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Coordinates - A survey on higher geometry

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Abstract

Under the name "Higher Geometry" usually those different geometries are summarized which in the sense of F. Klein's Erlangen program (1872) are isomorphic to subgeometries of projective geometry. In the following we give a brief survey on such geometries like multi-dimensional projective, affine and Euclidean geometry, the geometry of lines and the geometry of oriented spheres in the three-dimensional case. It is a goal of this paper to demonstrate both the elegance of the classical analytical treatment and its applicability for various tasks, e.g. in the field of CAGD. The latter however has been reduced to the presentation of references only. © 1997 Elsevier Science B.V.

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1. Introduction

Coordinates are used everywhere in Computer-graphics in order to bridge the gap between geometric objects and their graphic representation. Coordinates must describe the objects uniquely. Conversely neither each geometric object needs to define its coordinates uniquely nor each possible choice of coordinates needs to correspond to any object.

Usually coordinates are seen as numbers selected from any field or at least from any algebra $\mathbb{A}$. But in view of the determination of a solid either by its CSG-tree or by its boundary representation, it is obvious that much more complex structures than numbers can be necessary for describing geometric objects precisely. However, in the following we restrict ourselves in the different geometries to basic objects like points, lines, spheres, affine or collinear transformations. And here n-tuples of real numbers or of elements of any algebra $\mathbb{A}$ over $\mathbb{R}$ will be sufficient.

This means finally that all coordinatizations γ under consideration are based on an injective mapping of a set S of geometric objects s into the set $\mathbb{A}^n$ of n-tuples over $\mathbb{A}$, to say

$$\gamma : S \to \mathbb{A}^n, \quad s \mapsto (x_1, \dots, x_n).$$

In the standard book [16] no clear definition of "Higher Geometry" is included, but the following sections are certainly part of it (see also the preface in [5]). After a brief survey on the basic elements and the automorphisms of n-dimensional affine, projective and Euclidean spaces we present some selected topics. We introduce a type of redundant coordinates in the Euclidean space $\mathbf{E}^n$, the kinematic mapping of spherical kinematics in $\mathbf{E}^3$, the geometry of lines in the projective space $\mathbf{P}^3$ together with its Plücker representation and also Study's concept of Euclidean line geometry together with the representation of spatial motions in $\mathbf{E}^3$ by dual quaternions. We conclude with

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an introduction into the group of Lie's sphere transformations in $\mathbf{E}^3$ enclosing the subgroups of Möbius and Laguerre. In each section we also present some applications, e.g. in the field of Computer Aided Geometric Design.

2. The affine space $\mathbf{A}^n$

The points of the *real n-dimensional affine space* $\mathbf{A}^n$ can be identified with the vectors of $\mathbb{R}^n$ (compare the definition in [1, 2.1.1]). For any $n+1$ *affinely independent* points $c, d_1, \ldots, d_n$ [2] there is an *affine coordinate system* $\mathcal{A}(c; d_1, \ldots, d_n)$ defined by

$$\gamma_{\mathcal{A}} : \mathbf{A}^n \to \mathbb{R}^n,$$

$$x = c + \sum_{i=1}^{n} x_i(d_i - c) \mapsto (x_1, \ldots, x_n)_{\mathcal{A}}.$$

Linear subspaces $L \subset \mathbf{A}^n$ can be represented in the form $L = a + V$, where V is a subspace of the vectorspace $\mathbb{R}^n$. This implies

$$x, y \in L = a + V \implies (x - y) \in V \quad \text{and}$$

$$z = rx + (1-r)y \in L \quad \text{for all } r \in \mathbb{R}.$$

We define the *dimension* $\dim L := \dim V$. Two linear spaces L_1, L_2 with $L_i = a_i + V_i$ are defined *parallel*, if $V_1 \subset V_2$ or $V_2 \subset V_1$.

A mapping $\alpha : L \to L'$ between two linear spaces L, L' is called *affine transformation*, if for all $x, y, z \in L$ [3]

$$z = rx + (1-r)y \implies$$

$$\alpha(z) = r\alpha(x) + (1-r)\alpha(y).$$

As a consequence the restriction of α to any linear subspace of L is again affine, and the composite $\alpha_2 \circ \alpha_1$ of any two affine transformations is affine too.

It can easily be proved that any affine transformation $\alpha : a + V \to a' + V'$ induces a linear mapping $l : V \to V'$ such that

$$\alpha(x) - \alpha(y) = l(x - y) \quad \text{for all } x, y \in L. \qquad (1)$$

[2] *Affinely independent* means that the difference vectors $\{d_1 - c, \ldots, d_n - c\}$ are linearly independent.

[3] A weaker definition can be found in [9, p. 8].

Hence for all pairs of coordinate systems $\mathcal{A} = (c; d_1, \ldots, d_r)$ in L and $\mathcal{A}' = (c'; d'_1, \ldots, d'_s)$ in L' the affine mapping $\alpha : L \to L'$ can be represented as

$$x = (x_1, \ldots, x_r)_{\mathcal{A}} \mapsto \alpha(x) = (x'_1, \ldots, x'_s)_{\mathcal{A}'},$$

where

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_s \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix} + \begin{pmatrix} a_{ij} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}. \qquad (2)$$

Therefore the set $\mathrm{A}(L, L')$ of affine mappings $\alpha : L \to L'$ can be coordinatized by

$$\gamma_{\mathcal{A},\mathcal{A}'} : \mathrm{A}(L, L') \to \mathbb{R}^s \times \mathbb{R}^{r \cdot s},$$

$$\alpha \mapsto (b_1, \ldots, b_s, a_{11}, \ldots, a_{sr}).$$

This can be seen as a mapping of $\mathrm{A}(L, L')$ onto an $s(r+1)$-dimensional affine space.

Due to standard results from Linear Algebra the *defect* of α

$$\operatorname{def} \alpha := \operatorname{def} l = \dim L - \dim \alpha(L)$$

equals the dimension of the *fibres* of α. Each fibre consists of all those points in L that share the image. Each two fibres of α are parallel.

An affine mapping $\pi : \mathbf{A}^n \to \mathbf{A}^n$ is called *parallel projection*, if π is idempotent, i.e. $\pi \circ \pi = \pi$. In this case the image $\pi(x)$ of each point x is obtained by intersecting the fibre passing through x with the image space $\pi(\mathbf{A}^n)$.

3. The projective space $\mathbf{P}^n$

Originally the concepts of Projective Geometry were developed in order to explain the distortions that arise from central projections (see e.g. [13]). We obtain the *real n-dimensional projective space* $\mathbf{P}^n$ as follows:

We extend the affine space $\mathbf{A}^n$ by adding *improper* points of intersection between parallel lines. For this purpose we replace the affine coordinates $(x_1, \ldots, x_n)$ by homogeneous coordinates $(1 : x_1 : \ldots : x_n)$ and we add all $(n+1)$-tuples $(0 : x_1 : \ldots : x_n)$ up to the zero vector o. So finally the points of $\mathbf{P}^n$ can be identified with the one-dimensional subspaces of an $(n+1)$-dimensional real vector space V^{n+1}. We denote the points of $\mathbf{P}^n$ by $\mathbb{R}x$ for any $x \in V^{n+1} \setminus \{o\}$.

Any $n+1$ *projectively independent* points $\mathbb{R}a_0, \ldots, \mathbb{R}a_n$ [4] together with a *unit point* $\mathbb{R}e$ define a *projective coordinate system* $\mathcal{P}(\mathbb{R}a_0, \ldots, \mathbb{R}a_n; \mathbb{R}e)$ by

$$\gamma_{\mathcal{P}} : \mathbf{P}^n \to \mathbb{R}^{n+1} \setminus \{o\} / \mathbb{R} \setminus \{0\},$$

$$\mathbb{R}x = \mathbb{R} \sum_{i=0}^{n} x_i a_i \mapsto (x_0 : \ldots : x_n)_{\mathcal{P}},$$

provided, point $\mathbb{R}e$ is independent from each n-tuple of *base points* $\mathbb{R}a_i$ and the vectors $a_0, \ldots, a_n$ representing the base points meet the norming condition

$$e = a_0 + \cdots + a_n.$$

Projective subspaces P of $\mathbf{P}^n$ are obtained by factorizing the at least one-dimensional subspaces V of V^{n+1}. We define $\dim P := \dim V - 1$. In this sense the intersection and the sum of any two sub-vectorspaces can be interpreted as the *intersection* and the *join* of the corresponding projective subspaces. This reveals that the structure of projective spaces is even simpler than that of affine spaces.

It turns out that in $\mathbf{P}^n$ the *hyperplanes*, i.e. the $(n-1)$-dimensional subspaces, form again an n-dimensional projective space $\widehat{\mathbf{P}}^n$, the *dual space*, which is isomorphic to $\mathbf{P}^n$. This implies that each valid result in $\mathbf{P}^n$ holds in the dual version too, where the terms "point" and "hyperplane" as well as "intersection" and "join" are exchanged. The dual of a k-dimensional subspace, seen as a point set, is the k-dimensional bundle of hyperplanes passing through an $(n-k)$-dimensional space.

A mapping κ from a projective space P into a projective space P' is called *projective transformation* or *collineation*, if it is induced by a linear mapping $l : V \to V'$ between the underlying vector spaces. *Regular* collineations can be characterized as bijections that preserve collinearity. *Singular* collineations [5] are never global as the kernel of l defines a projective subspace called *center* Z, consisting of points for which no image is defined under κ. Fibres of κ are projective spaces of dimension $\operatorname{def}\kappa := \operatorname{def} l$ through the center Z. For any four collinear points $\mathbb{R}p, \mathbb{R}q, \mathbb{R}r, \mathbb{R}s$ meeting $r = r_1 p + r_2 q$ and $s = s_1 p + s_2 q$ the *cross ratio*

$$(\mathbb{R}p, \mathbb{R}q, \mathbb{R}r, \mathbb{R}s) := \frac{r_2 s_1}{r_1 s_2}$$

is preserved under collineations, provided the four image points are defined and pairwise different.

Coordinate systems $\mathcal{P}(\mathbb{R}a_0, \ldots, \mathbb{R}a_r; \mathbb{R}e)$ in P and $\mathcal{P}'(\mathbb{R}a'_0, \ldots, \mathbb{R}a'_s; \mathbb{R}e')$ in P' enable to represent the collineation $\kappa : P \rightarrowtail P'$ as

$$\kappa : (x_0 : \ldots : x_r)_{\mathcal{P}} \mapsto (x'_0 : \ldots : x'_s)_{\mathcal{P}'},$$

where

$$\begin{pmatrix} x'_0 \\ \vdots \\ x'_s \end{pmatrix} = \begin{pmatrix} a_{00} & \ldots & a_{0r} \\ \vdots & \ddots & \vdots \\ a_{s1} & \ldots & a_{sr} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_r \end{pmatrix}. \qquad (3)$$

The matrix (a_{ij}) is unique up to a nonvanishing constant factor. Hence there is a bijection

$$\gamma_{\mathcal{P},\mathcal{P}'} : \mathrm{K}(P, P') \to \mathbf{P}^{(r+1)(s+1)-1},$$

$$\kappa \mapsto (a_{00} : \ldots : a_{sr})$$

of the set $\mathrm{K}(P, P')$ of projective transformations $\kappa : P \rightarrowtail P'$ onto a $[(r+1)(s+1)-1]$-dimensional projective space. [6]

Idempotent collineations $\kappa : \mathbf{P}^n \rightarrowtail \mathbf{P}^n$ are called *central projections*. Here again the image $\kappa(\mathbb{R}x)$ is the point of intersection between the image space $\kappa(\mathbf{P}^n)$ and the fibre of κ passing through $\mathbb{R}x$.

The regular collineations $\kappa : \mathbf{P}^n \to \mathbf{P}^n$ form a group $\mathrm{PGL}(n, \mathbb{R})$. Many important geometric groups are isomorphic to subgroups of $\mathrm{PGL}(n, \mathbb{R})$. In the sequel this will be shown in three-dimensional spaces for the fundamental groups of projective line geometry and of different sphere geometries.

Collineations that preserve a regular quadric Q_h^{n-1} of signature $n-1$ in $\mathbf{P}^n$ form a subgroup of $\mathrm{PGL}(n, \mathbb{R})$ which is isomorphic to the group of motions in the *hyperbolic space* $\mathbf{H}^n$ (see [10] or [1, 19.4], and note recent applications of hyperbolic geometry in [24]).

In the case $n = 2$ each collineation which preserves the conic $c : x_0 x_2 - x_1^2 = 0$ can be represented according to Eq. (3) by a matrix

$$(a_{ij}) = \begin{pmatrix} b_{00}^2 & 2b_{00}b_{01} & b_{01}^2 \\ b_{01}b_{10} & b_{00}b_{11} + b_{01}b_{10} & b_{01}b_{11} \\ b_{10}^2 & 2b_{10}b_{11} & b_{11}^2 \end{pmatrix}$$

[4] *Projectively independent* in $\mathbf{P}^n$ means that the vectors $\{a_0, \ldots, a_n\}$ are linearly independent in V^{n+1}.

[5] For a geometric definition see e.g. [4].

[6] In [5] this space is discussed as the span of the Segre manifold $P \times P'$ consisting of all pairs of points. In [19] this space is used in the case $r = s$ for obtaining results in projective kinematics.

obeying $b_{00}b_{11} - b_{01}b_{10} \neq 0$. This proves that the group of hyperbolic motions in the plane is isomorphic to $\mathrm{PGL}(1,\mathbb{R})$.

The group of collineations $\mathbf{P}^3 \to \mathbf{P}^3$ preserving each regulus of a ruled quadric Q_r^2 turns out to be isomorphic to the direct product $\mathrm{PGL}(1,\mathbb{R}) \otimes \mathrm{PGL}(1,\mathbb{R})$. In the case of a quadric Q_h^2 with signature 2, the corresponding subgroup of $\mathrm{PGL}(3,\mathbb{R})$ is isomorphic to the group of projective collineations $\mathrm{PGL}(1,\mathbb{C})$. This can be derived from the fact that under stereographic projection the direct automorphisms of Q_h^2 are mapped onto direct Möbius transformations in the plane (see last section).

4. The Euclidean space $\mathbf{E}^n$

We now define on the vectorspace $\mathbb{R}^n$ related to the affine space $\mathbf{A}^n$ a positive definite symmetric bilinear form

$$b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (\boldsymbol{u},\boldsymbol{v}) \mapsto \boldsymbol{u}\cdot\boldsymbol{v}.$$

Then $\mathbf{A}^n$ alters into the *n-dimensional Euclidean space* $\mathbf{E}^n$.

For any two points $\boldsymbol{a},\boldsymbol{b} \in \mathbf{E}^n$ there is a *distance* defined by

$$\overline{\boldsymbol{ab}} = \|\boldsymbol{a}-\boldsymbol{b}\| := \sqrt{(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})}.$$

Two linear spaces $\boldsymbol{a}+\boldsymbol{U}$ and $\boldsymbol{b}+\boldsymbol{V}$ are called *totally orthogonal*, if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$ for all $\boldsymbol{u} \in \boldsymbol{U}$ and $\boldsymbol{v} \in \boldsymbol{V}$.[7]

An affine coordinate system $\mathcal{A}(\boldsymbol{c};\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)$ is called *cartesian*, if

$$(\boldsymbol{d}_i - \boldsymbol{c})\cdot(\boldsymbol{d}_j - \boldsymbol{c}) = \delta_{ij} \quad \forall i,j \in \{1,\ldots,n\}.$$

In this case the line segments $\boldsymbol{cd}_1,\ldots,\boldsymbol{cd}_n$ are of unit length and pairwise orthogonal.

An affine transformation $\alpha : \boldsymbol{L} \to \boldsymbol{L}'$ is called *similarity*, if each two orthogonal lines in $\boldsymbol{L}$ are mapped onto orthogonal lines. This implies $\operatorname{def}\alpha = 0$. It turns out that the affine transformation $\alpha : \boldsymbol{a}+\boldsymbol{V} \to \boldsymbol{a}'+\boldsymbol{V}'$ is a similarity if and only, if the induced linear mapping $l : \boldsymbol{V} \to \boldsymbol{V}'$ obeys

$$l(\boldsymbol{u})\cdot l(\boldsymbol{v}) = c_\alpha(\boldsymbol{u}\cdot\boldsymbol{v}) \quad \text{for all } \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V} \qquad (4)$$

[7] In the case of two lines we say *orthogonal* only.

with any positive constant c_α. Hence for each pair of lines $\boldsymbol{a}+\mathbb{R}\boldsymbol{u}$, $\boldsymbol{b}+\mathbb{R}\boldsymbol{v}$ the enclosed angle ϕ is preserved under similarities, as ϕ is defined by

$$\cos\phi := \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}.$$

There is also a characterization of similarities based on distances only: A mapping $\sigma : \boldsymbol{L} \to \boldsymbol{L}'$ is a similarity if and only, if there is a constant *dilatation factor* $k > 0$ such that

$$\|\sigma(\boldsymbol{x}) - \sigma(\boldsymbol{y})\| = k\,\|\boldsymbol{x}-\boldsymbol{y}\|$$

for all points $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{L}$. A length-preserving mapping ($k = 1$) is called *isometry*. When cartesian coordinate systems $\mathcal{C}$ and $\mathcal{C}'$ are used in Eq. (2), then the affine transformation α is an isometry if and only, if

$$(a_{ij})^{\mathrm{T}} = (a_{ij})^{-1},$$

i.e. (a_{ij}) is an *orthogonal matrix*. Just in this case the matrix (a_{ij}) has *orthonormal* row vectors, to say, the row vectors are pairwise orthogonal unit vectors.

A parallel projection $\pi : \mathbf{E}^n \to \mathbf{E}^m$ is called *orthogonal projection*, if the fibres of π are totally orthogonal to the image space and $m < n$. Based on cartesian coordinate systems $\mathcal{C}$ in $\mathbf{E}^n$ and $\mathcal{C}'$ in $\pi(\mathbf{E}^n)$ the $(m \times n)$-matrix (a_{ij}) of an orthogonal projection according to Eq. (2) is characterized by m orthonormal row vectors.

A more general question reads: For which affine transformations α with $\operatorname{def}\alpha > 0$ each object in $\mathbf{E}^n$ has an image similar to that under a parallel projection? This is related to the classical Pohlke theorem. For the n-dimensional version see e.g. [20] and the references given there. The characterization of such α depends on the singular values of the corresponding matrix (a_{ij}).

In the projectively completed $\mathbf{E}^n$ there is a similar criterion for those singular collinear transformations κ which produce images congruent to that under a proper central projection (see [12]). For a sufficiently high defect of κ it can happen that even a continuum of different central projections produces congruent images [21].

5. Redundant coordinates in $\mathbf{E}^n$

Let $\mathcal{C}(\boldsymbol{c};\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)$ be a cartesian coordinate system in $\mathbf{E}^n$. Then the vectors

$$e_1 := d_1 - c,\ \ldots,\ e_n := d_n - c$$

form an orthonormal basis of the underlying vectorspace $\mathbb{R}^n$. As a consequence the coordinates $(x_1, \ldots, x_n)_C$ of each point $x \in \mathbf{E}^n$ obey

$$x_i = (x - c) \cdot e_i \quad \text{for } i = 1, \ldots, n.$$

Hence for each vector $v := x - c \in \mathbb{R}^n$ we get

$$v = \sum_{i=1}^{n} (v \cdot e_i) e_i = \sum_{i=1}^{n} v(e_i^{\mathrm{T}} e_i) = v \sum_{i=1}^{n} [e_i e_i]. \qquad (5)$$

Here we use the matrix notation: e_i is seen as row vector; the superscript $^{\mathrm{T}}$ indicates the transposed matrix. The matrix $[vw] := v^{\mathrm{T}} w$ is called *dyadic product* of vectors v, w.

We conclude from Eq. (5) that the sum of dyadic squares of the orthonormal basis equals the identity matrix, i.e.

$$\sum_{i=1}^{n} [e_i e_i] = I_n.$$

It is surprising that there are more general systems of vectors $f_1, \ldots, f_m$ with $m > n$ that still share many properties of an orthonormal basis. Such *almost-orthonormal* systems can be characterized by

$$\sum_{j=1}^{m} [f_j f_j] = I_n. \qquad (6)$$

This gives rise to *redundant coordinates* $(x_1, \ldots, x_m)$ for each $x = v + c \in \mathbf{E}^n$ such that

$$x - c = v = \sum_{j=1}^{m} x_j f_j.$$

Beside the *distinguished* solution $x_j := v \cdot f_j$ there is an $(m - n)$-dimensional set of solutions

$$(x_1 + r_1, \ldots, x_m + r_m) \quad \text{with} \sum_{j=1}^{m} r_j f_j = o.$$

Each linear mapping $l : \mathbb{R}^n \to \mathbb{R}^n$ has a *distinguished* matrix representation

$$(a_{jk}) = (f_j \cdot l(f_k)).$$

For selfadjoint linear mappings this distinguished matrix is always symmetric, just as in cartesian coordinates.

A more general *redundant* matrix representation of the same linear mapping l reads

$$(a_{jk} + r_{jk}) \quad \text{with} \sum_{j,k=1}^{m} r_{jk} [f_j f_k] = O_n,$$

where O_n denotes the zero-matrix. However, the multiplication with this matrix $(a_{jk} + r_{jk})$ transforms only the distinguished coordinates of v into redundant coordinates of $l(v)$. In many cases this matrix can even be diagonalized. Further it can be proved that each isometric mapping l obeying $l(u) \cdot l(v) = u \cdot v$ (cf. Eq. (4)) has also a redundant representation in form of an orthogonal matrix.

Systems of particularly normed vectors with rotational symmetry fulfill the condition in Eq. (6). Hence the use of such redundant coordinates is recommended whenever structures with high symmetry are to be described analytically. For further details the reader is referred to [11]. In this paper it is proved that a vector-system is almost-orthonormal if and only, if it coincides with the image of an orthonormal frame in $\mathbf{E}^m$ under orthogonal projection onto an n-dimensional subspace. This means that the distinguished redundant coordinates of any point in $\mathbf{E}^n$ equal the cartesian coordinates of the same point after the space $\mathbf{E}^n$ has been isometrically embedded into the higher-dimensional $\mathbf{E}^m$.

6. Spherical motions in $\mathbf{E}^3$

An isometry β in $\mathbf{E}^n$ is called *spherical*, if there is a point fixed under β. Then the unit sphere $\mathbf{S}^2$ centered at the fixed point c is mapped onto itself. We choose c as origin of our cartesian coordinate system $\mathcal{C}$. Then in the representation of β according to Eq. (2) only the orthogonal matrix (a_{ij}) shows up. Suppose the isometry is *direct*, i.e. orientation preserving, which is equivalent to

$$\det(a_{ij}) = +1.$$

Then β is called spherical *motion*. These motions form a group SO_3. The following quaternion representation of spherical motions in $\mathbf{E}^3$ reveals that there is a one-to-one mapping between SO_3 and the projective space $\mathbf{P}^3$:

The skew field $\mathbb{Q}$ of *quaternions* is a four-dimensional algebra over $\mathbb{R}$ with a basis $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. Each quaternion

$$Q := a + b\,\boldsymbol{i} + c\,\boldsymbol{j} + d\,\boldsymbol{k} =: a + \boldsymbol{q}$$

with $(a, \dots, d) \in \mathbb{R}^4$, is sum of the scalar part a and the vector part $\boldsymbol{q}$. Hence $\mathbb{Q}$ contains $\mathbb{R}$ and the three-dimensional vector space $\mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$ as subspaces. $\mathbb{R}$ is even a subfield.

The product of two quaternions $Q_i = a_i + \boldsymbol{q}_i$, $i = 1, 2$, is defined

$$Q_1 \circ Q_2 = [a_1 a_2 - (\boldsymbol{q}_1 \cdot \boldsymbol{q}_2)] + [a_1 \boldsymbol{q}_2 + a_2 \boldsymbol{q}_1 + (\boldsymbol{q}_1 \times \boldsymbol{q}_2)].$$

Also the dot product and cross product of two vectors ($a_1 = a_2 = 0$) can be expressed in terms of the quaternion product, since

$$\boldsymbol{x} \cdot \boldsymbol{y} = -\tfrac{1}{2}(\boldsymbol{x} \circ \boldsymbol{y} + \boldsymbol{y} \circ \boldsymbol{x}),$$
$$\boldsymbol{x} \times \boldsymbol{y} = \tfrac{1}{2}(\boldsymbol{x} \circ \boldsymbol{y} - \boldsymbol{y} \circ \boldsymbol{x}).$$

The conjugate quaternion of $Q = a + \boldsymbol{q}$ reads

$$\widetilde{Q} := a - b\,\boldsymbol{i} - c\,\boldsymbol{j} - d\,\boldsymbol{k} = a - \boldsymbol{q}.$$

It leads to the quaternion norm

$$\|Q\| := +\sqrt{Q \circ \widetilde{Q}} = +\sqrt{a^2 + b^2 + c^2 + d^2}$$

and – for nonvanishing quaternions – to the inverse

$$Q^{-1} = \frac{1}{\|Q\|^2}\widetilde{Q}.$$

Quaternions with norm 1 are called *unit quaternions.* They can be expressed as

$$E = \cos\alpha + \boldsymbol{e}\sin\alpha \quad \text{with } \|\boldsymbol{e}\| = 1 \tag{7}$$

and they fulfill $E^{-1} = \widetilde{E}$.

The mapping

$$\delta_E : \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{x} \mapsto \boldsymbol{x}' := E \circ \boldsymbol{x} \circ E^{-1} \tag{8}$$

is linear and preserves all dot products, as

$$-2\,\boldsymbol{x}' \cdot \boldsymbol{y}' = \boldsymbol{x}' \circ \boldsymbol{y}' + \boldsymbol{y}' \circ \boldsymbol{x}' = E \circ (\boldsymbol{x} \circ \boldsymbol{y} + \boldsymbol{y} \circ \boldsymbol{x}) \circ E^{-1} = -2\,\boldsymbol{x} \cdot \boldsymbol{y}.$$

Hence all distances are preserved. The isometry δ_E is direct for $\boldsymbol{z} := \boldsymbol{x} \times = \boldsymbol{y}$ is mapped on

$$\boldsymbol{z}' = E \circ \boldsymbol{z} \circ E^{-1} = E \circ (\boldsymbol{x} \times \boldsymbol{y}) \circ E^{-1} = \tfrac{1}{2} E \circ (\boldsymbol{x} \circ \boldsymbol{y} - \boldsymbol{y} \circ \boldsymbol{x}) \circ E^{-1} = \boldsymbol{x}' \times \boldsymbol{y}'.$$

It can be proved that for E according to Eq. (7) the spherical motion δ_E is the rotation about an axis parallel $\boldsymbol{e}$ through the angle 2α. Without changing δ_E we may replace E by any proportional $rE, r \neq 0$. Conversely each spherical motion is a rotation. Hence the mapping

$$\kappa_s : \mathrm{SO}_3 \to \mathbf{P}^3, \quad \delta_E \mapsto X = \mathbb{R}E \tag{9}$$

is bijective and called *kinematic mapping*. Now we can transfer the group structure of SO_3 into $\mathbf{P}^3$. This converts the projective space into a kinematic space with an invariant elliptic metric (see e.g. [3] or [17]).

Let a point $\boldsymbol{e}$ of the unit sphere $\mathbf{S}^2$ be fixed. Then for each point X in $\mathbf{P}^3$ the motion $\kappa_s^{-1}(X)$ carries $\boldsymbol{e}$ into a point $\boldsymbol{e}_X \in \mathbf{S}^2$. This gives rise to a rational mapping of second order

$$\lambda : \mathbf{P}^3 \to \mathbf{S}^2, \quad X \mapsto \boldsymbol{e}_X = \kappa_s^{-1}(X)(\boldsymbol{e}).$$

In homogeneous coordinates we obtain

$$\lambda : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ 2(x_0 x_2 + x_1 x_3) \\ 2(x_2 x_3 - x_0 x_1) \\ x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

This mapping has been used in order to produce rational B-spline-curves and surfaces on $\mathbf{S}^2$ (see e.g. [8]). The fibres of λ build a net of *left-parallel lines* in the elliptic space $\mathbf{P}^3$ (cf. [1, 19.1.4]).

7. Line geometry in $\mathbf{P}^3$ and $\mathbf{E}^3$

The h-dimensional subspaces of $\mathbf{P}^n$ form a *Grassmannian manifold* for $1 < h < n - 1$ (cf. [5]). The simplest example is the set $\mathcal{L}$ of lines in $\mathbf{P}^3$. Each line g can be spanned by two of its points, say by

$$\mathbb{R}\boldsymbol{a} = (a_0 : \ldots : a_3)_{\mathcal{P}} \quad \text{and} \quad \mathbb{R}\boldsymbol{b} = (b_0 : \ldots : b_3)_{\mathcal{P}}.$$

Then the so-called *Plücker coordinates* of g with respect to the projective coordinate system $\mathcal{P}$ are defined

$$g_{ij} := \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}, \quad i, j \in \{0, \ldots, 3\}.$$

These homogeneous coordinates of line g are independent from the choice of the points $\mathbb{R}\boldsymbol{a}, \mathbb{R}\boldsymbol{b} \in g$. However, only six of these coordinates are essential, e.g. those in

$$\mathbb{R}\boldsymbol{g} := (g_{01} : g_{02} : g_{03} : g_{12} : g_{13} : g_{23})_{\mathcal{P}},$$

and these obey the Plücker condition

$$\langle \boldsymbol{g}, \boldsymbol{g} \rangle_l := g_{01}g_{23} - g_{02}g_{13} + g_{03}g_{12} = 0. \qquad (10)$$

This reveals that there is a bijection

$$\lambda_l : \mathcal{L} \to Q_l^4 \subset \mathbf{P}^5 \qquad (11)$$

between the set $\mathcal{L}$ of lines in $\mathbf{P}^3$ and a quadric Q_l^4 of signature 0 in $\mathbf{P}^5$, the Plücker *quadric*.

When line g is seen as the intersection of the two planes $u_0x_0 + \cdots + u_3x_3 = 0$ and $v_0x_0 + \cdots + v_3x_3 = 0$, then the determinants

$$g'_{ij} := \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix}$$

are related to the previous g_{ij} according to

$$(g'_{01} : g'_{02} : g'_{03} : g'_{12} : g'_{13} : g'_{23}) = (g_{23} : -g_{13} : g_{12} : g_{03} : -g_{02} : g_{01}).$$

It turns out that two lines g, h are intersecting if and only, if the corresponding points $\mathbb{R}\boldsymbol{g}$, $\mathbb{R}\boldsymbol{h}$ on the Plücker quadric Q_l^4 are conjugate with respect to Q_l^4, to say

$$\langle \boldsymbol{g}, \boldsymbol{h} \rangle_l = g_{01}h_{23} - g_{02}h_{13} + g_{03}h_{12} + g_{12}h_{03} - g_{13}h_{02} + g_{23}h_{01} = 0. \qquad (12)$$

Due to Eq. (11) the geometry of lines in $\mathbf{P}^3$ is equivalent to the geometry of points at $Q_l^4 \subset \mathbf{P}^5$. There is an isomorphism between the group of collineations and correlations in $\mathbf{P}^3$ and the group of Q_l^4 preserving mappings of $\mathrm{PGL}(5, \mathbb{R})$. The linear one-, two- and three-dimensional sets of lines correspond to sections of Q_l^4 with projective 2-, 3- and 4-dimensional spaces in $\mathbf{P}^5$. Hence the dual space $\widehat{\mathbf{P}}^5$ can be identified with the set of linear line complexes in $\mathbf{P}^3$.

The Plücker quadric Q_l^4 is best for treating the projective geometry of lines. However, the Euclidean geometry of lines deals with angle measures and distances. For this purpose there is a more appropriate representation, which dates back to E. Study (see e.g. [2]):

Each *oriented line* $\overline{g}$ in $\mathbf{E}^3$ is uniquely defined by its normed direction vector $\boldsymbol{g}$ and by any point $\boldsymbol{a}$. It turns out that the *momentum vector* or 2nd Plücker vector

$$\widehat{\boldsymbol{g}} := \boldsymbol{a} \times \boldsymbol{g},$$

which obeys $\boldsymbol{g} \cdot \widehat{\boldsymbol{g}} = 0$, is independent from the choice of point $\boldsymbol{a} \in \overline{g}$. Conversely, a pair $(\boldsymbol{g}, \widehat{\boldsymbol{g}})$ of vectors with $\|\boldsymbol{g}\| = 1$ and $\boldsymbol{g} \cdot \widehat{\boldsymbol{g}} = 0$ determines an unique oriented line $\overline{g}$ since $\boldsymbol{p} := \boldsymbol{g} \times \widehat{\boldsymbol{g}}$ is the pedal point of $\overline{g}$ with respect to the origin. [8]

In the following we use the *dual unit* ε obeying $\varepsilon^2 = 0$ in order to combine the two vectors $\boldsymbol{g}, \widehat{\boldsymbol{g}}$ in a dual vector

$$\underline{\boldsymbol{g}} := \boldsymbol{g} + \varepsilon\widehat{\boldsymbol{g}}.$$

Because of the norming condition

$$\underline{\boldsymbol{g}} \cdot \underline{\boldsymbol{g}} = \boldsymbol{g} \cdot \boldsymbol{g} + 2\varepsilon\boldsymbol{g} \cdot \widehat{\boldsymbol{g}} = \underline{1} = 1 + \varepsilon 0$$

we call $\underline{\boldsymbol{g}}$ a *dual unit vector*. Hence there is a bijection

$$\lambda_{\bar{l}} : \overline{\mathcal{L}} \to \underline{\mathbf{S}}^2, \quad \overline{g} \mapsto \underline{\boldsymbol{g}} \qquad (13)$$

between the oriented lines $\overline{g}$ in $\mathbf{E}^3$ and the points $\underline{\boldsymbol{g}}$ of the *dual unit sphere* $\underline{\mathbf{S}}^2$, which is the extension of the unit sphere $\mathbf{S}^2$ into the ring $\mathbb{D}$ of dual numbers.

Actually this is more than a pure formalism. Many results of the geometry on the sphere $\mathbf{S}^2$ can immediately be extended into the geometry of oriented lines, for example the following two trigonometric formulas: [9]

For any two oriented lines $\underline{\boldsymbol{g}}, \underline{\boldsymbol{h}}$ the *dual angle* $\underline{\phi} := \phi + \varepsilon\widehat{\phi}$ combines the angle ϕ and the shortest distance $\widehat{\phi}$. This gives rise to a geometric interpretation of the *dot product* and *vector product* of $\underline{\boldsymbol{g}}, \underline{\boldsymbol{h}}$: [10]

$$\underline{\boldsymbol{g}} \cdot \underline{\boldsymbol{h}} := (\boldsymbol{g} \cdot \boldsymbol{h}) + \varepsilon(\widehat{\boldsymbol{g}} \cdot \boldsymbol{h} + \boldsymbol{g} \cdot \widehat{\boldsymbol{h}}) = \cos\phi - \varepsilon\widehat{\phi}\sin\phi =: \underline{\cos}\underline{\phi},$$

[8] Obviously the six coordinates of the pair $(\boldsymbol{g}, \widehat{\boldsymbol{g}})$ of vectors represent particularly normed Plücker coordinates.

[9] From now on we identify each oriented line with its dual unit vector.

[10] The dual extension $\underline{f}(\underline{x})$ of any analytic function $f(x)$ reads $\underline{f}(x + \varepsilon\widehat{x}) := f(x) + \varepsilon\widehat{x}f'(x)$. This can be seen as the beginning of a Taylor series.

$$\underline{g} \times \underline{h} := (g \times h) + \varepsilon[(\widehat{g} \times h) + (g \times \widehat{h})]$$
$$= (\sin\phi + \varepsilon\widehat{\phi}\cos\phi)(n + \varepsilon\widehat{n})$$
$$= \underline{\sin\phi}\,\underline{n}.$$

Here $\underline{n}$ denotes an oriented common perpendicular of the given lines $\underline{g}, \underline{h}$, and the signs of ϕ and $\widehat{\phi}$ are related to the orientation of $\underline{n}$. For details see [22].

The most impressive example for the efficiency of STUDYs principle of transference can be found in kinematics by extending the quaternion representation of spherical motions given in Eq. (7) and (8): When the dual unit quaternion

$$\underline{E} = \underline{\cos\alpha} + \underline{e}\underline{\sin\alpha} \quad \text{with } \underline{e} \cdot \underline{e} = \underline{1} \tag{14}$$

is used for the mapping

$$\underline{\delta_E} : \underline{\mathbf{S}}^2 \to \underline{\mathbf{S}}^2, \quad \underline{g} \mapsto \underline{g}' := \underline{E} \circ \underline{g} \circ \underline{\widetilde{E}}, \tag{15}$$

then the oriented line $\underline{g}'$ is the image of $\underline{g}$ under a screwing motion $\underline{\delta_E}$ with axis $\underline{e}$. The dual number $\underline{\alpha} := \alpha + \varepsilon\widehat{\alpha}$ combines half the angle of rotation and half the length of translation. The signs of these measures α and $\widehat{\alpha}$ are related to the orientation of the screwing axis $\underline{e}$.

This gives rise to a mapping of spatial motions $\underline{\delta_E}$ onto points $\mathbb{R}\underline{E}$ of a quadric of signature 0 in the projective space $\mathbf{P}^7$ (cf. [25]). This kinematic mapping is essential for the design of spatial B-spline-motions (cf. [15]). Another recent application on direct kinematics for platforms is given in [14].

8. Geometry of spheres in $\mathbf{E}^3$

Based on a cartesian coordinate system C the points of a sphere σ in $\mathbf{E}^3$ obey

$$x^2 + y^2 + z^2 - 2mx - 2ny - 2pz$$
$$+ m^2 + n^2 + p^2 - r^2 = 0$$

with constants $m, n, p, r \in \mathbb{R}$ under $r \neq 0$. We endow this sphere with an orientation by defining its radius r either positive or negative. This gives rise to the following mapping into $\mathbf{P}^5$

$$\sigma \mapsto \mathbb{R}s = (x_0 : \ldots : x_5)_C \quad \text{with}$$
$$x_0 = 1, \quad x_3 = p,$$
$$x_1 = m, \quad x_4 = m^2 + n^2 + p^2 - r^2,$$
$$x_2 = n, \quad x_5 = r.$$

The image point $\mathbb{R}s$ of σ fulfills

$$\langle s, s \rangle_s := x_1^2 + x_2^2 + x_3^2 - x_0x_4 - x_5^2 = 0 \tag{16}$$

and is therefore located on a quadric Q_s^4 of signature 2 in $\mathbf{P}^5$. Conversely, each point of Q_s^4 with $x_0x_5 \neq 0$ is the image of an oriented sphere in $\mathbf{E}^3$ with the so-called *hexaspherical coordinates* $(x_0 : \ldots : x_5)_C$.[11]

Each *zero-sphere* with $x_5 = r = 0, x_0 \neq 0$, can be identified with a point $(m, n, p)_C$ in $\mathbf{E}^3$. It turns out that the remaining points of Q_s^4 with $x_0 = 0$ can be seen as images of oriented planes ϕ in $\mathbf{E}^3$: For the plane ϕ in $\mathbf{E}^3$ with the Hesse equation

$$t + ux + vy + wz = 0, \quad u^2 + v^2 + w^2 = 1$$

the image point

$$\mathbb{R}s := (0 : -u : -v : -w : 2t : 1)_C$$

belongs again to Q_s^4. Thus all points of Q_s^4 with $x_0 = 0, x_5 \neq 0$ are obtained. The only remaining point $(0 : 0 : 0 : 0 : 1 : 0)_C$ is seen as the image of an *improper zero-sphere* representing at the same time also an improper plane, as $x_0 = 0$.

Hence we have obtained a bijection

$$\lambda_s : \mathcal{S} \to Q_s^4 \subset \mathbf{P}^5, \quad \sigma \mapsto \mathbb{R}s \tag{17}$$

of the union $\mathcal{S}$ of all oriented spheres and planes in $\mathbf{E}^3$, completed with the improper zero-sphere, onto the *Lie quadric* Q_s^4.[12]

Two points $\mathbb{R}s$, $\mathbb{R}s'$ are conjugate with respect to Q_s^4 if and only, if

[11] The analogous mapping exists for all dimensions $n \geq 2$. The coordinates for the planar version are called *pentacyclic*. Note that the *polyspheric* coordinates presented in [1, 20.7], refer to non-oriented spheres in $\mathbf{E}^n$. These coordinates are appropriate for Möbius sphere geometry only.

[12] The analogy to the Plücker model of the set $\mathcal{L}$ of lines in $\mathbf{P}^3$ is obvious. Over $\mathbb{C}$ the quadrics Q_l^4 and Q_s^4 can be identified. This gave rise to *Lie's line-sphere-transformation* (see e.g. [16, Sections 70 and 71], or [25, Chapter 4]). This transformations maps ruled quadrics onto Dupin's cyclides, for example.

$$\langle s, s' \rangle_s = x_1 x_1' + x_2 x_2' + x_3 x_3' - \tfrac{1}{2}(x_0 x_4' + x_4 x_0') - x_5 x_5' = 0. \quad (18)$$

When these two points are the images of two spheres or zero-spheres σ, σ' with centers $\boldsymbol{m}, \boldsymbol{m}'$ and radii r, r', then Eq. (18) is equivalent to

$$\|\boldsymbol{m} - \boldsymbol{m}'\| = |r - r'|.$$

This characterizes the *oriented contact* between σ and σ'. There is also such a contact when $\mathbb{R}s'$ is supposed to be the image of a plane ϕ', as due to Eq. (18) the radius r of σ equals the distance $\overline{\boldsymbol{m}\phi'}$. In the case of two planes this condition expresses direct parallelity. When finally $\mathbb{R}s$ is the image of the improper zero-sphere, then the conjugate $\mathbb{R}s'$ must be the image of any plane.

Due to this property the hexaspherical coordinates provide a useful tool for solving the *Apollonian sphere problem*: Determine all spheres σ that touch four given spheres $\sigma_1, \ldots, \sigma_4$. We suppose the spheres to be oriented. Then the oriented contact between all σ_i and σ is equivalent to four linear equations

$$\langle \boldsymbol{x}, s_i \rangle_s = 0 \quad \text{for } i = 1, \ldots, 4,$$

if $\mathbb{R}s_i = \lambda_s(\sigma_i)$ and $\mathbb{R}\boldsymbol{x} = \lambda_s(\sigma)$. Hence, in the generic case this problem has been reduced to the computation of the points of intersection between the Lie quadric Q_s^4 and a line. This line is polar with respect to Q_s^4 to the span $[\mathbb{R}s_1 \ldots \mathbb{R}s_4]$. For a recent application of this classical problem in the field of satellite geodesy see [23].

Transformations on $\mathcal{S}$, which correspond to collineations in $\mathbf{P}^5$ that map Q_s^4 onto itself, are called sc *Lie's sphere transformations*. These are bijections that preserve oriented contact. The subgroup of *Möbius transformations* is characterized by the condition that the corresponding collineation in $\mathbf{P}^5$ fixes the hyperplane $x_5 = 0$. Each Möbius transformation is a point transformation, and complementary orientations of a plane or sphere remain complementary. Since the angle ψ of intersection between two spheres or planes $\lambda_s^{-1}(\mathbb{R}s)$ and $\lambda_s^{-1}(\mathbb{R}s')$ obeys (cf. e.g. [7])

$$\sin^2 \frac{\psi}{2} = \frac{-\langle s, s' \rangle_s}{2 x_5 x_5'},$$

the Möbius transformations turn out to be *conformal*, i.e. angle-preserving. It can even be proved that all conformal transformations of $\mathbf{E}^3$ are of Möbius type [13].

Finally, Lie's sphere transformations which preserve the linear complex $x_0 = 0$ of planes are called *Laguerre transformations*. There is a subgroup of index 1 of *proper* Laguerre transformations which preserve the *tangential distance* for each two oriented spheres σ, σ'. The tangential distance t of the spheres $\sigma = \lambda_s^{-1}(\mathbb{R}s)$ and $\sigma' = \lambda_s^{-1}(\mathbb{R}s')$ obeys

$$t^2 = \frac{-2\langle s, s' \rangle_s}{x_0 x_0'}.$$

For details see e.g. [16] or [6, Chapters 1 and 2]. Applications of these transformations in the field of CAGD can be found in [18]. Only similarities are at the same time Möbius and Laguerre transformations.

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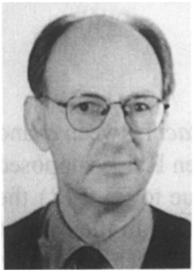

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