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VORONOI DIAGRAMS AND OFFSETS, AN ALGORITHM BASED ON DESCRIPTIVE GEOMETRY

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ABSTRACT

The Voronoi diagram V_B of any compact pointset B in an Euclidean space E^n is a sort of center curve of B. In the following B is supposed to be a connected polygon in E^2 or a connected polyhedron in E^3. Based on the concept of Voronoi cells of B we focus on the graph G_B of the distance function $d(X, \partial B)$ over B. One of our main goals is to provide some insight into the geometry of Voronoi cells and diagrams in E^3.

Moreover an algorithm is presented that computes the shape of V_B exactly with regard to any given precision of computation. The partition of B into its Voronoi cells is also utilized for computing the intersection-free offsets of ∂B in the interior of B.

Key Words: Descriptive Geometry, Computational Geometry, Voronoi diagram, Offsets.

0. Introduction

Voronoi diagrams have already been studied in different areas like discrete geometry, pattern recognition, biology or robotics. There are many papers, that provide algorithms for the computation of Voronoi diagrams (see e.g. [Wolter92] for references).

[Yap87] presents an $O(n \log n)$ algorithm for computing the Voronoi diagram of a set of n circular or straight line segments in the Euclidean plane. In [Hoffmann91] the relation between Voronoi diagrams and a classical descriptive geometry problem has first been pointed out. [Wolter92] focusses on topological properties of Voronoi diagrams.

1. TWO-DIMENSIONAL CASE

1.1 Definitions

Let $S = \{V_1, V_2, ..., V_n\}$ be a set of points in the Euclidean plane E^2. If $\overline{PQ}$ denotes the Euclidean distance, then the sets

$$C_i = \{ X \mid X \in E^2 \text{ and } \overline{XV_i} \leq \overline{XV_j} \text{ for } j \neq i \}, \quad i, j = 1, ..., n$$

are called *Voronoi cells* (Dirichlet cells) in the classical sense (see e.g. [Fejes-Tóth65]).

More generally let $B \subset E^2$ be a compact connected set with polygonal boundary ∂B. For each point $X \in B$ the *distance* (clearance) to ∂B is defined as

$$d(X, \partial B) = \min\{ \overline{XR} \mid R \in \partial B \}.$$

A point $X_f \in \partial B$ is called *foot* of X, if $\overline{XX_f} = d(X, \partial B)$. X_f can be either an interior point of a side of ∂B or a concave vertex. The line segment XX_f is called a *minimal path* of X.

Let $s_1, ..., s_m$ denote the closed sides of ∂B and $V_1, ..., V_n$ its concave vertices. Then we define a *Voronoi cell* of B as

$$C_i = \left\{ X \in B \mid \right.$$

$$\left. d(X, \partial B) = \begin{array}{ll} \overline{Xs_i} & \text{for } i \leq m \\ \overline{XV_{i-m}} & \text{for } m < i \leq m+n \end{array} \right\}.$$

There are two types of Voronoi cells C: A cell C is called of *type 0* or *type 1* according to the dimension of its *base* $C \cap \partial B$.

Remark: This definition of Voronoi cells includes the former definition for a finite point set S provided the points of S are seen as the limit of polygonal holes in the interior of a sufficiently large outward bound of B.

Let $\operatorname{int} B$ denote the interior $B \setminus \partial B$. Then the *Voronoi diagram* of B

$$V_B = \{ X \in \operatorname{int} B \mid X \in (C_i \cap C_j), i \neq j \}$$

is the union of boundaries of all Voronoi cells C_k, $k = 1, ..., m+n$ in the interior of B.

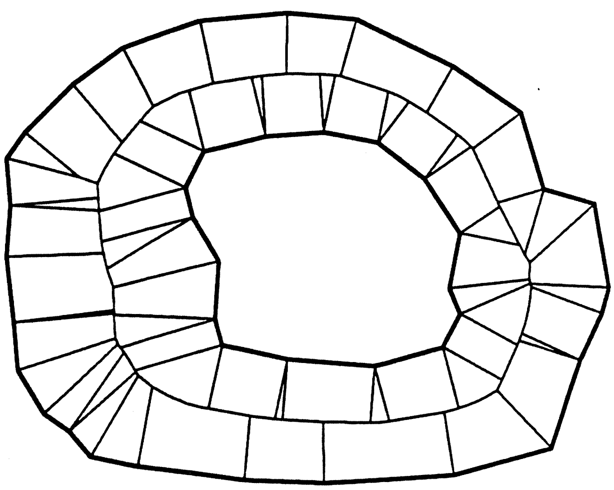

Figure 1. ∂B and Voronoi diagram V_B

V_B is a graph with open edges at the end vertices (see Fig. 1). The *vertices* ($\neq$ end vertices) of V_B are points of $\operatorname{int} B$ that belong to at least three different cells. Two different cells can share one or more edges of V_B.

The set of points of $\operatorname{int} B$ with at least two different feet,

$$M_B = \{ X \in \operatorname{int} B \mid \exists R_1, R_2 \in \partial B,$$
$$R_1 \neq R_2, \overline{XR_1} = \overline{XR_2} = d(X, \partial B) \}$$

is a subset of V_B. M_B is called *medial axis* (internal skeleton). The difference $V_B \setminus M_B$ consists of points that are shared by two neighbor cells of different type. M_B is also the set of centers of circular disks in B that meet the boundary ∂B at least two times.

1.2 Descriptive Geometry approach

We assume that E^2 with the given polygon B is the horizontal plane with equation $z = 0$ in the Euclidean 3-space E^3. Each point $X = (x, y, 0) \in \operatorname{int} B$ defines an unique space point $Y = (x, y, z)$ over X with $z = d(X, \partial B)$. The set of points Y is the *graph* G_B of the function $d(X, \partial B)$ over $\operatorname{int} B$. G_B has the form of a roof built over the horizontal drippings ∂B (compare [Ching-Hoffmann-Lynch91]).

If function $d(X, \partial B)$ is restricted to a cell C of type 0, then the corresponding graph is subset of a cone of revolution with its vertex at the base of C and with a slope of 45°. On the other hand each cell of type 1 is roofed by a 45°-inclined plane passing through its base line.

The edges of G_B are intersections between cones or planes and therefore of hyperbolic or parabolic shape or straight line segments. The top view of the union of these edges is exactly the medial axis M_B. It consists of line segments and of portions of parabolas with focus and directrix at the base of the intersecting cells. ∂B is the cyclographic image (see [Müller-Krames29] or [Hoffmann91]) of all edges of G_B. The Voronoi diagram V_B coincides with the top view of all edges and of all lines of contact between neighbor faces of this roof G_B.

The descriptive geometry approach of determining the Voronoi diagram consists in the following: Each line segment s_i, $i \in \{1, 2, ..., m\}$ of ∂B defines a portion γ_i of a plane of 45°-inclination over E^2. For the present this portion is bounded only by s_i and by lines normal to s_i passing through the endpoints of s_i. On the other hand each concave vertex V_j, $j \in \{1, 2, ..., n\}$ of ∂B defines a portion γ_{m+j} of a cone of revolution over E^2. At the beginning this portion is bounded only by generators in the vertical planes perpendicular to the sides of ∂B passing through R_j.

We have to determine all points Y of E^3 that belong to at least two different faces γ_i, γ_j

with $i, j \in \{1, ..., m+n\}$ and that match the following conditions:

1. $Y = (x, y, z)$ is located over E^2, to say $z > 0$.
2. The top view Y' of Y is an interior point of B.
3. Y is the lowest point over Y' that belongs to any face γ_k, $k \in \{1, ..., m+n\}$.

Remark: This method can also be used when the boundary of B contains circular arcs. Then the vertices of the corresponding cones are located under E^2.
The descriptive geometry approach can still be applied when the Euclidean metric is replaced (see [Yap87]). One has only to vary the shape of faces of G_B.

The top view of any contour line $z = d =$ const., $d > 0$ of G_B is an *offset* $c_{d,B}$ of ∂B ([Ching-Hoffmann-Lynch91], compare [Steiner90]). This is an intersection-free curve of constant distance

$$c_{d,B} = \{\, X \mid d(X, \partial B) = d \,\}$$

in the interior of B. Each offset is the union of straight line segments (in cells of type 1) and of circular arcs (type 0). This reveals that M_B is the set of corner points of all offsets in the interior of B. V_B consists of all corner points and all transition points between segments of different curvature for all offsets $c_{d,B}$ in $\operatorname{int} B$ (Fig. 2).

1.3 Geometric properties

Lemma 1: *For any connected polygon B the Voronoi diagram V_B is connected too.*

Proof: Let X_1, X_2 be two different points on V_B. Then there is a path c_0 in the interior of B, connecting X_1 with X_2. c_0 can be seen as the top view of a path d_0 on G_B. Now we continously raise each point $Y \in d_0$ inside of its face γ_i along a slope line until it reaches the boundary of γ_i. The top view c of this modified path $d \subset G_B$ is subset of V_B and it still connects X_1 with X_2. □

Remark: This variation from c_0 to c is used in [Stifter89] in order to prove the following theorem: A circular disk centered at X_1 can be moved within B to the position with center X_2, if and only if the displacement along V_B can be carried out within B.

Lemma 2: *Let $X_f \in C$ and $\overline{X_f} \in \overline{C}$ be different feet of point $X \in \operatorname{int} B$ in E^2. Then at X the tangent line of edge $e \subset (C \cap \overline{C})$ is perpendicular to line $X_f\overline{X_f}$.*

Proof: The corresponding faces γ and $\overline{\gamma}$ of G_B are equally inclined. □

Lemma 3: *If the foot X_f of X belongs to the Voronoi cell C, then the complete minimal path XX_f is subset of C.*

Proof: Suppose that there is a point Q between X and X_f with the foot $Q_f \neq X_f$ and $\overline{QQ_f} < \overline{QX_f}$. Then the triangle inequality implies the contradiction

$$\overline{XQ_f} \leq \overline{XQ} + \overline{QQ_f} < \overline{XQ} + \overline{QX_f} = \overline{XX_f}.$$

□

We avoid time consuming calculations of intersection points with the aid of

Lemma 4: *Let X_1 and X_2 be two different vertices of V_B or vertices of ∂B that both belong to the same two different cells C and $\overline{C}$. Then in each of the following cases $C \cap \overline{C}$ is an edge of V_B with endpoints X_1 and X_2:*

Case a) *B is simply connected.*
Case b) *C and $\overline{C}$ are both of type 0.*
Case c) *C and $\overline{C}$ are of different type and $X_1 \in \partial B$.*

Proof: The bases of C and $\overline{C}$ together with the minimal paths $X_iX_{if} \subset C$ and $X_i\overline{X}_{if} \subset \overline{C}$, $i = 1, 2$, bound a polygon D. If X_1 and X_2 are not the two endpoints of an edge in $C \cap \overline{C}$, then there exists a point $Q \in \operatorname{int} D$ belonging to another cell $C_j \neq C, \overline{C}$.
Suppose that Q has a foot Q_f outside of D. Then the minimal path QQ_f must cross any minimal path of X_1 or X_2, say $\{S\} = QQ_f \cap X_1X_{1f}$ with $\overline{QQ_f} \leq \overline{QX_{1f}}$. From Lemma 3 we deduce $\overline{SQ_f} = \overline{SX_{1f}}$, and the triangle inequality reveals the contradiction $\overline{QX_{1f}} < \overline{QS} + \overline{SX_{1f}} = \overline{QQ_f}$.
If there is a $Q \in \operatorname{int} D$ that belongs to another cell C_j, then the foot $Q_f \in C_j$ must be located in the interior of D. Hence B is not simply connected. Let K_i be the open circular disk with radius $d(X_i, \partial B)$ centered at X_i. Then the existence of any point $Q \in (C_j \cap \operatorname{int} D)$, $C_j \neq C, \overline{C}$ together with $\overline{X_iQ_f} \geq d(X_i, \partial B)$ implies

$$\operatorname{int} D \setminus (K_1 \cup K_2) \neq \{\}.$$

This cannot hold, if C and $\overline{C}$ are of type 0, because the two circles intersect in the base points of C and $\overline{C}$.
In case c) X_2X_1 is a minimal path of X_2 in C and in $\overline{C}$. □

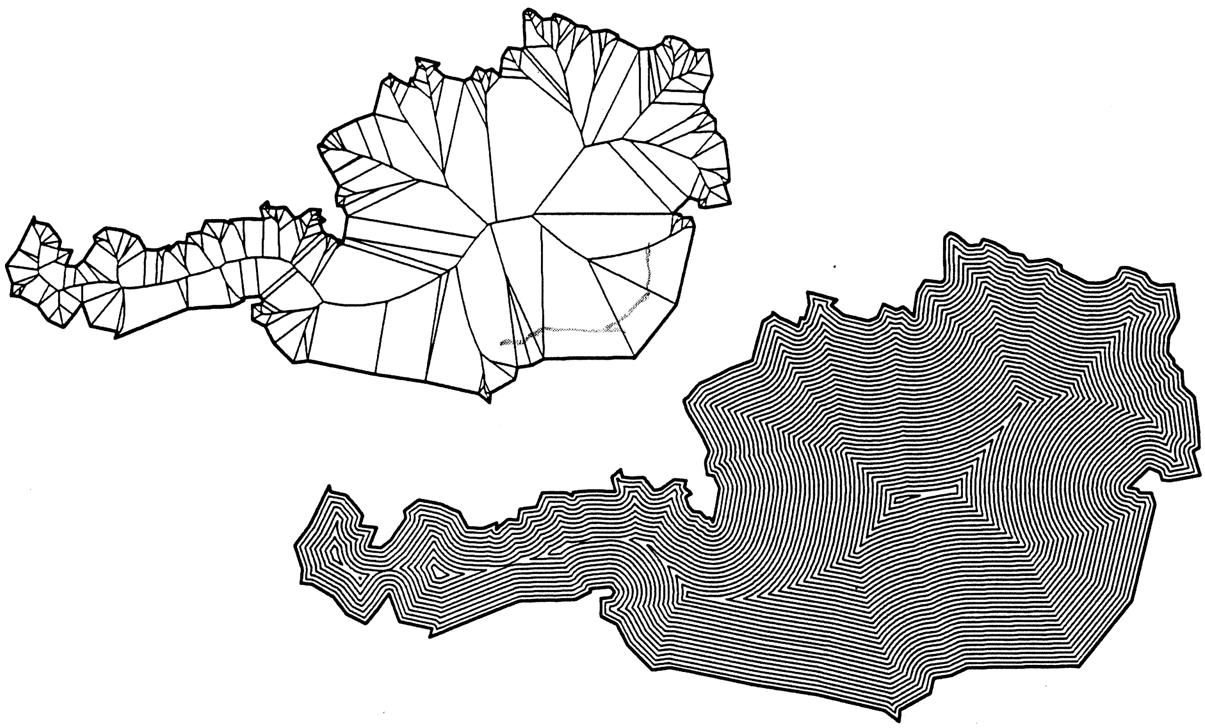

Figure 2. Voronoi diagram and offsets of "Austria"

1.4 Algorithm

The following algorithm computes the Voronoi diagram V_B cell by cell in order to prevent from overlooking any edge. No discretization of E^2 is used, no small modification of ∂B is allowed. This algorithm pays attention to all local symmetries of ∂B that may result in a special topological structure of V_B.

We compute the vertices of G_B by intersecting three faces. As all common points of two equally inclined cones of revolution with vertical axes belong to a vertical plane, we find the intersection points of a triplet of faces by solving a system of 3 linear equations or of two linear equations and a quadratic one. We can avoid any ambiguity in the quadratic case using the boundary conditions for each face together with Lemma 3.

We start with initializing all faces of G_B and with establishing their initial boundaries as mentioned above. The following is a brief outline of how to compute the edge loop in the current face γ_c. Let γ_l and γ_r denote the left and right neighbor face of γ_c, seen from outside.

Algorithm for finishing face γ_c :

1. Prepare list L of not finished faces that might share a vertex or an edge with γ_c.
2. Find endpoint E of edge $\gamma_c \cap \gamma_l$:

   a) Check first, if there is any point in the vertex list of G_B that is located in γ_c and γ_l simultanously. If additionally the conditions of Lemma 4 are given, then E has already been found.

   b) Otherwise E is the lowest point in the set $\{ (\gamma_c \cap \gamma_l \cap \gamma_i) \mid \gamma_i \in \mathrm{L}\}$.

   c) If E is a new point, store it in the vertex list and add the list of faces passing through E. If the edge from the base to E is new, store it in the edge list. Cancel γ_l from the list L of activated faces.
3. Find endpoint P of edge $\gamma_c \cap \gamma_r$ like E in steps 2a,b. If P is a new vertex, store it and add the list of incident faces. Store the edge from the base to P, provided it is new.
4. While $P \neq E$ do the following steps:

   a) If there are only three faces passing through P, then the third face beside γ_c and γ_r becomes the new right neighbor γ_r.

   b) Otherwise find face $\gamma \neq \gamma_c, \gamma_r$ passing through P with the property that near P the curve $\gamma_c \cap \gamma$ is not located over any face through P. Then γ becomes the new neighbor γ_r.

   c) Cancel all faces passing through P in list L.

d) Find endpoint Q of $\gamma_c \cap \gamma_r$ next to P and not on the right side (seen from outside) of the slope line of P in γ_c (Lemma 3). Before calculating the set of potential intersection points $\{ (\gamma_c \cap \gamma_r \cap \gamma_i) \mid \gamma_i \in \mathrm{L}\} \setminus \{P\}$ check the vertex list and the conditions of Lemma 4.

e) If Q is new, store it and add the list of incident faces. Store edge $PQ \subset (\gamma_c \cap \gamma_r)$, if it is new.

f) $P := Q$.

In step 4b we use the tangent vector $\mathbf{t}$ of $\gamma_c \cap \gamma$ at P according to Lemma 2. Let $\mathbf{p}$ denote the coordinate vector of P. Then the position of the points $\mathbf{p} + \varepsilon \mathbf{t}$ and $\mathbf{p} - \varepsilon \mathbf{t}$ with respect to all faces γ_k passing through P must be checked. ε has to be sufficiently small because the tangent line (or the curve $\gamma_c \cap \gamma$) can intersect γ_k two times.

How to keep the list L of potential meeting faces in step 1 as small as possible? The strategy is to build "bridges" of neighbor faces that allow to divide B into separate parts. On the other hand note that after face γ_c has been finished there are no more intersection points in γ_c to be calculated. Therefore in the case of a simply connected B we proceed as follows:

We select a starting face, say γ_i, and finish it. Then we look for a face γ_j that shares a vertex opposite to the base with γ_i. After finishing γ_j we break the list of remaining faces to two parts: In case $j > i$ the first list L_1 consists of $\gamma_{i+1}, \ldots, \gamma_{j-1}$, the second L_2 of $\gamma_{j+1}, \ldots, \gamma_{m+n}, \gamma_1, \ldots, \gamma_{i-1}$.
In each part, e.g. in L_1, we look for a medial face γ_k, $k = [\frac{i+j}{2}]$, and we finish it. If γ_k meets an already finished face, we separate $\gamma_{i+1}, \ldots, \gamma_{k-1}$ and $\gamma_{k+1}, \ldots, \gamma_{j-1}$. Otherwise we build a bridge in L_1. This iterative dividing process of any list ends when it contains one or two faces only.

If in the case of a not simply connected B we have built a bridge between different components of ∂B, then we merge them to one component. All other bridges divide B into separate subsets B_1, B_2. Then for finishing any face over B_1 we need only the list of unfinished faces adjacent to boundary elements of B_1.

2. THREE-DIMENSIONAL CASE

2.1 Definitions

Let $B \subset E^3$ be a compact connected set with polyhedral boundary ∂B. A point $X_f \in \partial B$ in minimal distance to any point $X \in \operatorname{int} B$ is either an interior point of any face or of any concave edge of ∂B or the *foot* X_f is a concave vertex. In this connection a vertex is called *concave*, if there are at least three concave edges of ∂B passing through.

Let $\varepsilon_1, \ldots, \varepsilon_l$ denote the closed faces of ∂B, $s_1, \ldots, s_m$ denote its closed concave edges and $V_1, \ldots, V_n$ denote its concave vertices. Then we define a *Voronoi cell* of B as

$$C_i = \left\{ X \in B \mid d(X, \partial B) = \right.$$

$$\left. = \begin{array}{ll} \overline{X\varepsilon_i} & \text{for } i \leq l \\ \overline{Xs_{i-l}} & \text{for } l < i \leq l+m \\ \overline{XV_{i-l-m}} & \text{for } l+m < i \leq l+m+n \end{array} \right\}.$$

In E^3 there are three types of Voronoi cells: A cell C is called of *type 0*, *type 1* or *type 2* according to the dimension of its *base* $C \cap \partial B$.

In analogy to the plane case the *Voronoi diagram* of B is defined as

$$V_B = \{ X \in \operatorname{int} B \mid X \in (C_i \cap C_j \cap C_k), \; i \neq j \neq k \neq i \}.$$

The vertices of V_B are interior points that belong to at least four different cells. Each edge of V_B is common for at least 3 cells.

V_B is again a superset of the *medial axis*

$$M_B = \{ X \in \operatorname{int} B \mid \exists R_1, R_2, R_3 \in \partial B, \; R_1 \neq R_2 \neq R_3 \neq R_1, \; \overline{XR_1} = \overline{XR_2} = \overline{XR_3} = d(X, \partial B) \}.$$

2.2 Geometry of 3d Voronoi cells

Let each common face of two different cells be called *cell wall*. Depending on the types (i:j) of the adjacent cells and on the relative position of the cell bases a cell wall is subset of a surface of following types: a plane (0:0, 0:1, 1:1, 1:2, 2:2), a quadratic cone (1:2), a parabolic cylinder (0:1, 1:2), a paraboloid of revolution (0:2) or a hyperbolic paraboloid (1:1). Hence the edges of V_B can be portions of lines, of conics or of curves of 4^{th} order.

Lemma 1 and 3 are still valid in E^3. There is a 3d analogon of lemma 4, cases a) and c). Lemma 2 has to be replaced by

Lemma 5: *Let $X_{fi} \in C_i$, $i=1,2,3$ be different feet of point $X \in \operatorname{int} B$ in E^3.*
If X_{f1}, X_{f2}, X_{f3} span a plane, then this is perpendicular to the tangent line t of edge $e \subset (C_1 \cap C_2 \cap C_3)$ at X.
Otherwise in the case $X_{f1} = X_{f2}$ the tangent line t is perpendicular to the line $X_{f2}X_{f3}$. If C_2 is of type 0, then C_1 is of type 1 and t is perpendicular to the base line of C_1 too. If C_2 is of type 2, then t belongs to the plane spanned by X and the base of C_1.

Proof: Let $\mathbf{x}(s)$ be a vector function representing the edge e and let $\mathbf{f}_i(s)$ denote the corresponding feet in C_i. Then the derivation of

$$(\mathbf{x}-\mathbf{f}_i).(\mathbf{x}-\mathbf{f}_i) = (\mathbf{x}-\mathbf{f}_j).(\mathbf{x}-\mathbf{f}_j)$$

and the equation $(\mathbf{x}-\mathbf{f}_i).\dot{\mathbf{f}}_i = 0$ reveal

$$\dot{\mathbf{x}}.(\mathbf{f}_i - \mathbf{f}_j) = 0.$$

This proves the stated perpendicularities. □

Lemma 5 gives raise to a pointwise computation of edges e of V_B by numeric integration. The derivation from the exact curve can be minimized in the following way: Let $\mathbf{t}$ denote the tangent vector of given small length at point $\mathbf{x}$. Then $\mathbf{x}+\mathbf{t}$ is the approximate new point. We recalculate its feet. If this feet span a plane, then we translate $\mathbf{x}+\mathbf{t}$ parallel to this plane in order to equal the distances to the feet. In the case of collinear feet we proceed analogously.

Lemma 6: *If C is a cell of type 0 with base point V, then the cell walls passing through V belong to planes perpendicular to concave edges adjacent to V. These planes either enclose a convex pyramid or* $\operatorname{int} C = \{\}$.

Proof: Let X be a point of $\partial C \backslash \{V\}$. Then X belongs also to a neighbor cell $\overline{C}$ of type 1 or type 2. In the first case X must be located in the plane ν through V perpendicular to the base line s of $\overline{C}$.

In the second case let ε denote the base plane of $\overline{C}$. The equation $\overline{XV} = \overline{X\varepsilon}$ reveals that X lies on the line n through V perpendicular to ε. If in ε an edge s of ∂B through V is concave, then n belongs to the bounding plane $\nu \perp s$ according to the first case.
It turns out to be impossible that both edges $s_1, s_2 \subset \varepsilon$ through V are convex: Let ε_i denote the second plane of ∂B through s_i, $i = 1,2$. Then at V the cell $\overline{C}$ is bounded by ε and by the planes σ_1, σ_2 of symmetry between ε and ε_1 or ε_2. The dihedral angle between σ_i and ε must be acute. Hence in this case V would be the only common point of C and $\overline{C}$.

Interior points of C are separated by these planes ν from interior points of the concave edges $s \perp \nu$ through V. Hence $\operatorname{int} C$ is subset of the intersection of halfspaces bounded by ν. For $\operatorname{int} C \neq \{\}$ it is necessary but not sufficient that there are at least three concave edges of ∂B through V. □

2.3 Descriptive Geometry approach

We embed E^3 as the hyperplane $x_4 = 0$ into the Euclidean 4-space E^4. Each point $X \in \operatorname{int} B$ with coordinates $X = (x_1, x_2, x_3)$ defines a point Y *"over"* X in E^4 with the coordinate vector

$$\underline{\mathbf{y}} = \big(x_1, x_2, x_3, d(X, \partial B)\big).$$

The set of points Y is the *graph* G_B of function $d(X, \partial B)$ over $\operatorname{int} B$.

$G_B \cup B$ is a sort of 4d polytope, containing eventually curved *3d-cells*, *2d-faces*, edges and vertices. The Voronoi diagram V_B is the *top view* of all edges and lines of contact between 2d-faces on G_B.

If the function $d(X, \partial B)$ is restricted to a Voronoi cell C, then the graph is a 3d-cell γ of G_B. For a cell C of type 2 γ is a portion of a hyperplane, that makes an angle of 45° with the hyperplane E^3. The initial boundary conditions for $Y \in \gamma$ require that orthogonal projection of E^4 into the base plane ε of C maps Y in a point of the base face $C \cap \partial B$.

For a cell of type 0 with base point V the 3d-cell γ is subset of a hypercone of revolution with 45°-inclination and with the vertex V. If $\underline{\mathbf{v}}$ denotes the 4d coordinate vector of V, then points Y of γ match

$$f(\underline{\mathbf{y}} - \underline{\mathbf{v}}, \underline{\mathbf{y}} - \underline{\mathbf{v}}) = 0.$$

Here f stands for the symmetric bilinearform over $\mathbb{R}^4$

$$f(\underline{\mathbf{a}}, \underline{\mathbf{b}}) := -a_1 b_1 - a_2 b_2 - a_3 b_3 + a_4 b_4.$$

Points Y of γ fullfil the initial boundary conditions, if the corresponding top view X belongs to the convex pyramid explained in lemma 6.

Cells C of type 1 are roofed by a quadratic hypercone γ with a one-dimensional vertex space at the base line s of C. Using the 4d

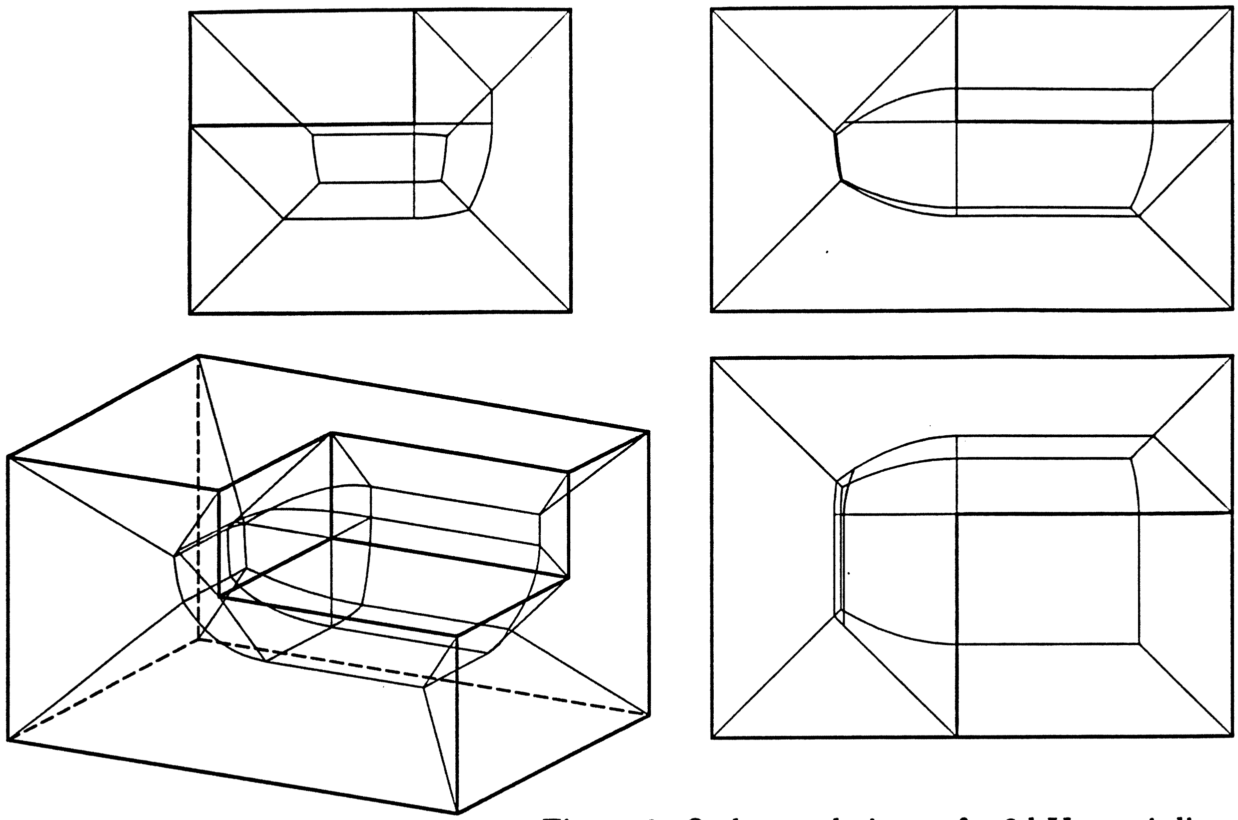

Figure 3. Orthogonal views of a 3d Voronoi diagram

coordinate vectors $\underline{v}_1, \underline{v}_2$ of the endpoints of s the equation of γ reads

$$f(\underline{v}_2 - \underline{v}_1, \underline{v}_2 - \underline{v}_1) f(\underline{y} - \underline{v}_1, \underline{y} - \underline{v}_1) - \\ -\left[f(\underline{v}_2 - \underline{v}_1, \underline{y} - \underline{v}_1)\right]^2 = 0.$$

This comes from the 3d formula

$$\overline{Xs}^2 = \|\mathbf{x} - \mathbf{v}_1\|^2 - \left[\frac{(\mathbf{x} - \mathbf{v}_1).(\mathbf{v}_2 - \mathbf{v}_1)}{\|\mathbf{v}_2 - \mathbf{v}_1\|}\right]^2$$

due to Pythagoras' theorem. The boundary conditions for γ consist of four linear inequalities. They assert that the top view X of $Y \in \gamma$ has a pedal point on side s and that there is no pedal point in the faces adjacent to s on ∂B.

In view of determining V_B we have to compute all points Y of E^4 over $B \subset E^3$ that belong to at least three different 3d-cells and that are not located over any 3d-cell.

The top view of any contour surface $x_4 = d =$ const., $d > 0$ of G_B is an *offset* $\Phi_{d,B}$ of ∂B in E^3. It consists of portions of planes, spheres and cylinders of revolution depending on the type of the enclosing Voronoi cell.

2.4 Brief outline of the 3d algorithm

We compute the vertices of G_B by intersecting four 3d-cells. Sometimes the corresponding system of equations can be solved explicitely, e.g. if there are only hyperplanes and hypercones of revolution to intersect.

Otherwise we use the controlled numeric integration for the intersection curve e of three cells as noted above. Then we are looking for intersection points between each line segment of e and the fourth cell. Finally we improve this approximate solution by standard methods based on linearized cell equations.

We again want to figure out the exact topological structure of V_B. Therefore at each new edge and at each new vertex of G_B we add the list of incident cells. Whenever we intersect three or four 3d-cells we check the edge list or the vertex list first. Moreover at each 3d-cell we add the lists of incident vertices, edges and 2d-faces. And at each 2d-face we add the loops of incident edges.

The 3d algorithm starts with initializing all 3d-cells of G_B and with establishing their initial boundary conditions. Then we finish cell by cell, beginning with cells of type 0. At each 3d-cell γ_c we start at the base.

Let γ_c be of type 0 with vertex V: According to lemma 6 each 2d-face of γ_c through V is the intersection of γ_c with a neighbor cell of type 1. If $e \subset \gamma_c$ is an edge of G_B through V, then e is the intersection of γ_c and two neighbor cells such that a point of $e \setminus \{V\}$ fulfills all boundary conditions of γ_c. If there are less than three such edges, then γ_c is to be cancelled from the list of 3d-cells.

If γ_c is a 3d-cell with base line s and endpoints V_1, V_2, then we start with the 2d-faces f_1, f_2 shared by γ_c and its neighbors γ_1, γ_2 of type 2 with base planes through s. The new edge e in f_i through V_j is the intersection of cells γ_c, γ_i and another cell through V_j such that
a) a point $Y \in (e \setminus \{V\})$ fulfills the boundary conditions of these three intersecting cells and
b) Y is not located over any cell through V.

In any cell γ_c of type 2 each edge b of the base polygon defines a 2d-face f shared by γ_c and a neighbor cell γ of type 1 or 2. If there are only three faces of ∂B that meet at the endpoint V of b, then the new edge e through V is the intersection of the corresponding 3d-cells. Otherwise e is the intersection of γ_c, γ and another cell through V such that again the previous conditions a) and b) are fullfilled.

Thus in any case we have defined 2d-faces in the current cell γ_c together with edges that meet the base of γ_c. In the next step we finish these faces analogously to the plane case. We summarize:

Algorithm for finishing the 3d-cell γ_c:

1. Define 2d-faces that meet the base of γ_c and finish these faces.
2. While there is an edge $e \subset \gamma_c$ that until now is subset of only one face $f_1 = \gamma_c \cap \gamma_1$ of γ_c, do the following steps:

   a) Define the second 2d-face f_2 through edge e.

   b) Finish this face f_2.

We find f_2 in step 2a in the following way: If there are only three cells passing through e, then f_2 is the intersection of γ_c with the third one. Otherwise we select γ_2 through e, $\gamma_2 \neq \gamma_c, \gamma_1$, such that a point $Z \in (\gamma_c \cap \gamma_2)$ at the "lower" side of e and sufficiently near to e is not located over any cell through e.

In order to compute Z we start with a medial point Y on edge e. The tangent plane of the corresponding cell-wall $C_c \cap C_2$ at the top view Y' of Y is perpendicular to the line connecting the feet of Y' in the cells C_c and C_2 (see proof of lemma 5). On the tangent line at Y' perpendicular to e' we select points Z_1', Z_2' near Y' on both sides of Y'. The x_4-coordinate of Z_i with top view Z_i' matches the equation of cell γ_c.

Fig. 3 shows the main views and an orthogonal view of V_B for the difference solid B of two boxes with parallel edges. Already this simple example demonstrates how hard it is to grasp the shape of a Voronoi diagram in E^3.

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