\mathbf{e}_j\mathbf{e}_0]$ are orthogonal for each $i, j \in \{1, ..., d\}$ with $i \neq j$. The associated affine coordinate system $S(\mathbf{e}_0; \mathbf{e}_1, ..., \mathbf{e}_d)$ in $\mathbf{S}^d$ is called *Cartesian*.

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[2] A surjective affine transformation α: $\mathbf{S} \to \mathbf{S}'$ is a parallel projection if and only if it is *idempotent*, that is, $\alpha^2 = \alpha$.

[3] More general: Two spaces $\mathbf{S}_i = \mathbf{p}_i + \mathbf{U}_i$, $i = 1, 2$ with $0 < \dim \mathbf{S}_1 \leq \dim \mathbf{S}_2$ are called *m-orthogonal*, if $m = \frac{\dim(\mathbf{U}_1 \cap \mathbf{U}_2^\perp)}{\dim \mathbf{U}_1}$.

Any affine transformation σ: $\mathbf{E^n} \to \mathbf{E^n}$ that preserves orthogonality is called a *similarity.* For any similarity σ there is a positive constant c_σ^2, where

$$l_\sigma(\mathbf{u}).l_\sigma(\mathbf{v}) = c_\sigma^2(\mathbf{u}.\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v} \in \Re^n.$$

Similarities in $\mathbf{E^n}$ can also be characterized as surjective maps σ: $\mathbf{E^n} \to \mathbf{E^n}$ with an associated constant c_σ such that

$$||\mathbf{p}\sigma - \mathbf{q}\sigma|| = c_\sigma ||\mathbf{p} - \mathbf{q}|| \quad \text{for all } \mathbf{p},\mathbf{q} \in \Re^n.$$

Similarities σ with $c_\sigma = 1$ preserve all distances and are called *congruences* in $\mathbf{E^n}$. In the matrix representation (2) of any congruence the n by n matrix A_{nn} is *orthogonal*, matching the equation $A_{nn}^T A_{nn} = A_{nn} A_{nn}^T = I_{nn}$, where A_{nn}^T denotes the transpose of A_{nn} and where I_{nn} is the identity matrix.

A parallel projection ι: $\mathbf{E^n} \to \mathbf{P^m}$ is called *orthogonal*, if the projectors of ι are $(n-m)$-dimensional and completely orthogonal to the image space $\mathbf{P^m}$.

THEOREM 1. For any affine transformation α: $\mathbf{E^n} \to \mathbf{P^m}$ of defect $d_\alpha = (n-m)$ there are Cartesian coordinate systems $S(\mathbf{e}_0; \mathbf{e}_1, ..., \mathbf{e}_n)$ in $\mathbf{E^n}$ and $S'(\mathbf{e}'_0; \mathbf{e}'_1, .., \mathbf{e}'_m)$ in $\mathbf{P^m}$ with the property that the coordinates $(\xi_1, ..., \xi_n)_S$ and $(\xi'_1, .., \xi'_m)_{S'}$ of corresponding points match the equation

$$\begin{pmatrix} \xi'_1 \\ \xi'_2 \\ \vdots \\ \xi'_m \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & \lambda_m & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \vdots \\ \xi_n \end{pmatrix}, \quad 0 < \lambda_1 \leq, ..., \leq \lambda_m.$$

PROOF. We obtain the required Cartesian bases in the following way: In an arbitrary fibre $\mathbf{F^{n-m}}$ we select a Cartesian base $\mathbf{e}_0$, $\mathbf{e}_{m+1}, ..., \mathbf{e}_n$. The restriction of α on the space $\mathbf{N^m}$ through $\mathbf{e}_0$ orthogonal to $\mathbf{F^{n-m}}$ is regular. The unit sphere $\Sigma^m \subset \mathbf{N^m}$ is mapped on an ellipsoid $\Sigma^{m\prime} \subset \mathbf{P^m}$ with the center $\mathbf{e}'_0 := \mathbf{e}_0\alpha$. Let $0 < \lambda_1 \leq, ..., \leq \lambda_m$ denote the semiaxes of $\Sigma^{m\prime}$ in increasing order. Then we specify the unit points $\mathbf{e}'_i$, $i = 1, ..., m$ of S' on the axes of $\Sigma^{m\prime}$. For each i the unit point $\mathbf{e}_i$ of S is the unique inverse of $\mathbf{e}'_0 + \lambda_i(\mathbf{e}'_i - \mathbf{e}'_0)$ in $\mathbf{N^m}$. ◇

Note that Σ^m is the *contour* of the n-dimensional unit sphere Σ^n, that is, the surface of contact between Σ^n and the projecting hypercylinder. $\Sigma^{m\prime} \subset \mathbf{P^m}$ is the *outline* of Σ^n.

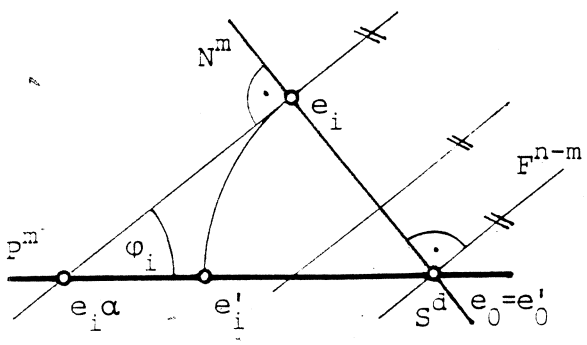

Figure 1. Standard base of α

If α: $\mathbf{E^n} \to \mathbf{P^m}$ is a parallel projection, then we can suppose $\mathbf{e}_0 \in \mathbf{P^m}$, hence $\mathbf{e}'_0 = \mathbf{e}_0$ (see Fig. 1). Let d be the dimension of the intersecting space $\mathbf{S^d} = \mathbf{N^m} \cap \mathbf{P^m}$. Due to the dimension theorem we have $d \geq 2m - n$. The points $\mathbf{e}_0, ..., \mathbf{e}_d$ form a Cartesian base in $\mathbf{S^d}$. These points remain fixed under α: therefore in case $d > 0$ we obtain $\lambda_1 = ... = \lambda_d = 1$. For $i = d+1, ..., m$ the rectangular triangles $\mathbf{e}_0\, \mathbf{e}_i\, \mathbf{e}_i\alpha$ reveal $\lambda_i = 1/\sin\varphi_i$, where φ_i is the interior angle at $e_i\alpha$. Due to the definition of angle measures (see e.g. [5]) between linear spaces in $\mathbf{E^n}$ we formulate

THEOREM 2. If the affine transformation considered in Theorem 1 is a parallel projection, then the diagonal elements in the m by n matrix read

$$\lambda_i = \begin{cases} 1 & \text{for } i = 1, ..., d, \ d \geq \max\{0, 2m - n\} \\ \frac{1}{\sin\varphi_i} > 1 & \text{for } i = d + 1, ..., m, \end{cases}$$

where $\varphi_{d+1}, ..., \varphi_m$ are the angle measures $\neq \frac{\pi}{2}$ made by the image space $\mathbf{P^m}$ and any projector $\mathbf{F^{n-m}}$. For any orthogonal projection we obtain $d = m$, hence $\lambda_1 = ... = \lambda_m = 1$.

Let L_{mn} denote the diagonal matrix in Theorem 1. The associated Gram's matrices $\overline{L}_{mm} := L_{mn}L_{mn}^T$ and $\overline{L}_{nn} := L_{mn}^T L_{mn}$ have again diagonal form. The sequences of eigenvalues read

$$\text{for } \overline{L}_{mm}: \ \lambda_1^2, ..., \lambda_m^2, \quad \text{for } \overline{L}_{nn}: \ \lambda_1^2, ..., \lambda_m^2, 0, ..., 0. \tag{4}$$

When we change the Cartesian *standard* coordinate systems S and S' of α, then the matrix L_{mn} has to be replaced by the product $A_{mn} = B_{mm}L_{mn}C_{nn}$, where B_{mm} and C_{nn} are orthogonal. Instead of the Gramians $\overline{L}_{mm}$ and $\overline{L}_{nn}$ we obtain

$$\overline{A}_{mm} = A_{mn}A_{mn}^T = B_{mm}\overline{L}_{mm}B_{mm}^T, \quad \overline{A}_{nn} = A_{mn}^T A_{mn} = C_{nn}^T \overline{L}_{nn} C_{nn}, \tag{5}$$

which are resp. similar to $\overline{L}_{mm}$ and $\overline{L}_{nn}$. Note that similar matrices share eigenvalues and trace.

The axonometric method in $\mathbf{E^n}$

In $\mathbf{E^m}$ any $(n+1)$ points $\mathbf{a}'_0, ..., \mathbf{a}'_n$ with $[\mathbf{a}'_0 ... \mathbf{a}'_n] = \mathbf{E^m}$, $n > m$ determine an *axonometric view* of $\mathbf{E^n}$ in the following way: The given points are seen as images of a Cartesian base $\mathbf{a}_0, ..., \mathbf{a}_n$ in $\mathbf{E^n}$ with the associated coordinate system S. For each point $\mathbf{p} \in \mathbf{E^m}$ with coordinates $(\xi_1, ..., \xi_n)_S$ the axonometric image $\mathbf{p}'$ is defined as

$$\mathbf{p}' = \mathbf{a}'_0 + \sum_{i=1}^n \xi_i(\mathbf{a}'_i - \mathbf{a}'_0).$$

Eqs. (1) and (3) reveal that the map $\mathbf{E^n} \rightarrow \mathbf{E^m}$, $\mathbf{p} \mapsto \mathbf{p}'$ is affine. Therefore each axonometric view of $\mathbf{E^n}$ can be derived as an affine transform of a parallel view of $\mathbf{E^n}$.

Let A_{mn} denote the matrix of the axonometric transformation $\mathbf{p} \mapsto \mathbf{p}'$ using the coordinate system S in $\mathbf{E^n}$ and any system S' in $\mathbf{E^m}$. Then the sum of squares

$$\sum_{i=1}^n \|\mathbf{a}'_i - \mathbf{a}'_0\|^2$$

is the trace of the Gramian $\overline{A}_{nn} := A_{mn}^T A_{mn}$. From (4), (5) and Theorem 2 we deduce

THEOREM 3. If points $\mathbf{a}'_0, ..., \mathbf{a}'_n \in \mathbf{E^m}$ define an axonometric view of $\mathbf{E^n}$ that is congruent to a parallel view of $\mathbf{E^n}$, then the associated *distortion factors* $\delta_i := \|\mathbf{a}'_i - \mathbf{a}'_0\| / \|\mathbf{a}_i - \mathbf{a}_0\|$, $i = 1, ..., n$ match the equation

$$\sum_{i=1}^{n} \delta_i^2 = \sum_{j=1}^{m} \lambda_j^2 = m + \sum_{k=d+1}^{m} \cot^2 \varphi_k \geq m.$$

Here $\lambda_1, \ldots, \lambda_m$ are the semiaxes of the ellipsoid $\Sigma^{m\prime}$ that is the outline of the unit sphere $\Sigma^n \subset \mathbf{E^n}$. And $\varphi_{d+1}, \ldots, \varphi_m$ denote the angle measures $\neq \frac{\pi}{2}$ made by any projector and the m-dimensional image space.

COROLLAR. In $\mathbf{E^m}$ a parallel view of $\mathbf{E^n}$ with $(n-m)$-dimensional projectors is an orthogonal view if and only if the images $\mathbf{a}_0', \ldots, \mathbf{a}_n'$ of any Cartesian base $\mathbf{a}_0, \ldots, \mathbf{a}_n$ match the equation

$$\sum_{i=1}^{n} \|\mathbf{a}_i' - \mathbf{a}_0'\|^2 = m.$$

If any axonometric view is congruent to an orthogonal view of $\mathbf{E^n}$ (see e.g. Fig. 2), then due to (5) for any associated matrix A_{mn} the Gramian $\overline{A}_{mm}$ is the identity matrix I_{mm}. This implies that the row vectors in A_{mn} are pairwise orthogonal unit-vectors; hence A_{mn} shares m row vectors with any orthogonal n by n matrix.

In $\mathbf{E^2}$ an orthogonal view of $\mathbf{E^3}$ can be defined by specifying the views $[\mathbf{a}_i'\mathbf{a}_0']$, $i = 1, 2, 3$ of the coordinate axes, provided these lines are the altitudes of any acute triangle. What can be said about the multidimensional case? Specifying the images of the coordinate axes means to specify each column vector of the associated matrix A_{mn} up to a factor γ_i, $i = 1, \ldots, n$. On the other hand there are $\frac{m(m+1)}{2}$ orthogonality conditions. Therefore we may expect unique solutions only if $2n = m(m+1)$. Necessary conditions for these images are given in the following theorem. We use the term *m-dimensional quarterspace* with *vertex* $\mathbf{p}$ to denote the intersection of two closed halfspaces in $\mathbf{E^m}$ with orthogonal boundary hyperplanes running through $\mathbf{p}$.

THEOREM 4. Let $\mathbf{a}_0, \ldots, \mathbf{a}_n$ be a Cartesian base in $\mathbf{E^n}$. If ι: $\mathbf{E^n} \to \mathbf{E^m}$, $\mathbf{a}_i \mapsto \mathbf{a}_i'$ is an orthogonal projection using $(n-m)$-dimensional projectors, then the images $[\mathbf{a}_1'\mathbf{a}_0'], \ldots, [\mathbf{a}_n'\mathbf{a}_0']$ of the coordinate axes must not avoid any m-dimensional quarterspace $\mathbf{Q}$ with vertex $\mathbf{a}_0'$. That is in detail: For each $\mathbf{Q}$ either there exists any $i \in \{1, \ldots, n\}$ such that $[\mathbf{a}_i'\mathbf{a}_0']$ contains interior points of $\mathbf{Q}$, or each image point $\mathbf{a}_i'$ is located in any boundary hyperplane of $\mathbf{Q}$.

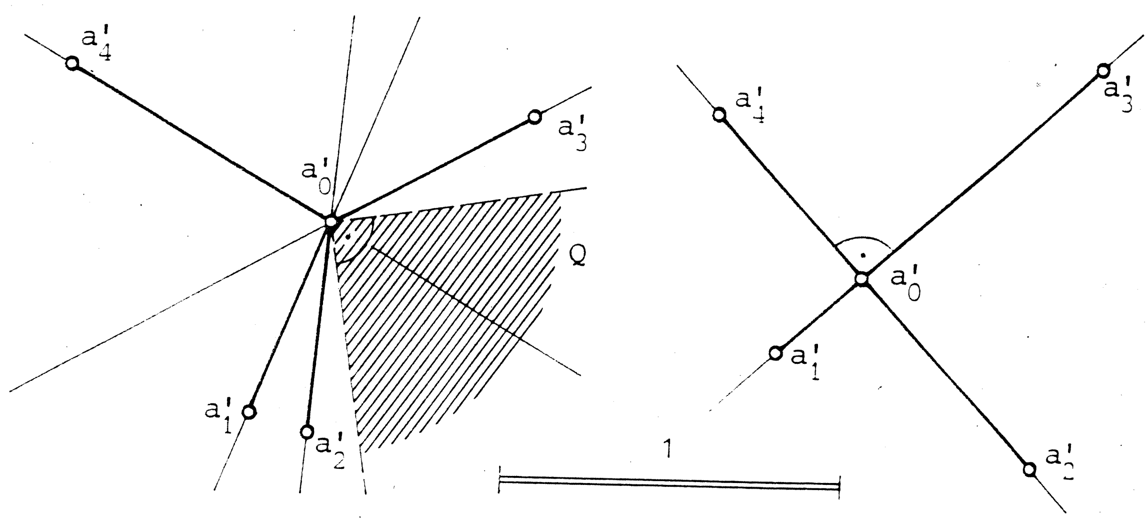

Figure 2. Orthogonal views of a Cartesian base of $\mathbf{E^4}$

PROOF. For any $\mathbf{Q}$ with vertex $\mathbf{a}_0'$ there is a coordinate system S' in $\mathbf{E^m}$ such that $\mathbf{Q} = \{\mathbf{x} = (\xi_1', \dots, \xi_m')_{S'} \mid \xi_1' \geq 0, \xi_2' \leq 0\}$. The coordinate vectors $(a_{1i}, \dots, a_{mi})_{S'}$ of $\mathbf{a}_i'$ are the column vectors in the associated matrix A_{mn}. Let us assume that the condition stated above doesn't hold for this $\mathbf{Q}$. Then after replacing some $\mathbf{a}_i$ by its reflection $(2\mathbf{a}_0 - \mathbf{a}_i)$ in $\mathbf{a}_0$ we obtain $a_{1i} \geq 0$ and $a_{2i} \geq 0$ for all $i = 1, \dots, n$, and additionally there exists any $j \in \{1, \dots, n\}$ with $a_{1j} > 0$ and $a_{2j} > 0$. This results in a positive dot product of the first and second row vector of A_{mn} in contradiction to their orthogonality. $\diamondsuit$

The n-dimensional version of POHLKE's theorem reads[4]

THEOREM 5. The axonometric view defined by $\mathbf{a}_0'$, ..., $\mathbf{a}_n'$ in $\mathbf{E^m}$ is similar to a parallel view of $\mathbf{E^n}$ if and only if either $m \leq \frac{n}{2} + 1$ or the smallest eigenvalue of the associated Gramian $\overline{A}_{mm}$ is at least $(2m - n)$-fold.

PROOF. Because of the similarity σ and due to (4), (5) and Theorem 2 the eigenvalues of $\overline{A}_{mm}$ in an increasing order read

$$c_\sigma^2, \dots, c_\sigma^2 \ (d \text{ times}), \ c_\sigma^2(1 + \cot^2\varphi_{d+1}), \dots, c_\sigma^2(1 + \cot^2\varphi_m),$$

where $d \geq \max\{0, 2m - n\}$. This proves the necessity of the condition stated above.

In order to prove its sufficiency we divide all eigenvalues of $\overline{A}_{mm}$ by the smallest one. Then they coincide with those of $\overline{L}_{mm}$ in (4). The given affine transformation α: $\mathbf{a}_i \mapsto \mathbf{a}_i'$ for $i = 0, \dots, n$ determines a standard base $\mathbf{e}_0$, ..., $\mathbf{e}_n$ in $\mathbf{E^n}$. Let 1 appear d times in the sequence of reduced eigenvalues, $d \geq 2m - n > 0$. Then the image space $\mathbf{P^m}$ in $\mathbf{E^n}$ is spanned by $\mathbf{e}_0'$, ..., $\mathbf{e}_m'$ with

$$\mathbf{e}_i' = \begin{cases} \mathbf{e}_i & \text{for } i = 0, \dots, d, \ d \geq \max\{1, 2m - n\} \\ \mathbf{e}_0 + (\mathbf{e}_i - \mathbf{e}_0)\sin\varphi_i - (\mathbf{e}_{m+i} - \mathbf{e}_0)\cos\varphi_i & \text{for } i = d+1, \dots, m. \ \diamondsuit \end{cases}$$

Applications of affine transformations in CAGD

In $\mathbf{E^m}$ let $\mathbf{p}_0$, ..., $\mathbf{p}_n$ be the control points for any Bézier curve of degree $n \geq m$

$$c_\mathbf{p}: \quad \mathbf{q}(t) = \textstyle\sum_{i=0}^n \mathbf{p}_i B_i^n(t), \quad B_i^n(t) := \binom{n}{i}(1-t)^{n-i}t^i.$$

On the other hand let us specify $(n+1)$ independet points $\mathbf{a}_0$, ..., $\mathbf{a}_n$ in $\mathbf{E^n}$, e.g. a Cartesian base. Then there is an associated *standard* Bézier curve

$$c_\mathbf{a}: \quad \mathbf{b}(t) = \textstyle\sum_{i=0}^n \mathbf{a}_i B_i^n(t).$$

Note that $\sum_{i=0}^n B_i^n(t) = 1$. Hence according to eq. (3) $c_\mathbf{p} \subset \mathbf{E^m}$ is the image of the standard curve $c_\mathbf{a} \subset \mathbf{E^n}$ under the affine transformation α: $\mathbf{E^n} \to \mathbf{E^m}$ that is uniquely determined by α: $\mathbf{a}_i \mapsto \mathbf{p}_i$ for $i = 0, \dots, n$.

The standard curve $c_\mathbf{a}$ is a *rational norm curve* (see [2]) of $\mathbf{E^n}$ containing

$$\mathbf{b}(0) = \mathbf{a}_0, \quad \mathbf{b}(1) = \mathbf{a}_n, \quad \mathbf{e} := \mathbf{b}(\tfrac{1}{2}) = \textstyle\sum_{i=0}^n \frac{1}{2^n}\binom{n}{i}\mathbf{a}_i.$$

The points $\mathbf{a}_0, \dots, \mathbf{a}_n$ represent an *osculating simplex* of $c_\mathbf{a}$, that is a simplex with the following property: For each $i \in \{1, \dots, n-1\}$ the span $[\mathbf{a}_0\mathbf{a}_1 \dots \mathbf{a}_i]$ is the osculating

[4] In [1] and [6] this theorem is formulated in a more geometric way using the inertia ellipsoid of points $\mathbf{a}_0'$, ..., $\mathbf{a}_n'$ in $\mathbf{E^m}$. See also the literature cited in [1] or [6].

i-space of $c_\mathbf{a}$ at $\mathbf{a}_0$, and $[\mathbf{a}_i\mathbf{a}_{i+1}...\mathbf{a}_n]$ is the osculating $(n-i)$-space at $\mathbf{a}_n$. The rational norm curve $c_\mathbf{a}$ is uniquely determined by this osculating simplex and by point $\mathbf{e}$.

The example of Bézier curves reveals a method to create families of curves or surfaces $\Phi_\mathbf{p}$ that are associated with their control points $\mathbf{p}_0...\mathbf{p}_n \in \mathbf{E}^\mathbf{m}$, $m \leq n$ in an affine-invariant manner: We define a "standard simplex" $\mathbf{a}_0...\mathbf{a}_n \in \mathbf{E}^\mathbf{n}$ and create any "standard curve" or "standard surface" $\Phi_\mathbf{a}$ in $\mathbf{E}^\mathbf{n}$. Then $\Phi_\mathbf{p}$ is the image of $\Phi_\mathbf{a}$ under the unique affine transformation α: $\mathbf{E}^\mathbf{n} \rightarrow \mathbf{E}^\mathbf{n}$, $\mathbf{a}_i \mapsto \mathbf{p}_i$ for $i = 0, ..., n$; for short $\Phi_\mathbf{p} = \Phi_\mathbf{a}\alpha$. In order to prove the affine-invariant connection between $\Phi_\mathbf{p}$ and $(\mathbf{p}_0, ..., \mathbf{p}_n)$ we apply any affine transformation β: $\mathbf{E}^\mathbf{m} \rightarrow \mathbf{E}^\mathbf{m}$. The image $\Phi_\mathbf{p}\beta = \Phi_\mathbf{a}\alpha\beta$ of $\Phi_\mathbf{p}$ is the image of the standard set $\Phi_\mathbf{a}$ under $\alpha\beta$: $\mathbf{E}^\mathbf{n} \rightarrow \mathbf{E}^\mathbf{m}$, $\mathbf{a}_i \mapsto \mathbf{p}_i\beta$ for $i = 0, ..., n$.

Let us assume a regular standard simplex $\mathbf{a}_0...\mathbf{a}_n \in \mathbf{E}^\mathbf{n}$. In this case each two different vertices have the same distance. From $\|\mathbf{a}_i - \mathbf{a}_0\| = \|\mathbf{a}_i - \mathbf{a}_1\|$ for $i = 2, ..., n$ we deduce that $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are contained in the hyperplane of symmetry of $\mathbf{a}_0$ and $\mathbf{a}_1$. The reflection σ_{01} in this hyperplane commutes $\mathbf{a}_0$ and $\mathbf{a}_1$ while $\mathbf{a}_2$, ..., $\mathbf{a}_n$ remain fixed. The product $\gamma = \sigma_{01}\sigma_{02}...\sigma_{0n}$ maps $\mathbf{a}_0 \mapsto \mathbf{a}_1$, $\mathbf{a}_1 \mapsto \mathbf{a}_2$, ..., $\mathbf{a}_n \mapsto \mathbf{a}_0$. γ is a congruence that fixes the center of the regular simplex. γ^{n+1} fixes each vertex of the simplex, hence this is the identity.

We now confine us on even n, say $n = 2k$. Then γ is the product on an even number of reflections and therefore a direct congruence with a fixpoint. Due to standard results of linear algebra (see e.g. [4]) γ is the product of k rotations about $(n-2)$-dimensional axes that are pairwise $\frac{2}{n-2}$-orthogonal. In an appropriate Cartesian coordinate system S the congruence γ can be written as

$$\gamma: \quad (\xi_1, ..., \xi_{2k})_S \quad \mapsto \quad (\xi'_1, ..., \xi'_{2k})_S, \quad \text{where}$$

$$\begin{aligned} \xi'_1 &= \xi_1 \cos\varphi_1 - \xi_2 \sin\varphi_1 & \quad \cdots\cdots \quad & \xi'_{2k-1} = \xi_{2k-1}\cos\varphi_k - \xi_{2k}\sin\varphi_k \\ \xi'_2 &= \xi_1 \sin\varphi_1 + \xi_2 \cos\varphi_1 & & \xi'_{2k} = \xi_{2k-1}\sin\varphi_k + \xi_{2k}\cos\varphi_k. \end{aligned}$$

Abbreviating we write $\gamma = \langle\varphi_1, ..., \varphi_k\rangle$. Then the i-th iteration reads $\gamma^i = \langle\varphi_1, ..., i\varphi_k\rangle$. From $\gamma^{2k+1} = id_{\mathbf{E}^\mathbf{n}}$ we deduce

$$\varphi_j = n_j\varphi, \text{ where } \varphi = \tfrac{2\pi}{2k+1} \text{ and } n_j \in \{0, ..., 2k\} \text{ for } j = 1, ..., k.$$

We may suppose that $\mathbf{a}_0 = (r_1, 0, r_2, 0, ..., r_k, 0)_S$, $r_j \geq 0$ for $j = 1, ..., k$. Then

$$\mathbf{a}_i = \mathbf{a}_0\gamma^i = (r_1 \cos n_1 i\varphi,\ r_1 \sin n_1 i\varphi,\ ...,\ r_k \cos n_k i\varphi,\ r_k \sin n_k i\varphi)_S \ \text{ for } \ i = 0, ..., 2k.$$

If $r_j = 0$, then the cooordinates of each $\mathbf{a}_i$ would match the equations $\xi_{2j-1} = \xi_{2j} = 0$. This contradicts the independence of $\mathbf{a}_0$, ..., $\mathbf{a}_{2k}$. If $n_j = n_l$ for $j \neq l$, then the coordinates of each $\mathbf{a}_i$ would match

$$r_j\xi_{2l-1} = r_l\xi_{2j-1}, \quad r_j\xi_{2l} = r_l\xi_{2j},$$

which agains expresses a contradiction. If $n_j = 2k+1-n_l$ for $j \neq l$, then the coordinates of each $\mathbf{a}_i$ would match

$$r_j\xi_{2l-1} = r_l\xi_{2j-1}, \quad r_j\xi_{2l} = -r_l\xi_{2j}$$

and thus disprove the presupposed independence. Hence we may specify $n_j = j$ for $j = 1, ..., k$. Straight forward calculation[5] shows that $r_1 = ... = r_k$ is sufficient for the regularity of the simplex $\mathbf{a}_0$, ..., $\mathbf{a}_{2k}$. We specify $r_0 = 1$ (edge length $\sqrt{2k+1}$).

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[5] It is useful to combine the real coordinates ξ_{2j-1} and ξ_{2j} to the complex number $\zeta_j = \xi_{2j-1} + i\xi_{2j}$, $i^2 = -1$.

THEOREM 6 ([3], page 245). Let $\mathbf{a}_0, \ldots, \mathbf{a}_{2k}$ in $\mathbf{E}^{2k}$ be a regular simplex whose radius of the circumsphere equals $\sqrt{k}$. Then there exists a Cartesian coordinate system S such that for $i = 0, \ldots, 2k$

$$\mathbf{a}_i = (\cos i\varphi,\ \sin i\varphi,\ \cos 2i\varphi,\ \sin 2i\varphi,\ \ldots,\ \cos ki\varphi,\ \sin ki\varphi)_S, \text{ where } \varphi = \tfrac{2\pi}{2k+1}.$$

The direct congruence γ: $\mathbf{a}_i \mapsto \mathbf{a}_{i+1}$ (indices modulo $2k+1$) can be embedded in a one-parametric group of congruences $G = \{\gamma(t) \mid 0 \le t < 2\pi\}$, $\gamma(t) = \langle t, 2t, \ldots, kt\rangle$ using the notation defined above. The path of $\mathbf{a}_0$ under G

$$c_{\mathbf{a}} := \{\mathbf{a}_0\gamma(t) \mid 0 \le t < 2\pi\} = (\cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \ldots,\ \cos kt,\ \sin kt)_S$$

is a closed analytic curve (compare [7]) of constant curvatures, that is even algebraic. $c_{\mathbf{a}}$ is located on the circumsphere Σ^n of the regular simplex $\mathbf{a}_0, \ldots, \mathbf{a}_{2k}$ and it contains all the vertices in the given order because of $\mathbf{a}_i = \mathbf{a}_0\gamma(i\varphi)$.

In order to apply the affinity α: $\mathbf{E}^{2k} \to \mathbf{E}^m$, $\mathbf{a}_i \mapsto \mathbf{p}_i$ on $c_{\mathbf{a}}$ (see Fig. 3) we have to compute the coordinates of $c_{\mathbf{a}}$ with respect to the affine coordinate system T based on $\mathbf{a}_0, \ldots, \mathbf{a}_{2k}$. These are barycentric coordinates $\mathbf{x} = (\eta_1, \ldots, \eta_{2k})_T$ in $\mathbf{E}^{2k}$, provided, we extend by $\eta_0 := 1 - \eta_1 - \ldots - \eta_{2k}$. When we express the equation

$$\mathbf{x} = (\xi_1, \ldots, \xi_{2k})_S = \sum_{i=0}^{2k} \eta_i \mathbf{a}_i$$

in coordinates of system S, then due to Theorem 6 we obtain the matrix equation

$$\begin{pmatrix} 1 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \\ \vdots \\ \xi_{2k} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \cos\varphi & \cos 2\varphi & \ldots & \cos 2k\varphi \\ 0 & \sin\varphi & \sin 2\varphi & \ldots & \sin 2k\varphi \\ 1 & \cos 2\varphi & \cos 4\varphi & \ldots & \cos 4k\varphi \\ 0 & \sin 2\varphi & \sin 4\varphi & \ldots & \sin 4k\varphi \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos k\varphi & \cos 2k\varphi & \ldots & \cos 2k^2\varphi \\ 0 & \sin k\varphi & \sin 2k\varphi & \ldots & \sin 2k^2\varphi \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \vdots \\ \vdots \\ \eta_{2k} \end{pmatrix} = M_{2k+1\,2k+1} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \vdots \\ \vdots \\ \eta_{2k} \end{pmatrix} \tag{6}$$

Then eq. (3) implies

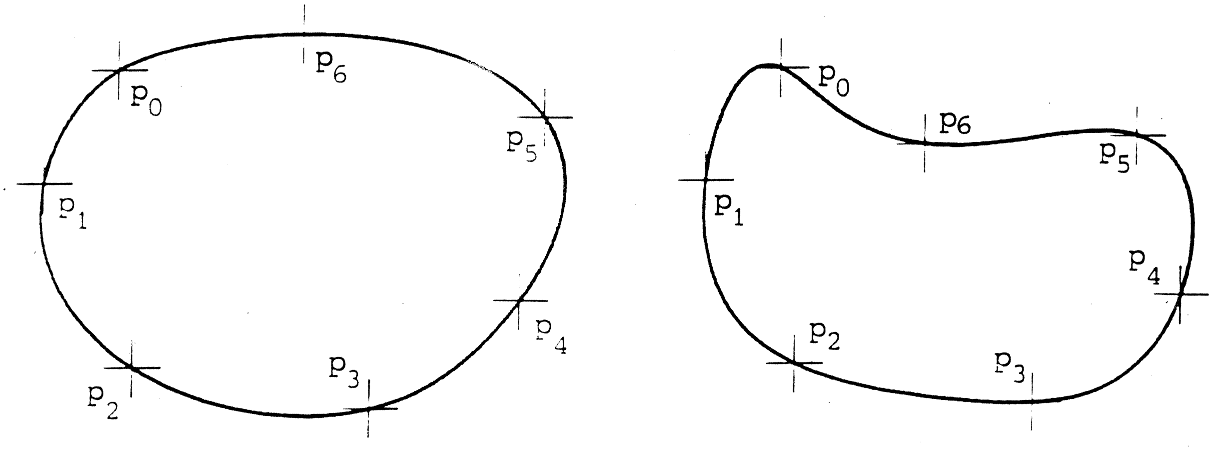

Figure 3. Two examples of $c_{\mathbf{p}}$, interpolating $\mathbf{p}_0, \ldots, \mathbf{p}_6$

THEOREM 7. Let an odd number of points $\mathbf{p}_0,\dots,\mathbf{p}_{2k} \in \mathbf{E}^m$, $m \leq 2k$ be given. Then the curve $c_\mathbf{p} = \sum_{i=0}^{2k} \eta_i \mathbf{p}_i$ with

$$\begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \vdots \\ \eta_{2k} \end{pmatrix} = \left(M_{2k+1\,2k+1}\right)^{-1} \begin{pmatrix} 1 \\ \cos t \\ \sin t \\ \vdots \\ \cos kt \\ \sin kt \end{pmatrix}$$

and matrix $M_{2k+1\,2k+1}$ according to eq. (6), is a finite closed algebraic curve that joins the given points in the given order. $c_\mathbf{p}$ has the following properties:

1) $c_\mathbf{p}$ is connected with $\mathbf{p}_0, \dots, \mathbf{p}_{2k}$ in an affine-invariant manner.

2) $c_\mathbf{p}$ remains unchanged under cyclic permutations of $(\mathbf{p}_0,\dots,\mathbf{p}_{2k})$.

3) $c_\mathbf{p}$ is at the same time associated with the reverse order $(\mathbf{p}_{2k},\dots,\mathbf{p}_0)$.

4) If $\mathbf{p}_0, \dots, \mathbf{p}_{2k}$ are consecutive vertices of a regular $2k$-gon, then $c_\mathbf{p}$ turns to the circumcircle.

PROOF. Ad 2) In $\mathbf{E}^{2k}$ the inverse $c_\mathbf{a}$ of $c_\mathbf{p}$ remains unchanged under γ.

Ad 3) Replace parameter t of $c_\mathbf{a}$ by $(2\pi - t)$.

Ad 4) For any Cartesian coordinate system S' in $\mathbf{E}^2$ the affinity

$$\alpha_j\colon\ \mathbf{E}^{2k} \to \mathbf{E}^2,\quad (\xi_1,\dots,\xi_{2k})_S \mapsto (\xi_{2j-1},\xi_{2j})_{S'} \quad j \in \{1,\dots,k\}$$

maps the regular simplex on a regular $2k$-gon and $c_\mathbf{a}$ on the j-fold covered circumcircle. Each regular $2k$-gon is similar to one of these k different image polygons. Note that the affinity α: $\mathbf{a}_i \mapsto \mathbf{p}_i$ is unique. $\diamondsuit$.

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