Journal for Geometry and Graphics
Volume 6 (2002), No. 2, 133–139.

Remarks on A. Hirsch's Paper concerning Villarceau Sections

Hellmuth Stachel

*Institute of Geometry, Vienna University of Technology
Wiedner Hauptstr. 8-10/113, A 1040 Vienna, Austria
email: stachel@geometrie.tuwien.ac.at*

Abstract. When a surface of revolution with a conic as meridian is intersected with a bitangential plane, then the curve of intersection splits into two congruent conics. Conversely a necessary and sufficient condition is presented such that the rotation of a conic about a non-coplanar axis gives a surface with conics as meridians. Both results are proved by direct computation.

Keywords: Villarceau section

MSC 2000: 51N05, 51N35

1. Introduction

On August 28th 1848 Yvon Villarceau reported in [6] that a bitangential plane intersects a ring torus along two circles.[1] In the previous paper [3] A. Hirsch generalized this result with the following

Theorem 1 *Let Ψ be a surface of revolution with a conic m as meridian. Then any bitangential plane τ intersects Ψ along a reducible curve v consisting of two congruent conics which either are imaginary or of the same affine type as m.*

A. Hirsch's proof is based on the standard result of Algebraic Geometry saying that the number of singularities of any irreducible planar algebraic curve of order 4 is less than four. The type of the components of v is figured out by intuitive arguments. Besides, A. Hirsch presented in [3] real representations of imaginary Villarceau sections, too.

The aim of this paper is to prove Theorem 1 without using the above mentioned result. Direct computation of v gives also rise to a certain converse which first has been proved in [2], p. 55.

[1] Y. Villarceau indicated an analytical proof in his short note [6]. An elementary proof can be found in [1]. The most remarkable property that the Villarceau circles are isogonal trajectories of the meridian circles has probably first been proved 1891 (see references in [7], footnotes 3 and 5).

ISSN 1433-8157/$ 2.50

2. Proof of Theorem 1

We introduce a cartesian coordinate system $(O; x_1, x_2, x_3)$ and use the axis a of Ψ as x_3-axis. We have to distinguish two cases whether the bitangential plane τ is parallel to the axis a of Ψ or not.

Case 1: τ not parallel a

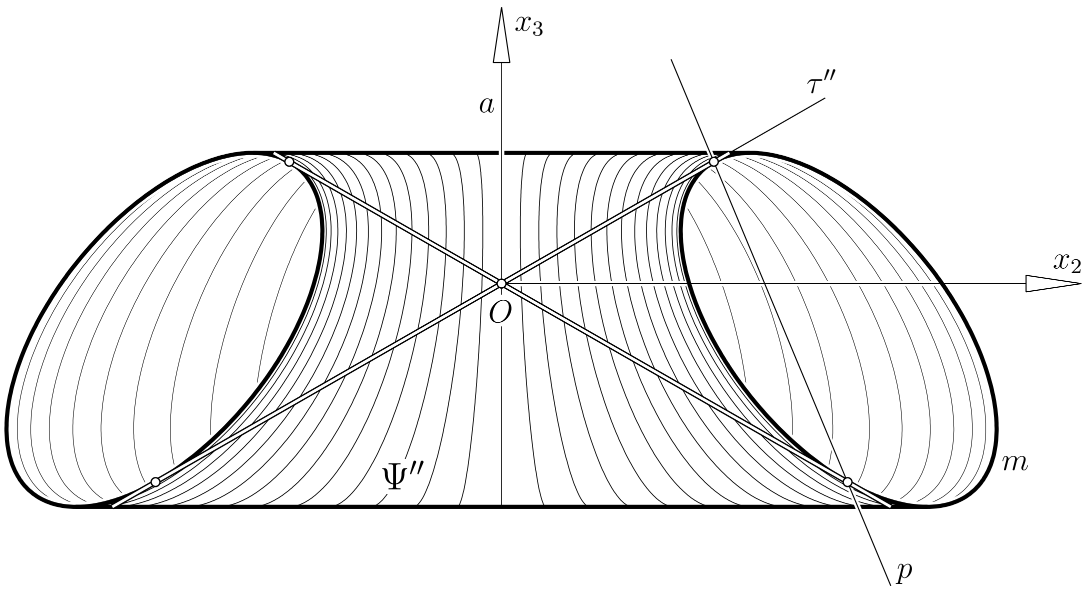

Figure 1: Case 1, the bitangential plane τ intersects the axis a at O

We specify the origin O of the coordinate system at the point of intersection between a and the bitangential plane τ by the definition that τ obeys the equation

$$\tau\colon\ x_3 = kx_2, \quad k \neq 0. \tag{1}$$

We furthermore assume for the meridian conic[2] m in the x_2x_3-plane, that the polar line p of O with respect to m (see Fig. 1 with front view in the x_2x_3-plane) has the equation

$$p\colon\ rx_2 + sx_3 = 1, \quad r, s \in \mathbb{R},\ r \neq 0. \tag{2}$$

Then m belongs to a pencil of conics which is spanned by the two tangent lines $x_3 = \pm kx_2$ and the polar line p with multiplicity 2. Therefore we can set up

$$m\colon\ l(x_3^2 - k^2x_2^2) + (rx_2 + sx_3 - 1)^2 = 0 \ \text{ for any } l \in \mathbb{R} \setminus \{0\}. \tag{3}$$

The condition $r \neq 0$ guarantees that m is not symmetric with respect to the x_3-axis; otherwise the swept surface would be part of a *quadric of revolution.* We assume $k^2 > 0$ thus confining ourselves to a real bitangential plane τ.

Eq. (3) results in the following equation of the surface of revolution

$$\Psi\colon\ \left[(r^2 - lk^2)(x_1^2 + x_2^2) + (l + s^2)x_3^2 - 2sx_3 + 1\right]^2 - 4r^2(sx_3 - 1)^2(x_1^2 + x_2^2) = 0. \tag{4}$$

[2]Under 'conic' we understand in the sequel an irreducible curve of second order.

Points of the '*Villarceau section*' $v = \Psi \cap \tau$ fulfill the equations of τ and Ψ simultaneously. Therefore by substituting x_3 from (1) in (4) we get the equation for the *top view* v' of v in the x_1x_2-plane:

$$v'\colon\ \big[(r^2 - lk^2)x_1^2 + (r^2 + k^2s^2)x_2^2 - 2ksx_2 + 1\big]^2 - 4r^2(ksx_2 - 1)^2(x_1^2 + x_2^2) = 0. \qquad (5)$$

This equation can be rewritten as

$$\big[(r^2 - lk^2)x_1^2 + (r^2 - k^2s^2)x_2^2 + 2ksx_2 - 1\big]^2 - 4lk^2(ksx_2 - 1)^2x_1^2 = 0. \qquad (6)$$

For verifying the identity of the left hand sides in (5) and (6) we combine the two terms with the brackets on one side, the rest on the other. Then we decompose on each side the difference of two squares into the product of the sum and the difference.

Over $\mathbb{C}$ the quartic v' with eq. (6) splits into the two conics c_1', c_2' obeying

$$c_j'\colon\ (r^2 - lk^2)x_1^2 + (r^2 - k^2s^2)x_2^2 \pm 2k\sqrt{l}(ksx_2 - 1)x_1 + 2ksx_2 - 1 = 0. \qquad (7)$$

c_2' is symmetric to c_1' with respect to the x_1-axis.

Only under $l > 0$ the equations of these conics have real coefficients. If τ is real, i.e., $k^2 > 0$, then the points of contact with Ψ are real, too. Hence, under $l > 0$ and $k^2 > 0$ the components c_1, c_2 of v are ellipses, parabolas or hyperbolas according to the sign of the discriminant D_c of the quadratic form included in the left hand side of eq. (7). We compute

$$D_c = r^2(k^2s^2 + k^2l - r^2).$$

On the other hand the discriminant D_m for the meridian conic with equation (3) reads

$$D_m = l(k^2s^2 + k^2l - r^2) = \frac{l}{r^2}\,D_c\,.$$

This reveals that the components of v are either imaginary ($l < 0$) or they share their affine type with m.

Case 2: τ parallel a

We set up the equation of the bitangential plane τ as

$$\tau\colon\ x_1 = k\,, \quad k \neq 0, \qquad (8)$$

thus excluding the trivial case with m tangent to a. The meridian m in the x_2x_3-plane belongs to a pencil of conics which is spanned by the pair of tangent lines $x_2 = \pm k$ and the double-line p polar to the point at infinity of the axis a (see Fig. 2). Therefore we can set up

$$m\colon\ l(x_2^2 - k^2) - (x_3 - rx_2)^2 = 0 \ \text{ for any } l, r \in \mathbb{R} \setminus \{0\}. \qquad (9)$$

In analogy to (4) we obtain the equation of Ψ,

$$\Psi\colon\ \big[l(x_1^2 + x_2^2 - k^2) - x_3^2 - r^2(x_1^2 + x_2^2)\big]^2 = 4r^2x_3^2(x_1^2 + x_2^2). \qquad (10)$$

By setting $x_1 = k$ we get for the Villarceau section $v = \Psi \cap \tau$

$$v\colon\ \big[(r^2 - l)x_2^2 + x_3^2 + r^2k^2\big]^2 - 4r^2(k^2 + x_2^2)x_3^2 = 0. \qquad (11)$$

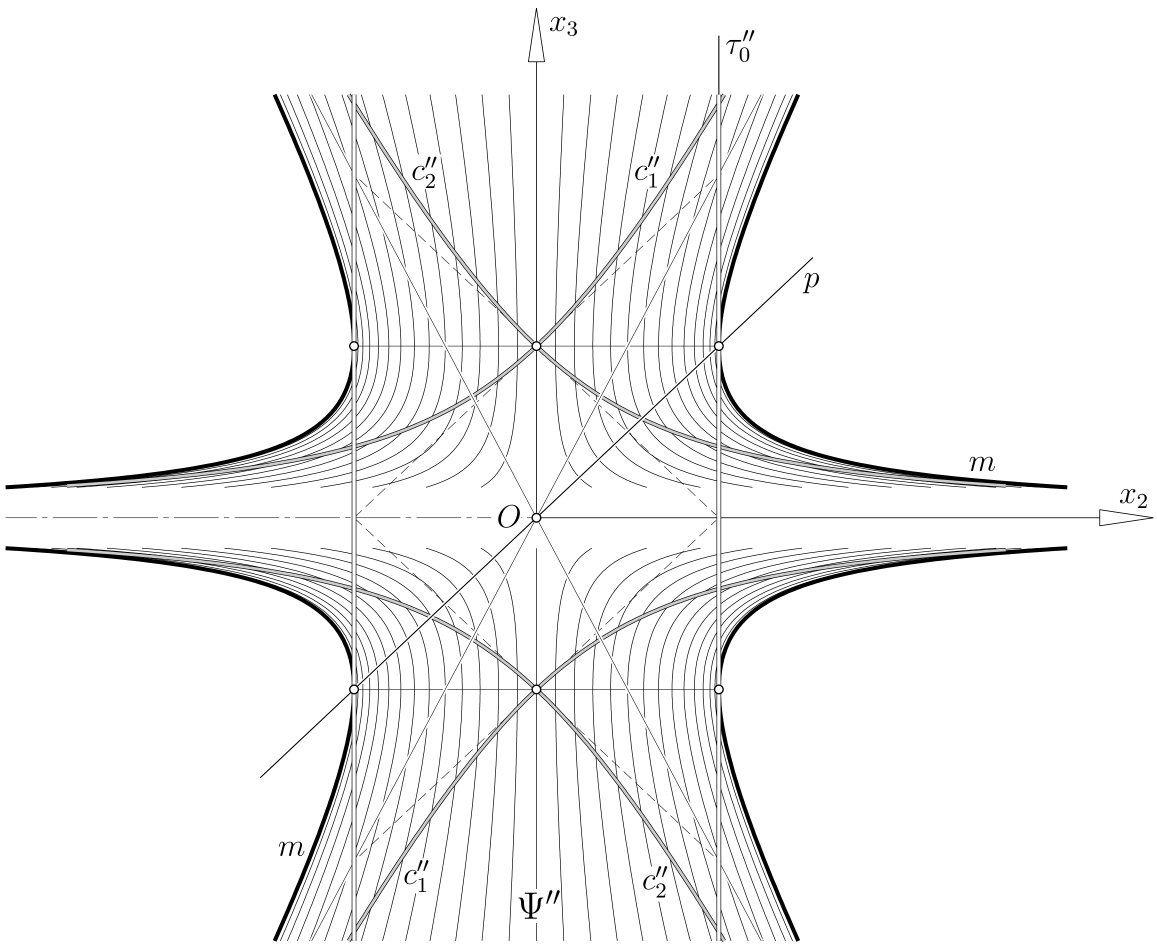

Figure 2: Case 2, the bitangential plane τ is parallel to a (special case $l = r^2$)

This can be rewritten as

$$v\colon\ \left[(r^2 - l)x_2^2 - x_3^2 + r^2k^2\right]^2 - 4lx_2^2x_3^2 = 0 \tag{12}$$

which proves that v splits into the two conics c_1, c_2 with equations

$$c_j\colon\ (r^2 - l)x_2^2 - x_3^2 + r^2k^2 \pm 2\sqrt{l}\, x_2x_3 = 0. \tag{13}$$

Only under $l > 0$ these equations have real coefficients, and in analogy to Case 1 the discriminants $D_m = l$ of m and $D_c = r^2$ of c_j obey again

$$D_m = \frac{l}{r^2}\, D_c\,.$$

c_j is either imaginary ($l < 0$) or it is a hyperbola like m. □

3. A converse of Theorem 1

The same quartic surface Ψ — or at least a portion of it — can also be generated when we rotate any component c_j of v about the axis a. Beside the parallel circles in planes orthogonal to a the surface Ψ contains *three families of conics*, the meridian curves in planes through a and the two families generated by c_1 and c_2, respectively. The latter two are mirror images from each other with respect to any meridian plane.

When in the generic case any conic is revolved about an axis a then it constitutes a surface of 4^{th} order with irreducible quartics in the meridian planes (see [2] or [8], vol. I, p. 143).[3] Hence the previously defined curves c_j must have any specific property. This is clarified in the following theorem which dates back to [2], p. 55:[4]

Theorem 2 *Let c be any conic in the plane τ. Suppose that c is rotated about an axis a which is neither perpendicular to τ nor located in τ. We further exclude that c is symmetric with respect to any plane through the axis a.*

Then the swept surface Ψ has conics as meridian curves if and only if both isotropic planes through a are tangent to c.[5] This means for Case 1 (τ, a intersecting) that orthogonal projection parallel to a maps c onto a conic c' with a focal point on the axis a.

Proof: We again discuss the two previously defined cases separately: In Case 1 we rewrite eq. (7) in the form

$$c_j'\colon\ r^2(x_1^2+x_2^2)-\left(\pm\sqrt{l}\,kx_1-ksx_2+1\right)^2=0 \tag{14}$$

which (in comparison with (3)) reveals that the top view c_j' of c_j in the x_1x_2-plane is tangent to the isotropic lines $x_2 \pm ix_2 = 0$, $i^2=-1$, passing through the origin O. This characterizes O as a *focal point* of c_j' and the line

$$f\colon\ \pm\sqrt{l}\,kx_1-ksx_2+1=0$$

as its polar, the so-called *directrix* of c_j'. Alternatively, we conclude from eq. (14) that the distances of any point $X\in c_j'$ from the focal point O and from the *directrix* f are proportional. More precisely, we obtain the *Apollonian property*

$$X\in c_j' \iff \overline{XO}=\left|\frac{k\sqrt{l+s^2}}{r}\right|\overline{Xf}.$$

In Case 2 we rewrite eq. (13) as

$$c_j\colon\ r^2(x_2^2+k^2)-(x_3\mp\sqrt{l}\,x_2)^2=0. \tag{15}$$

The isotropic planes through a intersect $\tau\colon\ x_1=k$ along the lines $x_2=\pm ik$. Eq. (15) reveals in analogy to (9) that c_j contacts these lines at points on the diameter $x_3=\pm\sqrt{l}\,x_2$. Note that the replacement of the constants (k^2,r) in (9) by $(-k^2,\pm\sqrt{l})$ exchanges m and c_j.

[3] In the generic case Ψ contains even *four real and two imaginary families of conics* (see [2], p. 17ff).

[4] The first part of Theorem 2 has already been stated in [4].

[5] This necessary and sufficient condition is equivalent to the statement: *Any pair of orthogonal planes through the axis a intersects the plane τ at lines which are conjugate with respect to c.* This formulation has the advantage to avoid the complex extension of the Euclidean space.

Conversely: Let a conic $c \subset \tau$ be given according to the conditions of Theorem 2. In Case 1 (τ not parallel a) we set up

$$\left.\begin{array}{r} c'\colon\ \lambda(x_1^2+x_2^2)-(\rho k x_1-\sigma k x_2+1)^2=0 \\ c\subset\tau\colon\quad x_3=kx_2 \end{array}\right\} k,\rho\neq 0, \lambda>0. \qquad (16)$$

The comparison with (14) gives

$$l=\rho^2, \quad r=\sqrt{\lambda}, \quad s=-\sigma.$$

Now we only need to follow the computation above in the reverse order. c generates the surface Ψ of revolution. Its complete meridian in the x_2x_3-plane consists of the conics

$$m_{1,2}\colon\ \rho^2(x_3^2-k^2x_2^2)+\left(\pm\sqrt{\lambda}\,x_2-\sigma x_3-1\right)^2=0. \qquad (17)$$

In Case 2 (τ parallel a) we can set up the equation of c as

$$c\colon\ \rho(x_2^2+k^2)-(x_3-\lambda x_2)^2=0, \quad \tau\colon\ x_1=k\,, \quad k,\lambda\neq 0,\ \rho>0\,. \qquad (18)$$

We compare this equation with (15) and notice that the swept surface Ψ has a meridian m obeying (9) for

$$r=\sqrt{\rho}, \quad l=\lambda^2. \qquad \square$$

Remarks: 1) By the conditions $\rho\neq 0$ in (16) and $\lambda\neq 0$ in (18) we exclude that a symmetry plane of c passes through the axis a of rotation. Otherwise the swept surface Ψ would be quadratic. And then proofs of DANDELIN type (see e.g. [8], vol. I, p. 65) give rise to metrical properties of conics (cf. [5]).

2) There is another way to conclude that the condition given in Theorem 2 is *necessary* for meridian conics:

Any rotation R about a[6] keeps the two isotropic planes $\mu_j\colon\ x_1\pm ix_2=0$ through a fixed. Each point in these planes preserves its x_3-coordinate under R and traces a line under the continuous rotation. When therefore the conic c intersects μ_j at the point $(\xi_1,\pm i\xi_1,\xi_3)$, $\xi_1,\xi_3\in\mathbb{C}$, then the swept surface Ψ meets the axis a at the point $(0,0,\xi_3)$.

For the required type of quartic surface Ψ the points of intersection with a must pairwise coincide with the points where any meridian conic m_j meets the axis. Since any symmetry has been excluded (cf. Remark 1), the x_3-coordinates of the points of intersection $c\cap\mu_1$ differ from those of $c\cap\mu_2$. Therefore c must intersect each μ_j at two coinciding points.

References

[1] Anonymous: *Section du tore. Par un plan tangent á cette surface et passant par son centre.* Nouvelles Annales de Mathématiques **18**, 258–261 (1859).

[2] J. DE LA GOURNERIE: *Mémoire sur la surface engendrée par la revolution d'une conique autour d'une droite.* Journal de l'École impériale Polytechnique **40**, 1–74 and 8 plates (1863).

[3] A. HIRSCH: *Extension of the 'Villarceau-Section' to Surfaces of Revolution with a Generating Conic.* J. Geometry Graphics **6**, 121–132 (2002).

[6] more exactly: The projective continuation of R into the complex affine space.

[4] M. Moutard: *Question 497.* Nouvelles Annales de Mathématiques **18**, 444–445 (1859).

[5] H. Stachel: *Einige metrische Kegelschnittseigenschaften.* Praxis der Mathematik **31**, 244–247 (1989).

[6] Y. Villarceau: *Note concernant un trosiéme système de sections circulaires qu'admet le tore circulaire ordinaire.* Comptes rendus hebdomadaires des Séances de l'Académie des Sciences **27**, 246 (1848).

[7] W. Wunderlich: *Über die Torusloxodromen.* Monatsh. Math. **56**, 313–334 (1952).

[8] W. Wunderlich: *Darstellende Geometrie I, II.* Bibliographisches Institut, Mannheim 1966, 1967.

Received November 30, 2002