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On the Tetrahedra in the Dodecahedron

Dedicated to Prof. Gerhard GEISE at the occasion of his 70th birthday

O tetraedrima u dodekaedru

SAŽETAK

60 bridova od 10 tetraedara upisanih u pravilan dodekaedar čine tzv. GRÜNBAUMOVU mrežu. Poznato je da je ta struktura fleksibilna. Postoje jednoparametarska gibanja koja čuvaju simetriju s obzirom na os stranice ili na os koja prolazi vrhom. Rad se bavi analitičkim reprezentacijama takvih gibanja. Osim toga dokazano je da se oba gibanja mogu spojiti u dvoparametarska gibanja koja ne čuvaju simetriju.

On the Tetrahedra in the Dodecahedron

ABSTRACT

The 60 edges of the ten tetrahedra inscribed in a regular pentagondodecahedron form the so-called GRÜNBAUM framework. It is already known that this structure is flexible. There are one-parameter motions which preserve the symmetry with respect either to a face axis or to a vertex axis. The paper treats analytical representations of these motions. Furthermore it is proved that both motions can blend into two-parametric motions which do not preserve any symmetry.

MSC 1994: 53A17, 51M20

1 Introduction

Since ancient times it is known that with the vertices of a regular pentagondodecahedron **D** one can build five cubes $\mathbf{C}_1, \dots, \mathbf{C}_5$. It was EUCLID's strategy (cf. [3], p. 69) for constructing a dodecahedron by adding 'roofs' to each face of a cube (see Fig. 2). The edges of such an inscribed cube $\mathbf{C}_i$ are diagonals of the faces of **D**. With respect to **D** we can distinguish between right and left vertices of $\mathbf{C}_i$ depending on whether the edges through any vertex A are the right or the left diagonals of **D** in the faces through A, if seen from outside (see Fig. 1). For each edge of $\mathbf{C}_i$ the two endpoints A, B are of different type. So, the right vertices of $\mathbf{C}_i$ form a *right* tetrahedron $\mathbf{R}_i$, the left vertices a *left* tetrahedron $\mathbf{L}_i$. And of course both $\mathbf{L}_i$ and $\mathbf{R}_i$ are inscribed in $\mathbf{C}_i$ and therefore in **D**.

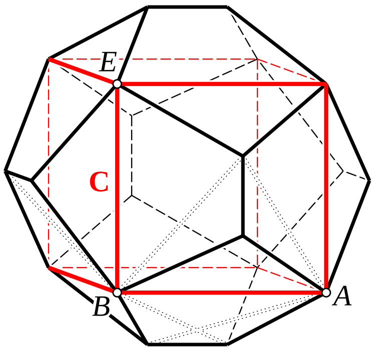

Fig. 1: Dodecahedron **D** with one inscribed cube **C**. Points A and E are *right* vertices, B is a *left* vertex of **C**.

Another way to identify two vertices AE of **D** as endpoints of an edge of any inscribed tetrahedron is as follows: There must be a path from A to E along three edges of **D** – via the 'roof' displayed in Fig. 2. If at the first crossing point you take the *right* edge and at the second vertex the left one, then AE belongs to a *right tetrahedron*. The left choice at the first vertex and the right one afterwards results in an edge of a left tetrahedron.

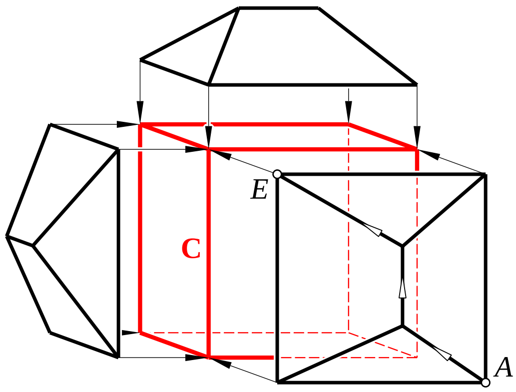

Fig. 2: Constructing a dodecahedron **D** by adding roofs on a cube **C**.

Each vertex of **D** is a left vertex of any inscribed cube $\mathbf{C}_i$, therefore vertex of $\mathbf{L}_i$ and at the same time vertex of any right tetrahedron $\mathbf{R}_j$. We have $i \neq j$ as $\mathbf{L}_i$ and $\mathbf{R}_i$ are complementary tetrahedra of the cube $\mathbf{C}_i$. In the sequel we denote each vertex of $\mathbf{D}$ by the ordered pair ij of different indices of the left and the right tetrahedron meeting there.

Suppose the tetrahedra are solids. Then the union of the right tetrahedra as well as the union of the left ones are well known stellated icosahedra (see [2], note cover page of [8]). They are mirror images from each other which surprisingly share all vertices and all oriented planes spanned by their faces (see [7]). Also the union of all ten tetrahedra is a stellated icosahedron.

Let us now focus on wire-frame models of the tetrahedra: The ten tetrahedra can be seen as the links of a kinematic chain. Each link $\mathbf{L}_i$ ($\mathbf{R}_j$) is connected with four $\mathbf{R}_j$ ($\mathbf{L}_i$), $i \neq j$, by a spherical joint at the common vertex ij. Due to [4] this structure is called GRÜNBAUM *framework*. Surprisingly it is finitely movable, though the structure formula gives -6 as the degree of mobility. We start with representing the well known one-parameter motions of types I and II of GRÜNBAUM's framework.

2 Motions of type I

Theorem 1 (R. CONNELLY et al., 1991)***:*** *For each face axis f of the regular dodecahedron* $\mathbf{D}$ *there is a one-parameter motion of* GRÜNBAUM*'s framework preserving the five-fold symmetry about f.*

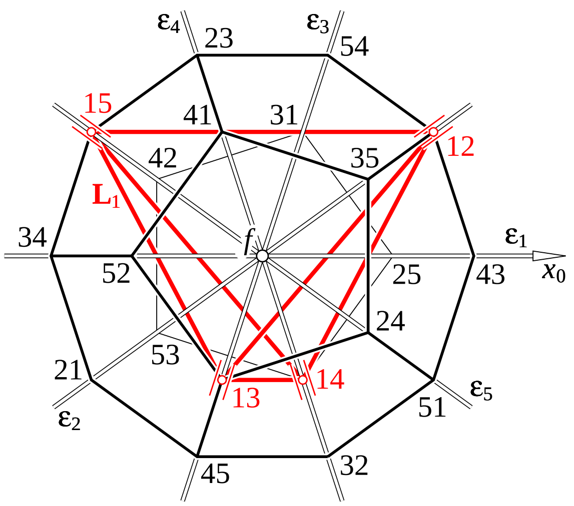

Fig. 3: Motion of type I: The vertices 12, ...,15 of $\mathbf{L}_1$ move in the planes $\varepsilon_2,\dots,\varepsilon_5$ of symmetry.

According to R. CONNELLY et al. [1] (compare also [6]) the motion of one tetrahedron, say $\mathbf{L}_1$, is defined as follows: Let $\varepsilon_1,\dots,\varepsilon_5$ be the five planes of symmetry through the face axis f (see view in direction of f in Fig. 3). Then the vertices 12, 13, 14, 15 of $\mathbf{L}_1$ move within $\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$, respectively. The trivial translation in direction of f can be ruled out by the additional condition that the center of $\mathbf{L}_1$ remains in a plane orthogonal to f. The resulting one-parameter motion of $\mathbf{L}_1$ turns out to split into two rational motions of order 4. By iterated 72°-rotations about f these motions of $\mathbf{L}_1$ are transformed into those of $\mathbf{L}_2,\dots,\mathbf{L}_5$, resp., provided the notation is specified according to Fig. 3. The motions of $\mathbf{R}_1,\dots,\mathbf{R}_5$ are obtained by reflecting those of $\mathbf{L}_1$ in $\varepsilon_1,\dots,\varepsilon_5$, respectively.

Let (x,y,z) be Cartesian coordinates in the moving frame — attached to $\mathbf{L}_1$ — and let (x_0,y_0,z_0) be coordinates in the fixed frame. Then we can set up the motion of $\mathbf{L}_1$ as

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (1)$$

with an orthogonal matrix (a_{ij}). For the representation of (a_{ij}) we use quaternions: Each nontrivial homogeneous quadruple $(q_0,\dots,q_3)$ defines an orthogonal matrix according to

$$(a_{ik}) = \frac{1}{q_0^2+q_1^2+q_2^2+q_3^2} \cdot \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix} \quad (2)$$

and vice versa.

With regard to the fixed frame, we specify the z_0-axis on f and the x_0-axis in ε_1. By the condition $w=0$ the origin of the moving frame remains in the x_0y_0-plane.

The vertices of the moving tetrahedron $\mathbf{L}_1$ with edge length $2\sqrt{2}$ can be set up as

$$\begin{aligned} 12 &= (1,-1,-1), & 14 &= (-1,\ \ 1,-1), \\ 13 &= (1,\ \ 1,\ \ 1), & 15 &= (-1,-1,\ \ 1). \end{aligned} \quad (3)$$

Now for $j=2,\dots,5$ the vertex $1j$ is supposed to move in the plane ε_j of symmetry. According to (1) this results in four linear equations for the coordinates $(u,v,0)$ of the translation vector and the entries a_{ij} of the orthogonal matrix. Setting

$$s_i := \sin\frac{2i\pi}{5}, \quad c_i := \cos\frac{2i\pi}{5} \quad \text{for } i=1,2 \quad (4)$$

we obtain

$$\begin{aligned} & s_2(u+a_{11}-a_{12}-a_{13})+ \\ & \quad +c_2(v+a_{21}-a_{22}-a_{23})=0, \\ -& s_1(u+a_{11}+a_{12}+a_{13})+ \\ & \quad +c_1(v+a_{21}+a_{22}+a_{23})=0, \\ & s_1(u-a_{11}+a_{12}-a_{13})+ \\ & \quad +c_1(v-a_{21}+a_{22}-a_{23})=0, \\ -& s_2(u-a_{11}-a_{12}+a_{13})+ \\ & \quad +c_2(v-a_{21}-a_{22}+a_{23})=0. \end{aligned} \quad (5)$$

We eliminate u, v by multiplying the equations in (5) either with $-s_1$, $-s_2$, s_2, s_1 or with $-c_1$, c_2, c_2, $-c_1$, resp., and summing up. This gives two linear equations

$$\begin{aligned} &4s_1c_1a_{12} + 2c_1a_{21} - a_{23} = 0 \\ &4s_1c_1a_{11} + 2s_1a_{13} - a_{22} = 0 \end{aligned} \tag{6}$$

when we pay attention to the identities

$$\begin{aligned} &4c_i^2 + 2c_i - 1 = 0, && c_i^2 + s_i^2 = 1, \\ &s_1c_2 + s_2c_1 = -s_2, && s_1c_2 - s_2c_1 = -s_1, \\ &s_2 = 2s_1c_1, && c_1c_2 = -\tfrac{1}{4}, \\ &4c_1^2(4s_1^2 - 1) = (1-2c_1)(2+2c_1) = 1. \end{aligned} \tag{7}$$

Now we substitute in (6) the representation (2) of the a_{ij}. By setting $k := 1 - 4s_1c_1$ we obtain

$$[2c_1(1-2s_1)q_0 - q_2]\,[2c_1(1+2s_1)q_1 - q_3] = 0, \tag{8}$$

$$\begin{aligned} &[kq_0 - (2s_1+2)q_2]\,[kq_0 - (2s_1-2)q_2] - \\ &\quad -[(2s_1+2)q_1 + kq_3]\,[(2s_1-2)q_1 + kq_3] = 0. \end{aligned} \tag{9}$$

We start with the second factor in (8) and express in (9) q_3 in terms of q_1. This results in a quadratic equation for $(q_0 : q_1 : q_2)$

$$\begin{aligned} &[kq_0 - (2s_1+2)q_2]\,[kq_0 - (2s_1-2)q_2] + \\ &\qquad +4(8c_1 - 4s_1 + 16s_1c_1 - 3)q_1^2 = 0. \end{aligned} \tag{10}$$

We introduce homogeneous motion parameters $(\sigma,\tau) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for this "conic" by setting

$$\begin{aligned} &[kq_0 - (2s_1+2)q_2] \cdot c_1^2(2s_1-1)^2 = \sigma^2, \\ &[kq_0 - (2s_1-2)q_2] = \\ &\qquad = 4(-8c_1 + 4s_1 - 16s_1c_1 + 3)\tau^2, \\ &c_1(2s_1-1)q_1 = \sigma\tau, \end{aligned} \tag{11}$$

and obtain after some simplifications due to (7) the representation

$$\begin{aligned} q_0 &= -(2s_1-1)\sigma^2 + 2s_1(1+4s_1c_1)\tau^2, \\ q_1 &= \quad 2(2s_1-1)\sigma\tau, \\ q_2 &= -(2c_1+1)\sigma^2 + 2s_1(2s_1-2)\tau^2, \\ q_3 &= \quad 2(2c_1+1)\sigma\tau. \end{aligned} \tag{12}$$

From (2) we get the orthogonal matrix in (1). The translation vector $(u, v, 0)$ is given by

$$\begin{aligned} u &= -a_{12} + c_1(a_{21} + a_{32})/s_1, \\ v &= -a_{22} + s_1(a_{11} + a_{13})/c_1 \end{aligned} \tag{13}$$

resulting from (5).

Due to the last equation in (7), the involutive projective transformation

$$(q_0 : q_1 : q_2 : q_3) \mapsto (q_0' : q_1' : q_2' : q_3')$$

obeying

$$\begin{aligned} &q_0' = -q_3, && q_2' = q_1, \\ &q_1' = -q_2, && q_3' = q_0 \end{aligned}$$

switches the two factors in (8) while (9) is preserved. On the other hand the first two rows in the orthogonal matrix (2) change signs. So a 180°-rotation about the z_0-axis transforms the one-parameter motion represented in (12) and (13) into that stemming from the first factor in (8).

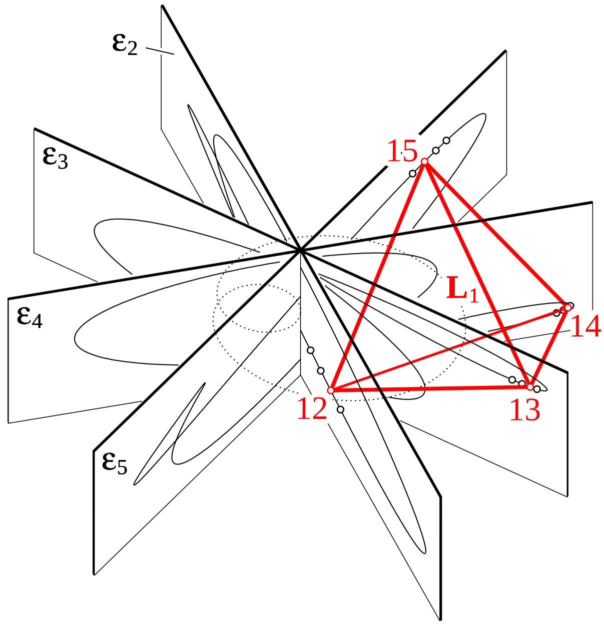

Fig. 4: The trajectories of the vertices of $\mathbf{L}_1$ under the motion of type I as defined in Fig. 3 and represented in (12) and (13).

Theorem 2: *For a given face axis f of* **D** *the motion of type I of any tetrahedron splits into two rational motions which can be transformed into each other by a halfturn about f. Both components are of order 4 and type a) according to the classification given by* O. RÖSCHEL *in* [5].

Fig. 4 shows the trajectories of the vertices of $\mathbf{L}_1$ under the motion with the quaternion representation (12). When the motion parameters $(\sigma : \tau)$ are replaced by $(-\sigma : \tau)$, then q_1 and q_3 change signs. The same effect appears when the moving frame performs a halfturn about the y-axis (switching 12 with 15 and 13 with 14) while at the same time the fixed frame rotates about the y_0-axis through 180° exchanging ε_2 with ε_5 and ε_3 with ε_4. This is the reason why the trajectories of the vertices 12 and 15 are congruent as well as that of 13 and 14. The dotted curve in Fig. 4 shows the trajectory of the center of $\mathbf{L}_1$.

A real GRÜNBAUM framework will not perform the full motion since one vertex can't move "through" the other.

But one can dissolve some joints and reassamble the structure in another position.[1] Then one will realize that during the motion of type I the ten tetrahedra fall apart and form a ring with a diameter which approximately doubles that of the initial position in the interior of the dodecahedron **D**. Fig. 5 shows the extended position in comparison with **D**.

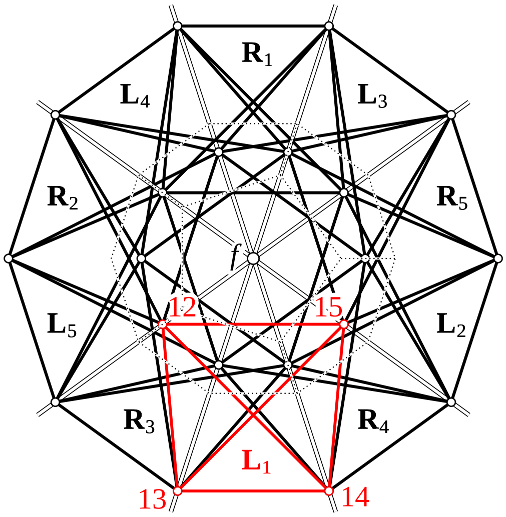

Fig. 5: GRÜNBAUM's framework forming a ring during the motion of type I. Note the size of **D** (dotted lines) which encloses the initial position of the framework.

3 Motions of type II

Theorem 3 (H. S., 1991)***:*** *For each vertex axis v of the regular dodecahedron there is a one-parameter motion of* GRÜNBAUM*'s framework preserving the three-fold symmetry about v.*

Suppose v connects the vertices 45 and 54. Then v is a common axis of symmetry for $\mathbf{L}_4$, $\mathbf{R}_4$, $\mathbf{L}_5$ and $\mathbf{R}_5$. When this axis v is kept fixed, then the vertices 41, 42, 43, 51, 52, 53, 14, 15, 24, 25, 34 and 35 can only move on a cylinder Φ of rotation with axis v (see view in Fig. 6 showing v as a point). According to [6] we define the motion of $\mathbf{L}_1$ by the additional condition that 12 and 13 remain in the planes φ_2, φ_3 of symmetry passing through v . After ruling out the translations along v we again end up with a one-parameter motion of $\mathbf{L}_1$.

The movements of $\mathbf{L}_2$ and $\mathbf{L}_3$ are obtained by iterated 120°-rotations about v. Reflections in $\varphi_1, \varphi_2, \varphi_3$ transform the motion of $\mathbf{L}_1$ into those of $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, resp., provided the notation is specified as in Fig. 6.

In order to represent the motion of type II analytically we define the z_0-axis on v and the x_0-axis in φ_1. We specify the moving frame (x,y,z) attached to $\mathbf{L}_1$ by

$$\begin{aligned} 12 &= (\ \sqrt{2}, 0, 1), & 14 &= (0,\ \sqrt{2}, -1), \\ 13 &= (-\sqrt{2}, 0, 1), & 15 &= (0, -\sqrt{2}, -1). \end{aligned} \tag{14}$$

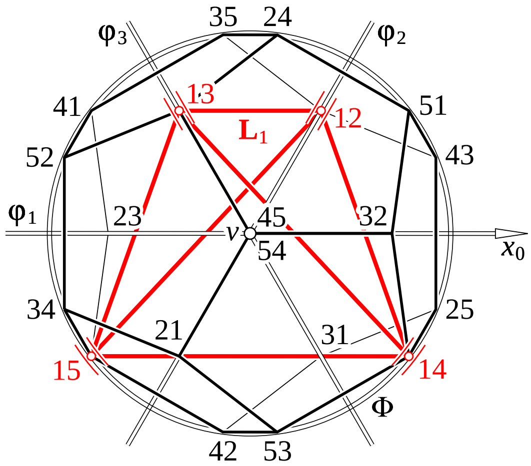

Fig. 6: Motion of type II: The vertices 12, 13 of $\mathbf{L}_1$ move in the planes φ_2, φ_3 of symmetry. 14 and 15 remain on the cylinder Φ of rotation.

Then the constraints defined above result in four equations

$$\begin{aligned} &(u+a_{11}\sqrt{2}+a_{13})\sqrt{3} - (v+a_{21}\sqrt{2}+a_{23}) = 0, \\ &(u-a_{11}\sqrt{2}+a_{13})\sqrt{3} + (v-a_{21}\sqrt{2}+a_{23}) = 0, \\ &(u+a_{12}\sqrt{2}-a_{13})^2 + (v+a_{22}\sqrt{2}-a_{23})^2 = \tfrac{8}{3}, \\ &(u-a_{12}\sqrt{2}-a_{13})^2 + (v-a_{22}\sqrt{2}-a_{23})^2 = \tfrac{8}{3}. \end{aligned} \tag{15}$$

From the two linear equations we obtain

$$u = a_{21}\sqrt{\tfrac{2}{3}} - a_{13}, \qquad v = a_{11}\sqrt{6} - a_{23}. \tag{16}$$

We substitute this in the difference and sum of the last two equations in (15). So we end up with

$$\begin{aligned} &A a_{12} + B a_{22} = 0, \\ &A^2 + B^2 + 2a_{12}^2 + 2a_{22}^2 = \tfrac{8}{3} \ \text{ for} \\ &\quad A := a_{21}\sqrt{\tfrac{2}{3}} - 2a_{13}, \quad B := a_{11}\sqrt{6} - 2a_{23}. \end{aligned} \tag{17}$$

Together with the orthogonality conditions

$$\begin{aligned} &a_{11}^2 + a_{12}^2 + a_{13}^2 = a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \\ &a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \end{aligned}$$

we have five equations for the six entries in the first two rows of the matrix (a_{ik}). When in (17) the a_{ij} are replaced by $q_0, \dots, q_3$ according to (2) we obtain two homogeneous equations of degree 4.

However, explicit representations for the motions of type II have not yet been found.

[1] The author thanks Elisabeth ZACH for producing a model of GRÜNBAUM's framework. This was the key for dedecting the two-parametric mobility presented in Section 4.

4 Two-parametric motions of GRÜNBAUM's framework

During the motion of type I as defined in Fig. 3 the moving tetrahedron $\mathbf{L}_1$ reaches positions which are symmetric with respect to ε_3 (see Fig. 7): Since ε_3 bisects the angle between ε_2 and ε_4, we can choose $12 \in \varepsilon_2$ and $14 \in \varepsilon_4$ in symmetric position. This implies $13, 15 \in \varepsilon_3$, hence $15 \in f$.

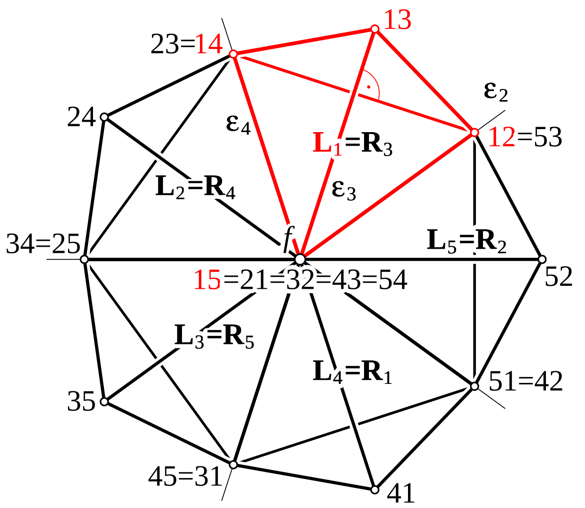

Fig. 7: Position of GRÜNBAUM's framework with pairwise coinciding tetrahedra and two-parametric mobility.

In this particular position the reflection of $\mathbf{L}_1$ in ε_3 gives $\mathbf{R}_3 = \mathbf{L}_1$. The iterated 72°-rotations about f and the reflections in ε_i, $i = 2, \ldots, 5$, reveal that all tetrahedra are pairwise concident: We have $\mathbf{R}_4 = \mathbf{L}_2$, $\mathbf{R}_5 = \mathbf{L}_3$, $\mathbf{R}_1 = \mathbf{L}_4$, and $\mathbf{R}_2 = \mathbf{L}_5$. All tetrahedra share the vertex $S := 15 = 21 = 32 = 43 = 54$.

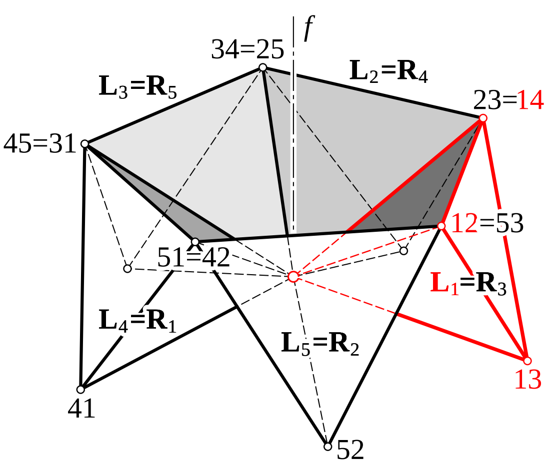

Fig. 8: GRÜNBAUM framework seen as a flexing pyramid with five triangular faces as bases for tetrahedra.

Because of $14 = 23$, $25 = 34$, $31 = 45$, $42 = 51$, and $53 = 12$ any two consecutive tetrahedra in the cycle $(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3\mathbf{L}_4\mathbf{L}_5)$ have an edge through S in common.

So the whole structure can be seen as a five-sided pyramid built of five regular triangles which are the bases for the tetrahedra (Fig. 8). Such a pyramid with revolute joints at its edges flexes with mobility 2 like a spherical pentagon with hinges at its vertices.

During motions of type I the tetrahedron $\mathbf{L}_1$ also occupies positions symmetric with respect to ε_2 (Fig. 9). This time 14 and 15 are mutual mirror images in ε_2 which implies $13 \in f$. Hence the ten tetrahedra are again pairwise coinciding: $\mathbf{L}_1 = \mathbf{R}_2$, $\mathbf{L}_2 = \mathbf{R}_3$, $\mathbf{L}_3 = \mathbf{R}_4$, $\mathbf{L}_4 = \mathbf{R}_5$, and $\mathbf{L}_5 = \mathbf{R}_1$. The five tetrahedra share one vertex $(13 = 24 = 35 = 41 = 52)$; any two consecutive tetrahedra in this closed kinematic chain with five links $\mathbf{L}_1, \ldots, \mathbf{L}_5$ share an edge, passing through $14 = 32$, $25 = 43$, $31 = 54$, $42 = 15$, and $53 = 21$, respectively. This time the five-sided pyramid formed by the revolute axes is two-times wound around f (see Fig. 9).

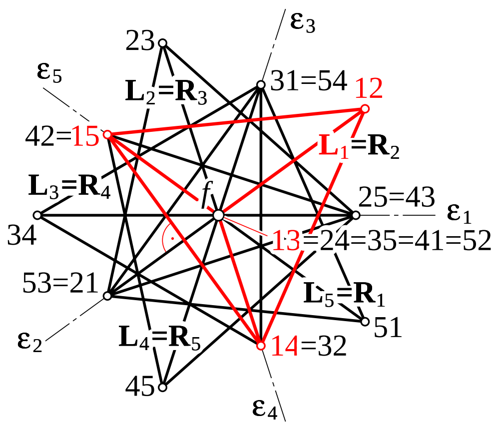

Fig. 9: Another position of GRÜNBAUM's framework with pairwise coinciding tetrahedra and degree 2 of mobility.

After interchanging 12 with 15 and 13 with 14 we obtain analogous cases where $\mathbf{L}_1$ occupies positions symmetric with respect to ε_4 or ε_5, hence $\mathbf{L}_1 = \mathbf{R}_4$ or $\mathbf{L}_1 = \mathbf{R}_5$.

Also the motions of type II can blend into a two-parametric mobility: When during the motion displayed in Fig. 6 the vertex 12 crosses the axis v of symmetry, then because of the given edge length of $\mathbf{L}_1$ the vertices 14 and 15 are located on the same circle of the cylinder Φ. Hence 13 must be located on Φ too. This implies that in this position $\mathbf{L}_1$ is symmetric with respect to φ_1, φ_2 and φ_3. We get $\mathbf{L}_1 = \mathbf{L}_2 = \mathbf{L}_3 = \mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}_3$ (see Fig. 10). Even two of the remaining tetrahedra, say $\mathbf{L}_4$ and $\mathbf{R}_5$, coincide with $\mathbf{L}_1$. The two last coinciding tetrahedra $\mathbf{L}_5 = \mathbf{R}_4$ share a face with $\mathbf{L}_1$.

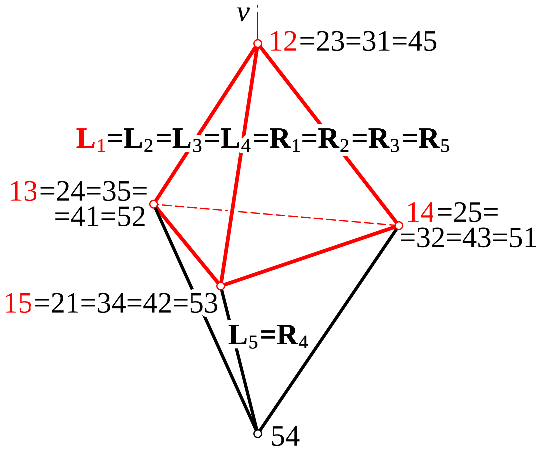

Fig. 10: In this position of GRÜNBAUM's framework a motion of type II can blend into two-parametric flexibility.

The same collapsed position can be reached when the flexing five-sided pyramid (Fig. 8) is folded such that three faces coincide.

Theorem 4: *The* GRÜNBAUM *framework admits also two-parametric motions: Here tetrahedra of different type are coupled into pairs such that – in cyclic order – each two consecutive tetrahedra share an edge, and all these edges pass through a common vertex.*
The motions of type I can blend into this two-parametric flexes whenever one tetrahedron occupies a position symmetric with respect to any fixed plane ε_i *of symmetry. Motions of type II can bifurcate into two-parametric mobility when one vertex of* $\mathbf{L}_1$ *crosses the fixed axis* v *of symmetry. Then even eight of the ten tetrahedra are coinciding.*

It is still open whether these two-parametric motions complete the list of nontrivial flexes of the GRÜNBAUM framework.

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