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Euclidean line geometry and kinematics in the 3-space

Hellmuth Stachel

Institute for Geometry, University of Technology Vienna

Wiedner Hauptstraße 8-10/113, A-1040 Wien

email: stachel@geometrie.tuwien.ac.at

Abstract

Corresponding to the extension of the field of real numbers into the ring of dual numbers results on spherical geometry can be transferred into the geometry of directed lines in the Euclidean 3-space. This wellknown principle of transference is applied to two overconstrained spherical mechanisms of DIXON's type which gives rise to new overconstrained spatial structures with two-parametric mobility.

MSC 1991: 53A17

1 Introduction

Due to E. STUDY the directed lines in the Euclidean $\mathbf{E}^3$ can be identified with threedimensional unit vectors $\underline{\mathfrak{g}} = \mathfrak{g}+\varepsilon\widehat{\mathfrak{g}}$ over the ring $\mathbb{D}$ of dual numbers ($\varepsilon^2=0$). This is the basis of STUDY's *principle of transference* (German: *Übertragungsprinzip*) that allows to transfer the results of spherical geometry immediately into the metric line geometry of $\mathbf{E}^3$. W. BLASCHKE was one of the first who used this way for an elegant approach to spatial kinematics in [3]. And it seems that this is the most natural way of treating spatial kinematics (see also [8]).

The present paper starts with a brief summary of STUDY's representation of directed lines together with some formulas dealing with dot product, vector product and scalar triple product of dual vectors. We continue with conditions for the linear dependence of dual vectors that recently has also been studied by D.P. CHEVALLIER in [5]. After that we follow W. BLASCHKE in [3] and present his proof for the HJELMSLEV-MORLEY-theorem in order to demonstrate the power of STUDY's principle of transference. Finally this principle will be applied to two spherical mechanisms whose planar versions date back to A.C. DIXON [6]. This leads to new examples of flexible spatial structures consisting of an arbitrary number of links and cylindrical joints only.

2 Directed lines and dual unit vectors

At the beginning we briefly introduce in STUDY's representation of directed lines (*spears*) in the Euclidean 3-space $\mathbf{E}^3$ (cf. W. BLASCHKE [2]):

Let $\mathfrak{a}$ be any point of line g with given direction vector $\mathfrak{g}$ matching $\|\mathfrak{g}\| = 1$. Then the corresponding *momentum vector* (2^{nd} PLÜCKERvector) $\widehat{\mathfrak{g}} := \mathfrak{a} \times \mathfrak{g}$ is independent from the choice of point $\mathfrak{a}$ at line g and it matches $\mathfrak{g} \cdot \widehat{\mathfrak{g}} = 0$. Conversely, a pair $(\mathfrak{g}, \widehat{\mathfrak{g}})$ of vectors with $\|\mathfrak{g}\| = 1$ and $\mathfrak{g} \cdot \widehat{\mathfrak{g}} = 0$ determines a unique directed line since $\mathfrak{p} := \mathfrak{g} \times \widehat{\mathfrak{g}}$ is the coordinate vector of the pedal point of this line with respect to the origin. Hence there is a bijection between the set of directed lines in $\mathbf{E}^3$ and the set of *dual unit vectors*

$$\underline{\mathfrak{g}} := \mathfrak{g} + \varepsilon\widehat{\mathfrak{g}} \quad \textit{with} \quad \underline{\mathfrak{g}} \cdot \underline{\mathfrak{g}} = \mathfrak{g} \cdot \mathfrak{g} + 2\varepsilon \mathfrak{g} \cdot \widehat{\mathfrak{g}} = \underline{1}\,^{1)}, \quad \varepsilon^2 = 0\,. \tag{2.1}$$

In the following we identify directed lines with their dual unit vector.

For two directed lines $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}$ the *dual angle* $\underline{\varphi} := \varphi + \varepsilon\widehat{\varphi}$ combines the angle φ and the shortest distance $\widehat{\varphi}$ (compare Figure 3). This gives rise to a geometric interpretation for the *dot product* and *vector product* of dual unit vectors $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}$:

$$\begin{aligned} \underline{\mathfrak{g}} \cdot \underline{\mathfrak{h}} &:= \mathfrak{g} \cdot \mathfrak{h} + \varepsilon(\widehat{\mathfrak{g}} \cdot \mathfrak{h} + \mathfrak{g} \cdot \widehat{\mathfrak{h}}) = \underline{\cos}\,\underline{\varphi} := \cos\varphi - \varepsilon\widehat{\varphi}\sin\varphi \\ \underline{\mathfrak{g}} \times \underline{\mathfrak{h}} &:= \mathfrak{g} \times \mathfrak{h} + \varepsilon\left[(\widehat{\mathfrak{g}} \times \mathfrak{h}) + (\mathfrak{g} \times \widehat{\mathfrak{h}})\right] = \\ &= \underline{\sin}\,\underline{\varphi}\,\underline{\mathfrak{n}} = (\sin\varphi + \varepsilon\widehat{\varphi}\cos\varphi)(\mathfrak{n} + \varepsilon\widehat{\mathfrak{n}})\,. \end{aligned} \tag{2.2}$$

Here $\underline{\mathfrak{n}}$ represents a directed common perpendicular of the given lines $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}$, and the signs of φ and $\widehat{\varphi}$ are corresponding to the orientation of $\underline{\mathfrak{n}}$.

The definition of dot product and vector product in (2.2) holds for any dual vectors, not only for dual unit vectors. Each dual vector $\underline{\mathfrak{v}} = \mathfrak{v} + \varepsilon\widehat{\mathfrak{v}}$ is a multiple of a unit vector $\underline{\mathfrak{g}}$, i.e. $\underline{\mathfrak{v}} = \underline{\lambda}\,\underline{\mathfrak{g}}$. Only under $\mathfrak{v} \neq \mathfrak{o}$ the line of $\underline{\mathfrak{g}}\,^{2)}$ is uniquely determined by $\underline{\mathfrak{v}}$.

The dual *scalar triple product* of three dual vectors is defined

$$\begin{aligned} \underline{\det}(\underline{\mathfrak{v}}_1, \underline{\mathfrak{v}}_2, \underline{\mathfrak{v}}_3) &:= \underline{\mathfrak{v}}_1 \cdot (\underline{\mathfrak{v}}_2 \times \underline{\mathfrak{v}}_3) = \\ &= \det(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3) + \\ &\quad + \varepsilon\left[\det(\widehat{\mathfrak{v}}_1, \mathfrak{v}_2, \mathfrak{v}_3) + \det(\mathfrak{v}_1, \widehat{\mathfrak{v}}_2, \mathfrak{v}_3) + \det(\mathfrak{v}_1, \mathfrak{v}_2, \widehat{\mathfrak{v}}_3)\right]. \end{aligned}$$

Dot product, vector product and scalar triple product of dual vectors are linear over $\mathbb{D}$ in each component, to say e.g.

$$\underline{\lambda}\,\underline{\mathfrak{v}}_1 \cdot \underline{\mathfrak{v}}_2 = \underline{\mathfrak{v}}_1 \cdot \underline{\lambda}\,\underline{\mathfrak{v}}_2 = \underline{\lambda}\,(\underline{\mathfrak{v}}_1 \cdot \underline{\mathfrak{v}}_2), \quad \underline{\lambda}\,\underline{\mathfrak{v}}_1 \times \underline{\mathfrak{v}}_2 = \underline{\mathfrak{v}}_1 \times \underline{\lambda}\,\underline{\mathfrak{v}}_2 = \underline{\lambda}\,(\underline{\mathfrak{v}}_1 \times \underline{\mathfrak{v}}_2) \quad \textit{for} \quad \underline{\lambda} \in \mathbb{D}.$$

1) The underbar indicates dual vectors as well as dual numbers. Note that only for dual numbers $\underline{\lambda} = \lambda + \varepsilon\widehat{\lambda}$ with $\lambda \neq 0$ the inverse $\underline{\lambda}^{-1}$ is defined. The pure dual number $\underline{\lambda} = 0 + \varepsilon\widehat{\lambda} \neq \underline{0}$ is a zero divisor in the ring $\mathbb{D}$ of dual numbers.

2) Line $\underline{\mathfrak{g}}$ is the axis of the linear line complex whose finite lines $\underline{\mathfrak{x}} = \mathfrak{x} + \varepsilon\widehat{\mathfrak{x}}$ match $\widehat{\mathfrak{v}} \cdot \mathfrak{x} + \mathfrak{v} \cdot \widehat{\mathfrak{x}} = 0$. These are the path normals of the screwing motion which is associated to the dual vector (=*screw*) $\underline{\mathfrak{v}}$ (compare [5]).

Vanishing products of dual unit vectors characterize the following situations:

$$\begin{array}{rcl} \underline{\mathfrak{g}}\cdot\underline{\mathfrak{h}} = \underline{0} & \iff & \underline{\mathfrak{g}} \textit{ and } \underline{\mathfrak{h}} \textit{ intersect perpendicularily} \\ \underline{\mathfrak{g}}\times\underline{\mathfrak{h}} = \underline{\mathfrak{o}} & \iff & \underline{\mathfrak{g}}, \underline{\mathfrak{h}} \textit{ are located on the same line, i.e. } \underline{\mathfrak{g}} = \pm\underline{\mathfrak{h}} \\ \underline{\det}(\underline{\mathfrak{g}},\underline{\mathfrak{h}},\underline{\mathfrak{k}}) = \underline{0} & \iff & \left\{ \begin{array}{l} \underline{\mathfrak{g}}, \underline{\mathfrak{h}} \textit{ and } \underline{\mathfrak{k}} \textit{ are located on parallel lines or} \\ \textit{they all intersect a line perpendicularily.} \end{array} \right. \end{array} \tag{2.3}$$

When two lines $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}$ match $\underline{\mathfrak{g}}\cdot\underline{\mathfrak{h}} = \underline{0}$, then

$$\underline{\mathfrak{k}} := \underline{\cos}\,\underline{\varphi}\,\underline{\mathfrak{g}} + \underline{\sin}\,\underline{\varphi}\,\underline{\mathfrak{h}} \tag{2.4}$$

defines a spear which is the image of $\underline{\mathfrak{g}}$ under a screwing displacement about the axis $(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}})$ with the *dual screw angle* $\underline{\varphi}$.

Proof: We obtain from $(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}})\cdot(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}}) = (\underline{\mathfrak{g}}\cdot\underline{\mathfrak{g}})(\underline{\mathfrak{h}}\cdot\underline{\mathfrak{h}}) - (\underline{\mathfrak{g}}\cdot\underline{\mathfrak{h}})^2 = 1$ and

$$\underline{\mathfrak{k}}\cdot\underline{\mathfrak{k}} = \underline{\cos}^2\underline{\varphi} + \underline{\sin}^2\underline{\varphi} + 2\,\underline{\cos}\,\underline{\varphi}\,\underline{\sin}\,\underline{\varphi}\,(\underline{\mathfrak{g}}\cdot\underline{\mathfrak{h}}) = 1$$

that $(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}})$ and $\underline{\mathfrak{k}}$ are directed lines. The equation $\underline{\mathfrak{g}}\times\underline{\mathfrak{k}} = \underline{\sin}\,\underline{\varphi}\,(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}})$ reveals the stated property.
The proof could also be carried out by use of the dual quaternion

$$\underline{\mathfrak{Q}} = \underline{\cos}\frac{\underline{\varphi}}{2} + \underline{\sin}\frac{\underline{\varphi}}{2}\,(\underline{\mathfrak{g}}\times\underline{\mathfrak{h}})$$

which represents the screwing displacement stated above. □

3 Linear dependencies

Two dual vectors $\underline{\mathfrak{v}}_1, \underline{\mathfrak{v}}_2$ are *linearly dependent* over the ring $\mathbb{D}$ of dual numbers, if and only if their real parts are linearly dependent over $\mathbb{R}$, i.e.[3])

$$\underline{\lambda}_1\,\underline{\mathfrak{v}}_1 + \underline{\lambda}_2\,\underline{\mathfrak{v}}_2 = \underline{\mathfrak{o}} \ \textit{ and } \ (\underline{\lambda}_1,\underline{\lambda}_2) \neq (\underline{0},\underline{0}) \iff \mathfrak{v}_1\times\mathfrak{v}_2 = \mathfrak{o}\,. \tag{3.1}$$

Proof ad "⇒": From

$$\sum \underline{\lambda}_i\,\underline{\mathfrak{v}}_i = \sum \lambda_i\,\mathfrak{v}_i + \varepsilon\left(\sum \lambda_i\,\widehat{\mathfrak{v}}_i + \sum \widehat{\lambda}_i\,\mathfrak{v}_i\right) = \underline{\mathfrak{o}} = \mathfrak{o} + \varepsilon\mathfrak{o}$$

we conclude the linear dependence of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in both cases, for $(\lambda_1,\lambda_2)\neq(0,0)$ and also for $(\lambda_1,\lambda_2) = (0,0)$ and hence $(\widehat{\lambda}_1,\widehat{\lambda}_2)\neq(0,0)$.

Ad "⇐": If $\mu_1\mathfrak{v}_1 + \mu_2\mathfrak{v}_2 = \mathfrak{o}$ with $(\mu_1,\mu_2)\neq(0,0)$, then for $\underline{\lambda}_i := 0 + \varepsilon\mu_i$ we obtain $(\underline{\lambda}_1,\underline{\lambda}_2)\neq(\underline{0},\underline{0})$ and

$$\sum \underline{\lambda}_i\,\underline{\mathfrak{v}}_i = \sum 0\cdot\mathfrak{v}_i + \varepsilon\left(\sum 0\cdot\widehat{\mathfrak{v}}_i + \sum \mu_i\mathfrak{v}_i\right) = \mathfrak{o}\,. \quad □$$

[3]) Compare [5], Proposition 2.2 i), ii), iii).

Additionally we state for two dual *unit* vectors $\underline{\mathfrak{g}}_1, \underline{\mathfrak{g}}_2$

$$\begin{aligned} \underline{\lambda}_1 \underline{\mathfrak{g}}_1 + \underline{\lambda}_2 \underline{\mathfrak{g}}_2 = \underline{\mathfrak{o}} \ \ \textit{with real parts} \ \ (\lambda_1, \lambda_2) \neq (0,0) \iff \\ \iff \underline{\mathfrak{g}}_1 \times \underline{\mathfrak{g}}_2 = \underline{\mathfrak{o}} \,. \end{aligned} \tag{3.2}$$

Proof ad "$\Rightarrow$": Under the condition $\lambda_1 \neq 0$ we can solve the given linear equation for $\underline{\mathfrak{g}}_1$ by

$$\underline{\mathfrak{g}}_1 = \mathfrak{g}_1 + \varepsilon \widehat{\mathfrak{g}}_1 = \underline{\mu}\, \underline{\mathfrak{g}}_2 = \mu \mathfrak{g}_2 + \varepsilon(\mu \widehat{\mathfrak{g}}_2 + \widehat{\mu} \mathfrak{g}_2).$$

Because of $\|\mathfrak{g}_1\| = \|\mathfrak{g}_2\| = 1$ we obtain $\mathfrak{g}_1 = \pm \mathfrak{g}_2$ and $\mu = \pm 1$. The lines $\underline{\mathfrak{g}}_1$ and $\underline{\mathfrak{g}}_2$ have coinciding pedal points $\mathfrak{p}_1, \mathfrak{p}_2$ with respect to the origin as

$$\mathfrak{p}_1 = \mathfrak{g}_1 \times \widehat{\mathfrak{g}}_1 = \mu \mathfrak{g}_2 \times (\mu \widehat{\mathfrak{g}}_2 + \widehat{\mu} \mathfrak{g}_2) = \mathfrak{g}_2 \times \widehat{\mathfrak{g}}_2 = \mathfrak{p}_2$$

due to $\mu^2 = 1$. From (2.3) we deduce $\underline{\mathfrak{g}}_1 \times \underline{\mathfrak{g}}_2 = \underline{\mathfrak{o}}$.

Ad "$\Leftarrow$": Two directions $\underline{\mathfrak{g}}_1, \underline{\mathfrak{g}}_2$ of the same line match $\underline{\mathfrak{g}}_1 \pm \underline{\mathfrak{g}}_2 = \underline{\mathfrak{o}}$. □

Analogously to (3.1) we can prove that three dual vectors $\underline{\mathfrak{v}}_1, \underline{\mathfrak{v}}_2, \underline{\mathfrak{v}}_3$ are linearly dependent over $\mathbb{D}$, if and only if their real parts are linearly dependent over $\mathbb{R}$, i.e.[4])

$$\exists\, (\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3) \neq (\underline{0}, \underline{0}, \underline{0}) \ \ \textit{with} \ \ \sum_{i=1}^{3} \underline{\lambda}_i \underline{\mathfrak{v}}_i = \underline{\mathfrak{o}} \iff \underline{\det}(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3) = 0 \,. \tag{3.3}$$

For dual *unit* vectors $\underline{\mathfrak{g}}_1, \underline{\mathfrak{g}}_2, \underline{\mathfrak{g}}_3$ we get additionally

$$\begin{aligned} \exists\, (\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3) \ \ \textit{with} \ \ (\lambda_1, \lambda_2, \lambda_3) \neq (0,0,0) \ \ \textit{and} \ \ \sum_{i=1}^{3} \underline{\lambda}_i \underline{\mathfrak{g}}_i = \underline{\mathfrak{o}} \iff \\ \iff \textit{the lines } \underline{\mathfrak{g}}_1,\ \underline{\mathfrak{g}}_2,\ \underline{\mathfrak{g}}_3 \textit{ intersect a line } \underline{\mathfrak{n}} \textit{ perpendicularily.} \end{aligned} \tag{3.4}$$

Proof ad "$\Rightarrow$": If $\lambda_3 \neq 0$, then we get $\underline{\mathfrak{g}}_3 = \underline{\mu}_1 \underline{\mathfrak{g}}_1 + \underline{\mu}_2 \underline{\mathfrak{g}}_2$. There is a common perpendicular $\underline{\mathfrak{n}}$ of $\underline{\mathfrak{g}}_1$ and $\underline{\mathfrak{g}}_2$, and $\underline{\mathfrak{n}} \cdot \underline{\mathfrak{g}}_1 = \underline{\mathfrak{n}} \cdot \underline{\mathfrak{g}}_2 = \underline{0}$ due to (2.3) imply $\underline{\mathfrak{n}} \cdot \underline{\mathfrak{g}}_3 = \underline{0}$ too.

Ad "$\Leftarrow$": By use of $\underline{\mathfrak{h}} := \underline{\mathfrak{n}} \times \underline{\mathfrak{g}}_1$ and according to (2.4) the dual angle $\underline{\varphi}_{ij}$ between $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{g}}_j$ gives rise to representations

$$\underline{\mathfrak{g}}_j = \underline{\cos}\, \underline{\varphi}_{1j}\, \underline{\mathfrak{g}}_1 + \underline{\sin}\, \underline{\varphi}_{1j}\, \underline{\mathfrak{h}} \ \ \text{for} \ \ j = 2, 3 \,.$$

We eliminate $\underline{\mathfrak{h}}$ and obtain

$$\underline{\sin}\, \underline{\varphi}_{12}\, \underline{\mathfrak{g}}_3 + \underline{\sin}\, \underline{\varphi}_{23}\, \underline{\mathfrak{g}}_1 + \underline{\sin}\, \underline{\varphi}_{31}\, \underline{\mathfrak{g}}_2 = \underline{\mathfrak{o}} \,. \tag{3.5}$$

[4]) Compare [5], Proposition 2.1 ii), iii), iv).

If there is a any nonvanishing $\sin\varphi_{ij}$, then (3.5) is already the required equation. Else the three spears $\underline{\mathfrak{g}}_1$, $\underline{\mathfrak{g}}_2$, $\underline{\mathfrak{g}}_3$ are located on parallel lines, to say

$$\mathfrak{g}_2 = v_2\,\mathfrak{g}_1 \quad\text{and}\quad \mathfrak{g}_3 = v_3\,\mathfrak{g}_1 \quad\text{with}\quad v_2, v_3 \in \{1, -1\}.$$

When $\underline{\mathfrak{g}}_1$ and $\underline{\mathfrak{g}}_2$ are on the same line, then we get the stated equation with $\underline{\lambda}_3 = \underline{0}$ from (2.3) and (3.2). Otherwise there is a $\lambda \in \mathbb{R}$ such that for $i = 1,2,3$ the points $\mathfrak{x}_i$ of intersection between $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{n}}$ match

$$\mathfrak{x}_3 = (1-\lambda)\,\mathfrak{x}_1 + \lambda\,\mathfrak{x}_2\,.$$

Because of $v_2^2 = 1$ we obtain

$$\begin{aligned}\widehat{\mathfrak{g}}_1 &= \mathfrak{x}_1 \times \mathfrak{g}_1\,, \quad \widehat{\mathfrak{g}}_2 = v_2(\mathfrak{x}_2 \times \mathfrak{g}_1)\,,\\ \widehat{\mathfrak{g}}_2 &= v_3[(1-\lambda)\mathfrak{x}_1 + \lambda\,\mathfrak{x}_2] \times \mathfrak{g}_1 = v_3(1-\lambda)\,\widehat{\mathfrak{g}}_1 + v_2 v_3 \lambda\,\widehat{\mathfrak{g}}_2\,.\end{aligned}$$

At the same time the real parts match

$$v_3(1-\lambda)\,\mathfrak{g}_1 + v_2 v_3 \lambda\,\mathfrak{g}_2 = [v_3(1-\lambda) + v_3\lambda]\,\mathfrak{g}_1 = v_3\,\mathfrak{g}_1 = \mathfrak{g}_3$$

which leads to

$$[v_3(1-\lambda) + \varepsilon\cdot 0]\,\underline{\mathfrak{g}}_1 + [v_2 v_3 \lambda + \varepsilon\cdot 0]\,\underline{\mathfrak{g}}_2 - [1 + \varepsilon\cdot 0]\,\underline{\mathfrak{g}}_3 = \underline{\mathfrak{o}}\,. \quad \square$$

A comparison of the third equivalence in (2.3) with (3.3) and (3.4) reveals the following implications:

$$\begin{aligned}&\underline{\lambda}_1\,\underline{\mathfrak{g}}_1 + \underline{\lambda}_2\,\underline{\mathfrak{g}}_2 + \underline{\lambda}_3\,\underline{\mathfrak{g}}_3 = \underline{\mathfrak{o}} \quad \textit{with real parts} \quad (\lambda_1, \lambda_2, \lambda_3) \neq (0,0,0)\\ &\qquad \Longrightarrow \quad \underline{\det}(\underline{\mathfrak{g}}_1, \underline{\mathfrak{g}}_2, \underline{\mathfrak{g}}_3) = \underline{0} \quad \Longrightarrow\\ &\Longrightarrow \underline{\lambda}_1\,\underline{\mathfrak{g}}_1 + \underline{\lambda}_2\,\underline{\mathfrak{g}}_2 + \underline{\lambda}_3\,\underline{\mathfrak{g}}_3 = \underline{\mathfrak{o}} \quad \textit{with} \quad (\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3) \neq (0,0,0)\,.\end{aligned} \tag{3.6}$$

4 A standard example for STUDY's principle

In [2], p. 75, W. BLASCHKE presented the following application of STUDYs principle of transference: Let $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ denote the vertices of a triangle on the unit sphere $\mathbf{S}^2$. Then the spherical centers $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ of the sides (great circles) $\mathfrak{g}_2\mathfrak{g}_3, \mathfrak{g}_3\mathfrak{g}_1, \mathfrak{g}_1\mathfrak{g}_2$ can be expressed as

$$\mathfrak{n}_i := r_i\,(\mathfrak{g}_j \times \mathfrak{g}_k) \quad\text{with}\quad r_i \in \mathbb{R}$$

for any even permutation (i,j,k).

We suppose that for each $i \in \{1,2,3\}$ the vectors $\mathfrak{g}_i$ and $\mathfrak{n}_i$ are linearly independent. Then there are nonvanishing real numbers s_1, s_2, s_3 such that the center $\mathfrak{h}_i$ of the i-th spherical altitude in the triangle matches

$$\mathfrak{h}_i = s_i(\mathfrak{g}_j \times \mathfrak{g}_k) \times \mathfrak{g}_i = s_i\,[(\mathfrak{g}_j\cdot\mathfrak{g}_i)\mathfrak{g}_k - (\mathfrak{g}_k\cdot\mathfrak{g}_i)\mathfrak{g}_j]\,.$$

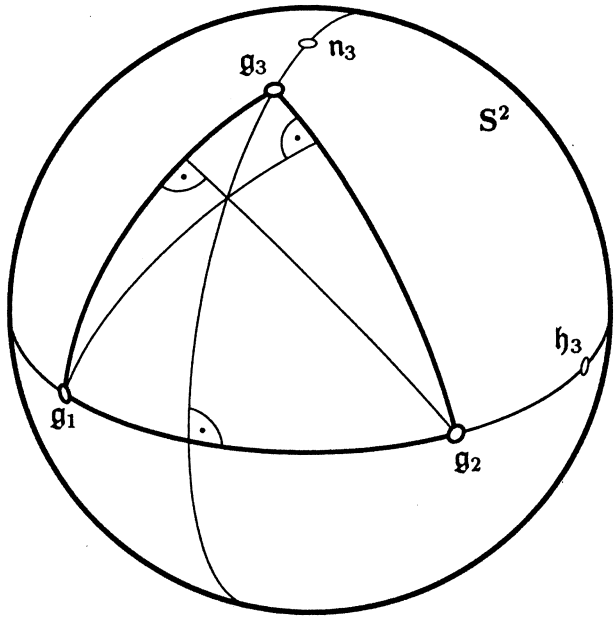

Figure 1: Spherical orthocenter

This results in the equation

$$s_2 s_3 \mathfrak{h}_1 + s_3 s_1 \mathfrak{h}_2 + s_1 s_2 \mathfrak{h}_3 = \mathfrak{o} \tag{4.1}$$

which expresses the linear dependence of $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ as well as the concurrence of the three altitudes in any spherical triangle with atmost one right angle.

Now we extend this computation into $\mathbb{D}$. For this purpose we replace the vertices $\mathfrak{g}_1, \ldots$ by three pairwise distinct directed lines $\underline{\mathfrak{g}}_1, \ldots$ according to (2.3). Instead of the spherical centers $\mathfrak{n}_1, \ldots$ we use common perpendiculars $\underline{\mathfrak{n}}_1, \ldots$ of the pairs $(\underline{\mathfrak{g}}_2, \underline{\mathfrak{g}}_3), \ldots$ according to (2.2). Under the condition $\underline{\mathfrak{g}}_i \neq \pm \underline{\mathfrak{n}}_i$ the dual unit vector $\underline{\mathfrak{h}}_i := \underline{s}_i(\underline{\mathfrak{g}}_j \times \underline{\mathfrak{g}}_k) \times \underline{\mathfrak{g}}_i$ defines a common perpendicular between $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{n}}_i$.

The dual version of (4.1) expresses a linear dependence of type (3.3), if $(\underline{s}_2\underline{s}_3, \underline{s}_3\underline{s}_1, \underline{s}_1\underline{s}_2) \neq (\underline{0}, \underline{0}, \underline{0})$. It is necessary and sufficient that at least one coefficient $\underline{s}_i$ is no zero divisor. This means that the lines $\underline{\mathfrak{n}}_i$ and $\underline{\mathfrak{g}}_i$ are neither parallel nor antiparallel. We meet the more restrictive condition (3.4) with real parts $(s_2 s_3, s_3 s_1, s_1 s_2) \neq (0, 0, 0)$, if and only if at most one $\underline{s}_i$ is a zero divisor. In this way we obtain

Theorem 1: (J. Hjelmslev and F. Morley, 1898)
At a skew hexagon with only right angles the common perpendiculars $\underline{\mathfrak{h}}_1, \underline{\mathfrak{h}}_2, \underline{\mathfrak{h}}_3$ of opposite sides intersect a common line $\underline{\mathfrak{n}}$ perpendicularily, provided atmost one pair of opposite sides is parallel[5]).

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[5]) We exclude the case of coincident opposite sides $\underline{\mathfrak{g}}_i, \underline{\mathfrak{n}}_i$ by the implied assumption that no side length in our hexagon may vanish.

5 Transference of DIXON's first mechanism

Let c, d be two perpendicular great circles on the unit sphere $\mathbf{S}^2$. Then for any three or more points $\mathfrak{g}_1, \ldots$ of c and at least three other points $\mathfrak{h}_1, \ldots$ of d there is a one-parameter transformation of all $\mathfrak{g}_i \in c$ and $\mathfrak{h}_j \in d$ that preserves all distances $\mathfrak{g}_i\mathfrak{h}_j$. We get a flexing spherical framework (see Figure 2) when each of these distances is materialized by a circular rod with pivots at the knots $\mathfrak{g}_1, \ldots, \mathfrak{h}_1, \ldots$. This is the spherical version of A.C. DIXON's first planar mechanism in [6]. Its finite mobility has already been pointed out by W. WUNDERLICH in [10]. We prove it anew in the following way:

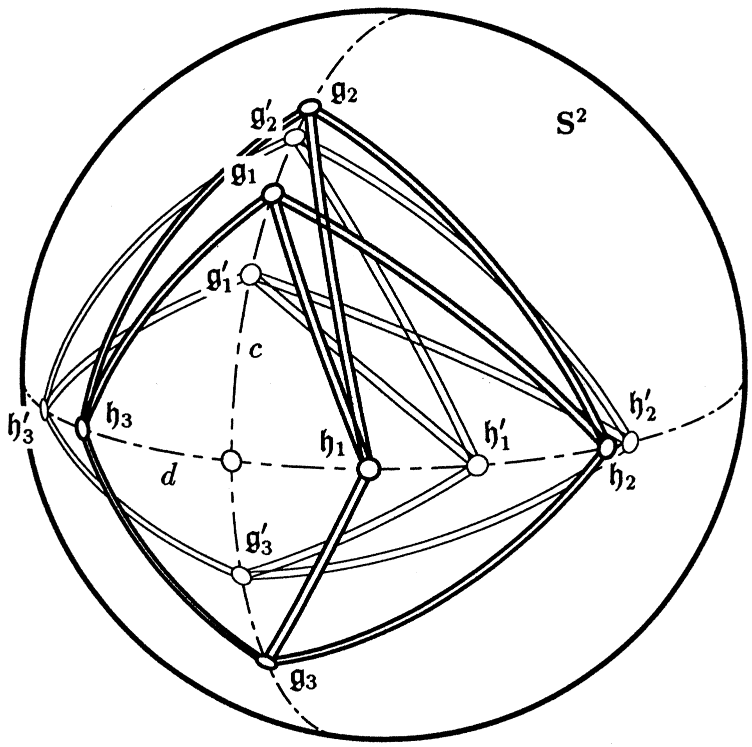

Figure 2: DIXON's first spherical mechanism

Let $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$ be a Cartesian base with

$$\mathfrak{g}_i = \cos\alpha_i\,\mathfrak{e}_1 + \sin\alpha_i\,\mathfrak{e}_2\,, \quad \mathfrak{h}_j = \cos\beta_j\,\mathfrak{e}_1 + \sin\beta_j\,\mathfrak{e}_3\,.$$

Then all the dot products

$$c_{ij} := \cos \sphericalangle\, \mathfrak{g}_i\mathfrak{h}_j = \cos\alpha_i \cos\beta_j$$

are preserved, if we replace the vectors $\mathfrak{g}_i$ and $\mathfrak{h}_j$ by

$$\mathfrak{g}'_i := t\cos\alpha_i\,\mathfrak{e}_1 + \sqrt{1 - t^2\cos^2\alpha_i}\;\mathfrak{e}_2\,, \quad \mathfrak{h}'_j := \frac{1}{t}\,\cos\beta_j\,\mathfrak{e}_1 + \sqrt{1 - \frac{1}{t^2}\,\cos^2\beta_j}\;\mathfrak{e}_3$$

for a real parameter t sufficiently near to 1. The condition $\mathfrak{g}_i \neq \mathfrak{h}_j$ for all (i, j) guaranties an open intervall for t such that the radicands remain nonnegative.

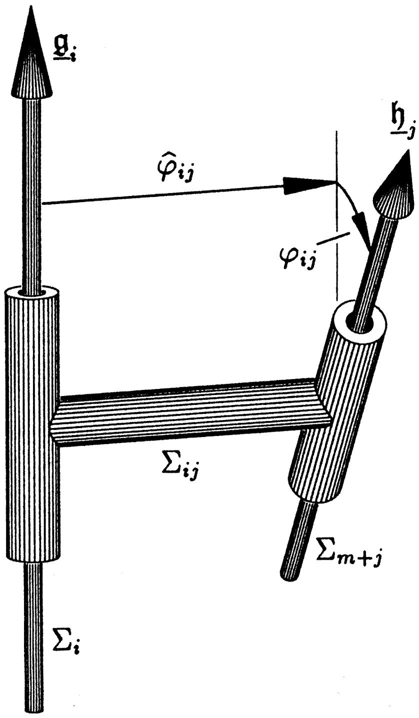

Figure 3: Link preserving the dual angle $\underline{\varphi}_{ij}$ between $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{h}}_j$

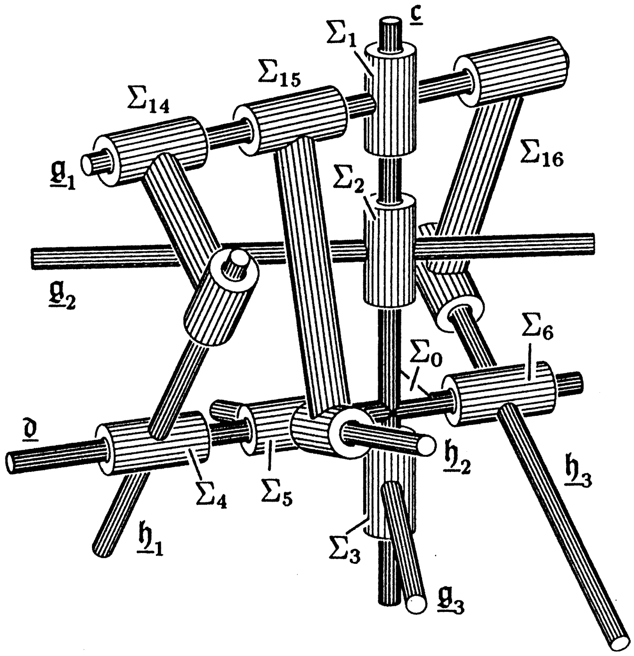

Figure 4: Parts of the dual version of DIXON's first mechanism, degree of freedom $dof = 2$ (Theorem 2)

In order to obtain the dual extension of this framework we replace the knots on the great circle c by dual unit vectors $\underline{\mathfrak{g}}_1, \ldots$ which have a vanishing dot product $\underline{\mathfrak{g}}_i \cdot \underline{\mathfrak{c}} = \underline{0}$ with a fixed vector $\underline{\mathfrak{c}}$. Therefore each directed line $\underline{\mathfrak{g}}_i$ intersects a fixed line $\underline{\mathfrak{c}}$ perpendicularily. We specify a second axis $\underline{\mathfrak{d}}$ with $\underline{\mathfrak{c}} \cdot \underline{\mathfrak{d}} = 0$ and a second set $\underline{\mathfrak{h}}_1, \ldots$ of directed lines that intersect $\underline{\mathfrak{d}}$ perpendicularily.

For the transference of the flexing spherical framework into $\mathbb{D}$ we keep the axes $\underline{\mathfrak{c}}$ and $\underline{\mathfrak{d}}$ fixed. All pairs $(\underline{\mathfrak{g}}_i, \underline{\mathfrak{c}})$ and $(\underline{\mathfrak{h}}_j, \underline{\mathfrak{d}})$ were coupled by cylindrical joints such that independently from each other all $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{h}}_j$ may perform screwing motions about $\underline{\mathfrak{c}}$ or $\underline{\mathfrak{d}}$, respectively. Now there is still a two-parametric mobility[6]) when for each pair $(\underline{\mathfrak{g}}_i, \underline{\mathfrak{h}}_j)$ the dual angle $\underline{\varphi}_{ij} = \varphi_{ij} + \varepsilon\widehat{\varphi}_{ij}$ is preserved by an intermediate link (see Figure 3) that has cylindrical joints with $\underline{\mathfrak{g}}_i$ and $\underline{\mathfrak{h}}_j$. Since the orientations of $\underline{\mathfrak{g}}_1, \ldots$ are unessential we can formulate our result in terms of the theory of mechanisms as follows:

[6]) We count the reals in the dual parameter $\underline{t} = t + \varepsilon\widehat{t}$.

Theorem 2: *Let a kinematic chain with* $1+m+n+mn$ *systems* $\Sigma_0, \Sigma_i, \Sigma_{m+j}, \Sigma_{ij}$ *for* $i \in \{1,..,m\}$, $j \in \{1,..,n\}$ *and* $m, n \geq 3$ *be given. All* $m+n+2mn$ *joints are cylindrical: Line* $\underline{\mathfrak{c}} \subset \Sigma_0$ *is the axis of the joints between* Σ_i *and* Σ_0, *line* $\underline{\mathfrak{d}} \subset \Sigma_0$ *plays the same role for* Σ_{m+j} *and* Σ_0. *The axis of the joints between* Σ_{ij} *and* Σ_i (Σ_{m+j}) *is denoted by* $\underline{\mathfrak{g}}_i$ ($\underline{\mathfrak{h}}_j$).
When all pairs of lines ($\underline{\mathfrak{c}}, \underline{\mathfrak{d}}$), ($\underline{\mathfrak{c}}, \underline{\mathfrak{g}}_i$), ($\underline{\mathfrak{d}}, \underline{\mathfrak{h}}_j$) *are perpendicularily intersecting, then this chain* (see Figure 4) *has twoparametric mobility.*

As a particular case we may specify the lines $\underline{\mathfrak{g}}_1, \dots$ and $\underline{\mathfrak{h}}_1, \dots$ as generators of different type at an orthogonal hyperbolic paraboloid Γ with vertex generators $\underline{\mathfrak{c}}$ and $\underline{\mathfrak{d}}$. This gives vanishing mutual distances $\widehat{\varphi}_{ij} = 0$. Therefore the flexing structure permits on the one hand similitudes. On the other hand there is still a mobility of degree 1 that interchanges the generators of the fixed hyperboloid Γ.

6 Transference of DIXON's second mechanism

DIXON's second mechanism on $\mathbf{S}^2$ consists of 16 rods $\mathfrak{g}_i\mathfrak{h}_j$, $i, j \in \{1,..,4\}$. In each position (see Figure 5) the quadrangles $\mathfrak{g}_1, .., \mathfrak{g}_4$ and $\mathfrak{h}_1, .., \mathfrak{h}_4$ of knots are symmetric with respect to two perpendicular great circles l_1, l_2.

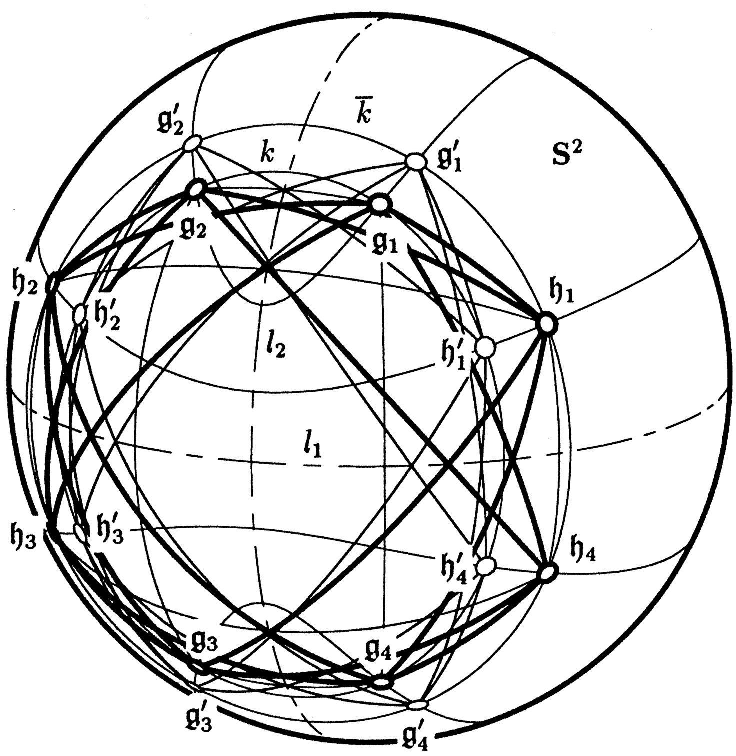

Figure 5: DIXON's second spherical mechanism

The shortest proof for its mobility[7]) is based on

Theorem 3: (IVORY's theorem, spherical version)
Let $k, \overline{k}$ be two confocal spherical conics for which there is an affine transformation α with $k \mapsto \overline{k}$ that fixes all three common axes of symmetry.
Then for each pair of points $\mathfrak{x}_1, \mathfrak{x}_2 \in k$ with α-images $\overline{\mathfrak{x}}_1, \overline{\mathfrak{x}}_2$[8]) the spherical distances $\mathfrak{x}_1 \overline{\mathfrak{x}}_2$ and $\overline{\mathfrak{x}}_1 \mathfrak{x}_1$ are equal (Figure 6).

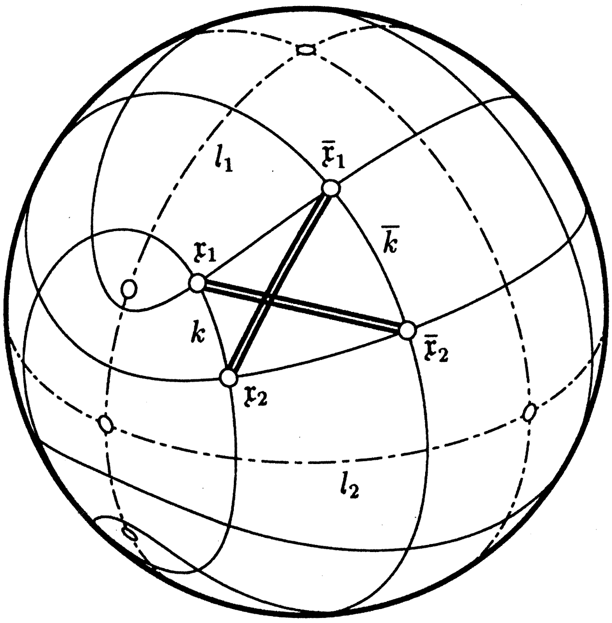

Figure 6: IVORY's theorem, spherical version

Proof:[9]) The spherical conics $k, \overline{k}$ can be supposed as curves of intersection between the unit sphere $\mathbf{S}^2$ and the confocal cones

$$\Phi\colon\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \text{and} \quad \overline{\Phi}\colon\ \frac{\overline{x}^2}{a^2 - k^2} + \frac{\overline{y}^2}{b^2 - k^2} - \frac{\overline{z}^2}{c^2 + k^2} = 0$$

with $a > b > 0, c \neq 0$. The existence of $\alpha\colon k \mapsto \overline{k}$ implies $b > k > 0$. Due to the homogeneity of a, b, c we may assume $k = 1$. Then the substitutions

$$a^2 := \frac{1}{1 - \lambda^2}, \quad b^2 := \frac{1}{1 - \mu^2}, \quad c^2 := \frac{-1}{1 - \nu^2}$$

[7]) References for the planar version are given in [10]. Another approach via multiply decomposable motions is presented in [9]. A true spatial generalization of DIXON's framework with 16 knots and 64 rods can be found in [7], Satz 3,1.

[8]) Each pair of points corresponding under the restriction $\alpha|_k$ is located on the same confocal conic, which intersects k and $\overline{k}$ perpendicularily.

[9]) Compare also [4], [1] or [7].

yield $\nu^2 > 1 > \lambda^2 > \mu^2 > 0$, and we obtain the representations

$$\begin{aligned}
\Phi:&\quad (1-\lambda^2)x^2 + (1-\mu^2)y^2 + (1-\nu^2)z^2 = 0\,,\\
\overline{\Phi}:&\quad \frac{1-\lambda^2}{\lambda^2}\overline{x}^2 + \frac{1-\mu^2}{\mu^2}\overline{y}^2 + \frac{1-\nu^2}{\nu^2}\overline{z}^2 = 0\,,\\
\alpha:&\quad (x,y,z) \mapsto (\overline{x},\overline{y},\overline{z}) = (\lambda x, \mu y, \nu z)\,.
\end{aligned}$$

For $k = \Phi \cap \mathbf{S}^2$ and $\overline{k} = \overline{\Phi} \cap \mathbf{S}^2$ we get

$$\mathfrak{x} \in k \iff \left\{ \begin{array}{l} \lambda^2 x^2 + \mu^2 y^2 + \nu^2 z^2 \\ = x^2 + y^2 + z^2 = 1 \end{array} \right. \iff \left\{ \begin{array}{l} \overline{x}^2 + \overline{y}^2 + \overline{z}^2 = \\ \dfrac{\overline{x}^2}{\lambda^2} + \dfrac{\overline{y}^2}{\mu^2} + \dfrac{\overline{z}^2}{\nu^2} = 1 \end{array} \right. \iff \overline{\mathfrak{x}} \in \overline{k}$$

which immediately reveals the the stated equality (Figure 6) as

$$\mathfrak{x}_1 \cdot \overline{\mathfrak{x}}_2 = \lambda x_1 x_2 + \mu y_1 y_2 + \nu z_1 z_2 = \overline{\mathfrak{x}}_1 \cdot \mathfrak{x}_2\,. \quad \square$$

In order to prove the mobility of DIXON's second spherical framework we argue as follows: We start with two quadrangles $\mathfrak{g}_1, .., \mathfrak{g}_4$ and $\mathfrak{h}_1, .., \mathfrak{h}_4$, both symmetric with respect to the orthogonal great circles l_1 and l_2. There is a one-parametric set of spherical conics k passing through $\mathfrak{g}_1, .., \mathfrak{g}_4$. Each k permits the reflections in l_1 and l_2. Therefore for each k there is a confocal spherical conic $\overline{k}$ passing through $\mathfrak{h}_1, .., \mathfrak{h}_4$ such that there is an affine transformation $\alpha\colon\ k \mapsto \overline{k}$ that fixes the common axes of symmetry. For each choice of k we denote the α-images of $\mathfrak{g}_1, .., \mathfrak{g}_4$ by $\mathfrak{g}_1', .., \mathfrak{g}_4'$ and the inverse images of $\mathfrak{h}_1, .., \mathfrak{h}_4$ by $\mathfrak{h}_1', .., \mathfrak{h}_4'$ (see Figure 5). This gives rise to a one-parametric set of points $\mathfrak{g}_1', \dots, \mathfrak{h}_4'$ with constant distances

$$\mathfrak{g}_i' \cdot \mathfrak{h}_j' = \mathfrak{g}_i \cdot \mathfrak{h}_j \quad \textit{for all} \quad i,j \in \{1, .., 4\}$$

due to IVORY's theorem. $\square$

STUDY's principle of transference leads to a mobile spatial structure consisting of 8 lines $\underline{\mathfrak{g}}_1, \dots, \underline{\mathfrak{h}}_4$ and 16 connecting links for all pairs $(\underline{\mathfrak{g}}_i, \underline{\mathfrak{h}}_j)$ as depicted in Figure 3. What is the dual analogon of the reflection of $\mathbf{S}^2$ in the plane spanned by a great circle l?

Let $\mathfrak{m}$ denote a spherical centre of l. Then the reflection in l can be expressed as

$$\mathfrak{g} \mapsto \mathfrak{g}^* = \mathfrak{g} - 2(\mathfrak{g}\cdot\mathfrak{m})\mathfrak{m} = \mathfrak{m} \circ \mathfrak{g} \circ \mathfrak{m}$$

using the quaternion product (cf. [3])

$$\mathfrak{u} \circ \mathfrak{v} := -(\mathfrak{u}\cdot\mathfrak{v}) + (\mathfrak{u} \times \mathfrak{v})$$

of vectors in the skew field of quaternions. Therefore the dual extension of the reflection in a plane reads

$$\underline{\mathfrak{g}} \mapsto \underline{\mathfrak{g}}^* = -(-\underline{\mathfrak{Q}} \circ \underline{\mathfrak{g}} \circ \underline{\mathfrak{Q}}) \quad \textit{for} \quad \underline{\mathfrak{Q}} := \underline{\cos}\left(\frac{\pi}{2} + \varepsilon 0\right) + \underline{\sin}\left(\frac{\pi}{2} + \varepsilon 0\right)\underline{\mathfrak{m}} = \underline{\mathfrak{m}}\,.$$

This means that the directed line $\underline{\mathfrak{g}}$ is carried into $\underline{\mathfrak{g}}^*$ by a reflection in $\underline{\mathfrak{m}}$ combined with the change of orientation.

STUDY's principle transfers each quadrangle of knots of DIXON's second mechanism into four directed lines that can be generated – up to their orientation – by iterated halfturns about two perpendicularly intersecting axes $\underline{\mathfrak{m}}_1, \underline{\mathfrak{m}}_2$.

On the other hand the dual extensions of spherical conics are quadratic line congruences which permit the halfturns about $\underline{\mathfrak{m}}_1$ and $\underline{\mathfrak{m}}_2$. The proofs for the dual versions of IVORY's theorem and for DIXON's mechanism work analogously. As finally the orientations of $\underline{\mathfrak{g}}_1, \ldots, \underline{\mathfrak{h}}_4$ turn out to be unessential, we obtain

Theorem 4: *Let* $\underline{\mathfrak{g}}_1, \ldots, \underline{\mathfrak{g}}_4$ *and* $\underline{\mathfrak{h}}_1, \ldots, \underline{\mathfrak{h}}_4$ *be quadrupels of lines, both symmetric with respect to two perpendicularily intersecting lines* $\underline{\mathfrak{m}}_1, \underline{\mathfrak{m}}_2$. *When for each* $i, j \in \{1, \ldots, 4\}$ *the lines* $\underline{\mathfrak{g}}_i$ *and* $\underline{\mathfrak{h}}_j$ *are connected by a link as depicted in* Figure 3, *then the structure has still the degree* 2 *of mobility.*

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Hellmuth Stachel

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