Remarks on Rigidity

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REMARKS ON RIGIDITY

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ABSTRACT

This paper treats flexible cross-polytopes in the Euclidean 4-space. It is shown that the examples presented 1998 by A. Walz are special cases of a more general class of flexible cross-polytopes. The proof is given by means of 4D descriptive geometry. Further, a parametrization of the one-parameter self-motions of Walz's polytopes is presented.

Key Words: Flexibility, polyhedra, 4D geometry.
MSC 2000: 52C25, 51N05, 53A17

1. INTRODUCTION

There is a basic and important question concerning the geometry of structures: Is a given structure rigid or is it not? In the engineering world there is a vigorous interest in rigidity, as bridges, buildings, mechanical gadgets and countless other things have to be built. This has been the background for interesting mathematical theories. And there is still a wide field of open problems left.

A long-standing problem is to prove if a smooth closed surface can continuously flex, i.e., one can find a continuous family of smooth surfaces each of which is isometric (in the intrinsic metric) to any other one and is not obtained from the initial surface by a rigid motion. A first piece-wise linear flexible embedding of the 2-sphere into the Euclidean 3-space was constructed by R. Connelly (Connelly (1978)). Two years later a simplified *"flexing sphere"* was presented by K. Steffen (see Dewdney (1992)). Both flexible polyhedra are based on Bricard's octahedra (Bricard (1897), compare Stachel (1987)).

A milestone in the theory of flexible polyhedra was recently the progress with the "bellows conjecture". This conjecture stated by R. Connelly says that any continuous flex that preserves the edge lengths of a closed triangulated polyhedron preserves its volume. A first proof in $\mathbb{E}^3$ was given by I. Sabitov (1995). A second proof by R. Connelly et al. (1997) followed two years later.

If a polyhedron admits a continuous flex then it admits also an analytical flex, i.e., for each vertex the trajectory under the flex can be expressed as an analytic function of the time t. One can weaken the continuous flexibility by limiting the Taylor series, i.e., by requiring that the edge lengths stay constant up to a given order of t, only. In this sense, flexibility of first order means that to each vertex a velocity vector can be assigned such that these are compatible with constant edge lengths. Additionally one must demand that these velocity vectors do not originate from a motion of the whole structure like a rigid body. When also acceleration vectors can be assigned then we get second order flexibility, and so on. Geometric characterizations of octahedra which are infinitesimally flexible of the orders 1 or 2 are given in Stachel (1999).

2. FLEXIBLE CROSS-POLYTOPES

In the Euclidean n-space $\mathbb{E}^n$ the analoga of octahedra are called *cross-polytopes* $\mathcal{C}_n$: These polytopes have $2n$ vertices coupled into pairs $(\mathbf{p}_1^i, \mathbf{p}_2^i)$ for $i = 1, \ldots, n$. The $4\binom{n}{2} = 2n(n-1)$ edges of $\mathcal{C}_n$ are $\mathbf{p}_{j_1}^i \mathbf{p}_{j_2}^k$ for $i \neq k$ and $j_1, j_2 \in \{1, 2\}$. The 2^n hyperfaces of $\mathcal{C}_n$ are the simplices $\mathbf{p}_{j_1}^1 \mathbf{p}_{j_2}^2 \ldots \mathbf{p}_{j_n}^n$ for any $j_1, \ldots, j_n \in \{1, 2\}$.

2.1 A. Walz's flexible cross-polytopes in $\mathbb{E}^4$

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{d}_2$ be the eight vertices of a four-dimensional cross-polytope $\mathcal{C}_4$. We partition the set of 24 edges into the edges of the quadrangles ($= \mathcal{C}_2$)

$$\mathcal{Q} := \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2, \quad \overline{\mathcal{Q}} := \mathbf{c}_1\mathbf{d}_1\mathbf{c}_2\mathbf{d}_2,$$

and the bipartite framework

$$\mathcal{F} := \{\, \mathbf{p}\,\overline{\mathbf{p}} \mid \mathbf{p} \in \mathcal{Q},\ \overline{\mathbf{p}} \in \overline{\mathcal{Q}}\,\}.$$

In 1998 at a conference in Canada[1] A. Walz presented a class of continously flexible cross-polytopes in $\mathbb{E}^4$. Following Walz, we visualize this polyhedron using two complementary orthogonal projections of $\mathbb{E}^4$ onto

[1] "Canadian Mathematical Society Winter 1998 Meeting" held at Queen's University and the Royal Military College, December 13-15, 1998. See `http://www.cms.math.ca/Events/winter98/w98-abs/node20.html` .

planes: Each point $\mathbf{x} = (x, y, z, t) \in \mathbb{E}^4$ is mapped onto its "top view" $\mathbf{x}' = (x, y)$ and the "front view" $\mathbf{x}'' = (z, t)$, thus representing $\mathbb{E}^4$ as $\mathbb{E}^2 \times \mathbb{E}^2$ (compare Stachel (1990)). Obviously, for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{E}^4$ the distance is given by

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}' - \mathbf{y}'\|^2 + \|\mathbf{x}'' - \mathbf{y}''\|^2. \tag{1}$$

At Walz's example the quadrangles $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are located in two totally-orthogonal planes, say, the xy-plane and the zt-plane, respectively. Therefore we obtain the true size of $\mathcal{Q}$ in the top view, the true size of $\overline{\mathcal{Q}}$ in the front view, and we have $\mathbf{c}_1' = \ldots = \mathbf{d}_2' = (0,0)$ and $\mathbf{a}_1'' = \ldots = \mathbf{b}_2'' = (0,0)$ (see Fig. 1). The quadrangles $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are *antiparallelograms*[2] with their circumcenter at the origin $(0,0,0,0)$. Let $\rho, \overline{\rho}$ denote the radii of the circumcircles. Then due to (1) all edges of $\mathcal{F}$ have the same length $r = \sqrt{\rho^2 + \overline{\rho}^2}$.

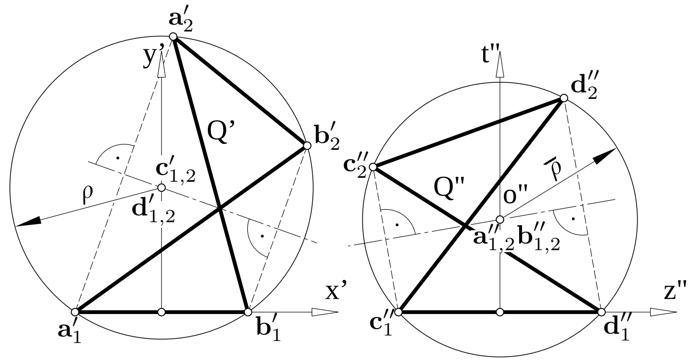

Figure 1: A. Walz's four-dimensional flexible cross-polytope $\mathcal{C}_4$ represented in top view and front view.

Suppose that both antiparallelograms $\mathcal{Q}, \overline{\mathcal{Q}}$ flex simultaneously like four-bar linkages in their planes such that for the circumcircles the centers remain fixed and the radii $\rho, \overline{\rho}$ obey the condition

$$r^2 := \rho^2 + \overline{\rho}^2 = \text{const.} \tag{2}$$

Then all edges of $\mathcal{C}_4$ preserve their lenghts. Since all 2-faces of $\mathcal{C}_4$ are triangles, the planar motions define a continuous selfmotion of the cross-polytope.[3]

For obtaining an analytic representation of this flex, we translate the coordinate frame such that

$$\begin{aligned} \mathbf{a}_1 &= (-\alpha, 0, 0, \tau), \qquad & \mathbf{b}_1 &= (\alpha, 0, 0, \tau), \\ \mathbf{c}_1 &= (0, \eta, -\gamma, 0), \qquad & \mathbf{d}_1 &= (0, \eta, \gamma, 0) \end{aligned}$$

with $\alpha, \gamma > 0$. We keep the top views of $\mathbf{a}_1$ and $\mathbf{b}_1$ fixed as well as the front views of $\mathbf{c}_1$ and $\mathbf{d}_1$. Hence α and γ are constant[4] while the coordinates η and τ vary.

Let

$$\begin{aligned} 2\beta &:= \|\mathbf{b}_2 - \mathbf{a}_1\| = \|\mathbf{b}_1 - \mathbf{a}_2\| > 2\alpha, \\ 2\delta &:= \|\mathbf{d}_2 - \mathbf{c}_1\| = \|\mathbf{d}_1 - \mathbf{c}_2\| > 2\gamma. \end{aligned}$$

It is well known (e.g. Wunderlich (1970)) that in any position of the four-bar linkage $\mathcal{Q}$ in the xy-plane the coupler $\mathbf{a}_2\mathbf{b}_2$ is the image of the frame link $\mathbf{a}_1\mathbf{b}_1$ under the reflection in any tangent line l of the ellipse e (=fixed polode) with focal points $\mathbf{a}_1, \mathbf{b}_1$ and semi-axes β and $\sqrt{\beta^2 - \alpha^2}$ (see Fig. 2).

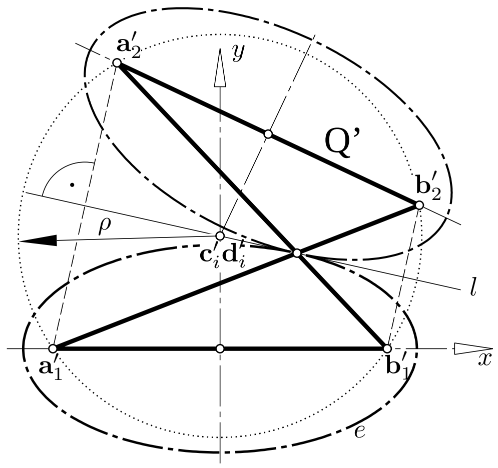

Figure 2: The antiparallelogram-motion as a symmetric rolling of ellipses.

Let in the xy-plane the tangent line l touch the ellipse e at the instantaneous pole

$$\left(\beta \sin\varphi, \sqrt{\beta^2 - \alpha^2}\,\cos\varphi\right). \tag{3}$$

Then l intersects the minor axis $(x = 0)$ at the point $(0, \eta)$ with

$$\eta = \sqrt{\beta^2 - \alpha^2}/\cos\varphi. \tag{4}$$

This point is the center of the circumcircle of $\mathcal{Q}$. Therefore the radius obeys

$$\rho^2 = \beta^2 + (\beta^2 - \alpha^2)\,\tan^2\varphi \geq \beta^2.$$

In the same way the flexes of $\overline{\mathcal{Q}}$ in the zt-plane are obtained under a symmetric rolling of ellipses with semi-axes δ and $\sqrt{\delta^2 - \gamma^2}$. We set

$$\left(\delta \sin\psi, \sqrt{\delta^2 - \gamma^2}\,\cos\psi\right). \tag{5}$$

Hence the center of the circumcircle of $\overline{\mathcal{Q}}$ is $(0, \tau)$ with

$$\tau = \sqrt{\delta^2 - \gamma^2}/\cos\psi, \tag{6}$$

[2]These are nonconvex quadrangles with opposite sides of equal lengths. Antiparallelograms have always a line of symmetry. If the four vertices are not aligned, there is a circumcircle.

[3]When we replace the condition (2) either by $\cos\rho\cos\overline{\rho} = $ const. or by $\cosh\rho\cosh\overline{\rho} = $ const., we obtain flexible cross-polytopes in the *elliptic* or *hyperbolic* 4-space, respectively.

[4]Under these conditions the hyperface $\mathcal{S}_1 := \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ of $\mathcal{C}_4$ is still movable in $\mathbb{E}^4$. It performs an elliptic motion parallel to the yt-plane. The trajectories of the vertices $\mathbf{a}_1, \ldots, \mathbf{d}_1$ are located on straight lines.

and the radius

$$\overline{\rho}^2 = \delta^2 + (\delta^2 - \gamma^2)\,\tan^2\psi \geq \delta^2.$$

The necessary condition (2) implies

$$r^2 - \beta^2 - \delta^2 = (\beta^2 - \alpha^2)\tan^2\varphi + \\ + (\delta^2 - \gamma^2)\tan^2\psi \geq 0. \tag{7}$$

This condition for φ and ψ gives rise to a *closed one-parameter flex of* $\mathcal{C}_4$: We set for $0 \leq t < 2\pi$

$$\begin{aligned} \varphi &= \arctan\left(\sqrt{\tfrac{r^2-\beta^2-\delta^2}{\beta^2-\alpha^2}}\,\cos t\right), \\ \psi &= \arctan\left(\sqrt{\tfrac{r^2-\beta^2-\delta^2}{\delta^2-\gamma^2}}\,\sin t\right). \end{aligned} \tag{8}$$

Then by reflecting $\mathbf{a}_1, \mathbf{b}_1$ in the tangent line l of the ellipse e at the pole (3) we obtain

$$\begin{aligned} \mathbf{a}_2 &= \left(-\alpha + \tfrac{2(\beta^2-\alpha^2)\sin\varphi}{\beta-\alpha\sin\varphi},\ \tfrac{2\beta\sqrt{\beta^2-\alpha^2}\,\cos\varphi}{\beta-\alpha\sin\varphi},\ 0,\ \tau\right), \\ \mathbf{b}_2 &= \left(\ \ \alpha + \tfrac{2(\beta^2-\alpha^2)\sin\varphi}{\beta+\alpha\sin\varphi},\ \tfrac{2\beta\sqrt{\beta^2-\alpha^2}\,\cos\varphi}{\beta+\alpha\sin\varphi},\ 0,\ \tau\right). \end{aligned} \tag{9}$$

In the same way (5) results in

$$\begin{aligned} \mathbf{c}_2 &= \left(0,\ \eta, -\gamma + \tfrac{2(\delta^2-\gamma^2)\sin\psi}{\delta-\gamma\sin\psi},\ \tfrac{2\delta\sqrt{\delta^2-\gamma^2}\,\cos\psi}{\delta-\gamma\sin\psi}\right), \\ \mathbf{d}_2 &= \left(0,\ \eta,\ \ \gamma + \tfrac{2(\delta^2-\gamma^2)\sin\psi}{\delta+\gamma\sin\psi},\ \tfrac{2\delta\sqrt{\delta^2-\gamma^2}\,\cos\psi}{\delta+\gamma\sin\psi}\right). \end{aligned} \tag{10}$$

The reflection of the xy-plane in the tangent line l can be extended to a reflection of the 4-space in a hyperplane L being orthogonal to the xy-plane and passing through l. As l contains the top views $\mathbf{c}_i' = \mathbf{d}_i'$ for $i = 1, 2$, the 4D-reflection maps

$$\mathbf{a}_1 \mapsto \mathbf{a}_2,\ \mathbf{b}_1 \mapsto \mathbf{b}_2,\ \mathbf{c}_i \mapsto \mathbf{c}_i,\ \mathbf{d}_i \mapsto \mathbf{d}_i.$$

In the same way the reflection of the zt-plane leads to a reflection of $\mathbb{E}^4$ in a hyperplane $\overline{\mathrm{L}}$ mapping

$$\mathbf{a}_i \mapsto \mathbf{a}_i,\ \mathbf{b}_i \mapsto \mathbf{b}_i,\ \mathbf{c}_1 \mapsto \mathbf{c}_2,\ \mathbf{d}_1 \mapsto \mathbf{d}_2.$$

$\overline{\mathrm{L}}$ ist orthogonal to L. Hence in any position of the flexing cross-polytope the two complementary hyperfaces $\mathcal{S}_1 = \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ and $\mathcal{S}_2 = \mathbf{a}_2\mathbf{b}_2\mathbf{c}_2\mathbf{d}_2$ of $\mathcal{C}_4$ are mirror images with respect to a plane $\mathrm{L} \cap \overline{\mathrm{L}}$.

2.2 Generalizing A. Walz's flexible cross-polytopes in $\mathbb{E}^4$

It turns out that Walz's polytopes are special cases in a larger class of flexible cross-polytopes:

Theorem 1: *Let* $\mathcal{C}_4$ *be a cross-polytope with the vertices* $\mathbf{a}_1, \ldots, \mathbf{b}_2$ *in the hyperplane* $z = 0$ *symmetric with respect to* $x = 0$*, and* $\mathbf{c}_1, \ldots, \mathbf{d}_2$ *in* $x = 0$ *symmetric with respect to* $z = 0$*, i.e.,*

$$\mathbf{a}_{1,2} = (\pm\alpha_1, \alpha_2, 0, \alpha_4),\ \mathbf{b}_{1,2} = (\pm\beta_1, \beta_2, 0, \beta_4)$$
for $\alpha_1, \beta_1 > 0$ *and* $|\beta_2 - \alpha_2| + |\beta_4 - \alpha_4| \neq 0$,
$$\mathbf{c}_{1,2} = (0, \gamma_2, \pm\gamma_3, \gamma_4),\ \mathbf{d}_{1,2} = (0, \delta_2, \pm\delta_3, \delta_4)$$
for $\gamma_3, \delta_3 > 0$ *and* $|\gamma_2 - \delta_2| + |\gamma_4 - \delta_4| \neq 0$.

Then $\mathcal{C}_4$ *can flex while the vertices remain in their hyperplanes and the symmetries are preserved.*

Remark 1: The vertices $\mathbf{a}_1, \ldots, \mathbf{b}_2$ in the 3-space $z = 0$ form a planar antiparallelogram $\mathcal{Q}$ because of the symmetry with respect to $x = 0$. Without loss of generality the affine span of $\mathcal{Q}$ can be defined as xy-plane. This implies $\alpha_4 = \beta_4 = 0$ in Theorem 1.

Remark 2: Also $\overline{\mathcal{Q}} := \mathbf{c}_1\mathbf{d}_1\mathbf{c}_2\mathbf{d}_2$ is an antiparallelogram. Its affine span within $x = 0$ is orthogonal to the affine span of $\mathcal{Q}$ but needs not be totally orthogonal as it is the case at Walz's example. Total orthogonality is characterized by

$$(\beta_2 - \alpha_2)(\delta_2 - \gamma_2) + (\beta_4 - \alpha_4)(\delta_4 - \gamma_4) = 0.$$

Proof of Theorem 1: We prefer a constructive proof based again on top view and front view. In the sense of Remark 1 we specify $\alpha_4 = \beta_4 = 0$. Hence, the top view shows $\mathcal{Q}$ in true size and $\mathbf{a}_i'' = \mathbf{b}_k''$ for all $i, k \in \{1, 2\}$.

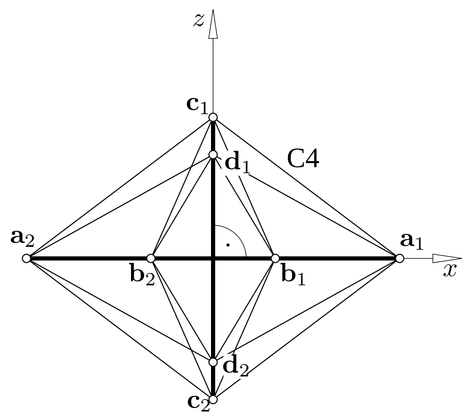

Figure 3: The flexible cross-polytope $\mathcal{C}_4$ of Theorem 1, orthogonally projected into the xz-plane.

There are eight edge lengths to distinguish at $\mathcal{C}_4$ (see Fig. 3):

$$\begin{aligned} &l_{ab} := \|\mathbf{a}_i - \mathbf{b}_i\|,\ \bar{l}_{ab} := \|\mathbf{a}_i - \mathbf{b}_j\|, \\ &l_{cd} := \|\mathbf{c}_i - \mathbf{d}_i\|,\ \bar{l}_{cd} := \|\mathbf{c}_i - \mathbf{d}_j\|, \end{aligned} \quad i \neq j,$$
$$\begin{aligned} &l_{ac} := \|\mathbf{a}_i - \mathbf{c}_k\|,\ l_{ad} = \|\mathbf{a}_i - \mathbf{d}_k\|, \\ &l_{bc} := \|\mathbf{b}_i - \mathbf{c}_k\|,\ l_{bd} := \|\mathbf{b}_i - \mathbf{d}_k\|. \end{aligned} \quad i, k \in \{1, 2\}.$$

We try to find for $\mathcal{C}_4$ a nontrivial flex $\widetilde{\mathcal{C}}_4$ with vertices $\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{d}}_2$, sufficiently near to the initial position. We start with a position $\widetilde{\mathbf{a}}_1\widetilde{\mathbf{b}}_1\widetilde{\mathbf{a}}_2\widetilde{\mathbf{b}}_2$ of the antiparallelogram $\mathcal{Q}$ in the xy-plane. The equations

$$\|\widetilde{\mathbf{a}}_i' - \widetilde{\mathbf{c}}_k'\|^2 + \|\widetilde{\mathbf{a}}_i'' - \widetilde{\mathbf{c}}_k''\|^2 = l_{ac}^2\,,$$
$$\|\widetilde{\mathbf{b}}_i' - \widetilde{\mathbf{c}}_k'\|^2 + \|\widetilde{\mathbf{b}}_i'' - \widetilde{\mathbf{c}}_k''\|^2 = l_{bc}^2$$

imply together with $\widetilde{\mathbf{a}}_i'' = \widetilde{\mathbf{b}}_i''$

$$\|\widetilde{\mathbf{a}}_i' - \widetilde{\mathbf{c}}_k'\|^2 - \|\widetilde{\mathbf{b}}_i' - \widetilde{\mathbf{c}}_k'\|^2 = l_{ac}^2 - l_{bc}^2\,. \tag{11}$$

Suppose that $\mathbf{a}_1$ and $\mathbf{b}_1$ are kept fixed, i.e., $\widetilde{\mathbf{a}}_1' = \mathbf{a}_1'$ and $\widetilde{\mathbf{b}}_1' = \mathbf{b}_1'$. Let $\mathbf{c}_0, \mathbf{d}_0$ denote the pedal points of $\mathbf{c}_i'$ and $\mathbf{d}_i'$ on the line $\mathbf{a}_1'\mathbf{b}_1'$ (see Fig. 4). Due to (1) and

$$\|\mathbf{a}_1' - \mathbf{c}_k'\|^2 - \|\mathbf{b}_1' - \mathbf{c}_k'\|^2 = \|\mathbf{a}_1' - \mathbf{c}_0\|^2 - \|\mathbf{b}_1' - \mathbf{c}_0\|^2$$

the points $\mathbf{c}_0$ and $\mathbf{d}_0$ must be also the pedal points of $\widetilde{\mathbf{c}}_i'$ and $\widetilde{\mathbf{d}}_i'$, respectively.

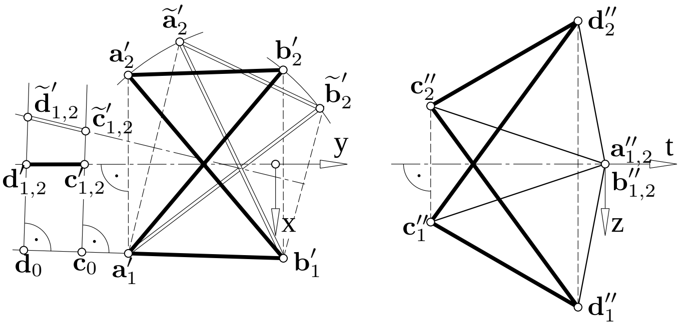

Figure 4: Generalized four-dimensional flexible cross-polytope $\mathcal{C}_4$ in top view and front view.

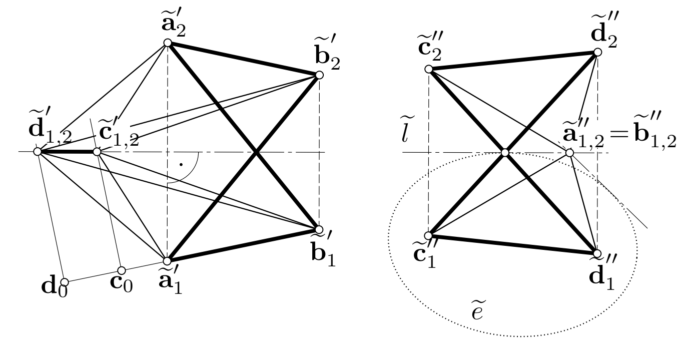

Figure 5: Flex $\widetilde{\mathcal{C}}_4$ of $\mathcal{C}_4$ from Fig. 4

After the top view of $\widetilde{\mathcal{C}}_4$ has been fixed, in the front view the dimensions of the antiparallelogram $\widetilde{\mathbf{c}}_1''\widetilde{\mathbf{d}}_1''\widetilde{\mathbf{c}}_2''\widetilde{\mathbf{d}}_2''$ as well as the distances $\|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{c}}_k''\|$ and $\|\widetilde{\mathbf{a}}_1'' - \widetilde{\mathbf{d}}_k''\|$ are defined. Because of $\gamma_3\delta_3 > 0$ in Theorem 1 we have $l_{cd} < \overline{l}_{cd}$, hence $\|\mathbf{c}_1'' - \mathbf{d}_1''\| < \|\mathbf{c}_1'' - \mathbf{d}_2''\|$ which implies also $\|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_1''\| < \|\widetilde{\mathbf{c}}_1'' - \widetilde{\mathbf{d}}_2''\|$.

We specify $\widetilde{\mathbf{c}}_1''$ and $\widetilde{\mathbf{d}}_1''$ and determine $\widetilde{\mathbf{a}}_1''$. As $\widetilde{\mathbf{a}}_1''$ has to be located on the axis of symmetry of the antiparallelogram $\widetilde{\mathbf{c}}_1''\widetilde{\mathbf{d}}_1''\widetilde{\mathbf{c}}_2''\widetilde{\mathbf{d}}_2''$, we construct a line $\widetilde{l}$ through $\widetilde{\mathbf{a}}_1''$ tangent to the ellipse $\widetilde{e}$, the fixed polode of the antiparallelogram motion (compare Fig. 2). Continuity guarantees uniqueness. The reflection in the tangent line $\widetilde{l}$ gives $\widetilde{\mathbf{c}}_2''$ and $\widetilde{\mathbf{d}}_2''$ (Fig. 5).

The limits for this flex $\widetilde{\mathcal{C}}_4$ are much more complex than that of Walz's example. On the one hand $\widetilde{\mathbf{a}}_1''$ must not be located in the interior of the ellipse $\widetilde{e}$. On the other hand all distances showing up in the top view must not be greater than the corresponding true lengths. A parametrization of the one-parameter motion of $\mathcal{C}_4$ is omitted here.

3. CONCLUSION

In this paper flexible cross-polytopes in $\mathbb{E}^4$ have been presented. There are many open problems left around this topic:

The characterization of first-order infinitesimal flexibility of cross-polytopes $\mathcal{C}_n$ in $\mathbb{E}^n$ seems to be similar to that in $\mathbb{E}^3$: Let $\mathcal{P}$ and $\overline{\mathcal{P}}$ be two complementary cross-polytopes of $\mathcal{C}_n$ of types $\mathcal{C}_{n/2}$ for even n and of type $\mathcal{C}_{(n+1)/2}$ and $\mathcal{C}_{(n-1)/2}$ for odd n. Then infinitesimal flexibility of order 1 is given if and only if the two complementary substructures $\mathcal{P}$ and $\overline{\mathcal{P}}$ are located on the same quadric, provided $\mathcal{P}$ is full-dimensional. However, a complete proof is open.

The cross-polytopes presented in Theorem 1 seem to be the only flexible cross polytopes in $\mathbb{E}^4$, and no non-trivially flexible cross-polytopes are expected for higher dimensions. However, a proof of these conjectures is left for future research, too.

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REFERENCES

Bricard, R., 1897, "Mémoire sur la théorie de l'octaèdre articulé", *J. math. pur. appl., Liouville*, Vol. 3, pp. 113–148.

Connelly, R., 1978, "A Flexible Sphere", *Math. Intell.*, Vol. 1, no. 3, pp. 130–131.

Connelly, R., Sabitov, I., and Walz, A., 1997, "The Bellows Conjecture", *Beitr. Algebra Geom.*, Vol. 38, pp. 1–10.

Dewdney, A.K., 1992, "Mathematische Unterhaltungen", *Spektrum der Wissenschaft*, März 1992, pp. 12–15.

Sabitov, I., 1995, "On the problem of invariance of the volume of a flexible polyhedron", *Russian Math. Surveys*, Vol. 50, no. 2, pp. 451–452.

Stachel, H., 1987, "Zur Einzigkeit der Bricardschen Oktaeder", *J. Geom.*, Vol. 28, pp. 41–56.

Stachel, H., 1990, "The Right-Angle-Theorem in Four Dimensions", *Journal of Theoretical Graphics and Computing*, Vol. 3, pp. 4–13.

Stachel, H., 1999, "Higher Order Flexibility of Octahedra", *Period. Math. Hung.*, Vol. 39, pp. 225–240.

Wunderlich, W., *1970, Ebene Kinematik*, BI–Hochschultaschenbücher, Band 447, Bibliographisches Institut, Mannheim.

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