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STROPHOIDS, A FAMILY OF CUBIC CURVES WITH REMARKABLE PROPERTIES

Abstract: *Strophoids are circular cubic curves which have a node with orthogonal tangents. These rational curves are characterized by a series or properties, and they show up as locus of points at various geometric problems in the Euclidean plane: Strophoids are pedal curves of parabolas if the corresponding pole lies on the parabola's directrix, and they are inverse to equilateral hyperbolas. Strophoids are focal curves of particular pencils of conics. Moreover, the locus of points where tangents through a given point contact the conics of a confocal family is a strophoid. In descriptive geometry, strophoids appear as perspective views of particular curves of intersection, e.g., of Viviani's curve. Bricard's flexible octahedra of type 3 admit two flat poses; and here, after fixing two opposite vertices, strophoids are the locus for the four remaining vertices. In plane kinematics they are the circle-point curves, i.e., the locus of points whose trajectories have instantaneously a stationary curvature. Moreover, they are projections of the spherical and hyperbolic analogues. For any given triangle ABC, the equicevian cubics are strophoids, i.e., the locus of points for which two of the three cevians have the same lengths.*

On each strophoid there is a symmetric relation of points, so-called 'associated' points, with a series of properties: The lines connecting associated points P and P' are tangent of the negative pedal curve. Tangents at associated points intersect at a point which again lies on the cubic. For all pairs (P, P') of associated points, the midpoints lie on a line through the node N. For any two pairs (P, P') and (Q, Q') of associated points, the points of intersection between the lines PQ and P'Q' as well as between PQ' and P'Q are again placed on the strophoid and mutually associated. The lines PQ and PQ' are symmetric with respect to the line connecting P with the node. Thus, strophoids generalize Apollonian circles: For given non-collinear points A, A' and N the locus of points X such that one angle bisector of the lines XA and XA' passes through N is a strophoid.

Key words: *Strophoid, rational cubic curves, plane kinematics, pedal curve, focal curve, equicevian curve, Viviani's curve.*

1. INTRODUCTION

The history of plane curves of degree 3 started most probably in the late 1600s with Newton's classification. Many prominent mathematicians studied cubics, e.g., Clairaut, Plücker, Hesse, Weierstrass, or Poincaré.

With respect to ('w.r.t.' in brief) singularities, there are three types of irreducible cubics to distinguish (see, e.g., [3, 5]), the cubics with a cusp (type 1), those with a node with either two real or two conjugate complex tangents (type 2) and those without any singularity (type 3). By virtue of Plücker's formulas, cubics of type 3 are of class 6 and have 9 real or conjugate complex inflection points; cubics with a node are of class 4 and have 3 inflection points; cubics with a cusp have the class 3 and one inflection point. Only cubics with a singularity are rational, i.e., they admit a rational parametrization.

We focus on a particular cubic of type 2, which surprisingly shows up as a geometric locus of points at many different geometric problems in Euclidean geometry. Most of the presented theorems can already be found in the literature.

Definition 1: An irreducible cubic is called *circular* if it passes through the absolute circle-points. A circular cubic is called *strophoid* if it has a node with orthogonal tangents. A strophoid without an axis of symmetry is called *oblique*, otherwise *right* (see, e.g., [15], pp. 37−39, [10], pp. 63−67, [9] or [12]).

In Section 2 we present some of the various properties of strophoids. In Section 3 we pick out some geometric problems where strophoids play an important role as a geometric locus.

2. PROPERTIES OF STROPHOIDS

2.1 Equations of Strophoids

We use cartesian coordinates (x, y) with the node N as origin and with the two tangents t_1, t_2 at N as coordinate axes (Fig. 1). Then we can set up the equation of the strophoid S as

$$(x^2 + y^2)(ax + by) - xy = 0 \qquad (1)$$

with constants $a, b \in \mathbb{R}$, $(a,b) \neq (0,0)$. In the cases $a = 0$ or $b = 0$ the cubic is reducible; it splits into a circle through N and the diameter line $y = 0$ or $x = 0$. In the case $a = b$ we obtain a right strophoid.

When using homogeneous coordinates $(X_0 : X_1 : X_2) = (1 : x : y)$, we obtain the homogeneous equation

$$(X_1^2 + X_2^2)(aX_1 + bX_2) - X_0 X_1 X_2 = 0 . \qquad (2)$$

It reveals that S intersects the line at infinity $X_0 = 0$ at the absolute circle-points $(0 : 1 : \pm i)$ and the real point $F' = (0 : b : a)$ with the (real) asymptote satisfying

$$a(a^2+b^2)x+b(a^2+b^2)y+ab=0. \qquad (3)$$

The conjugate complex tangents to S at the absolute circle-points obey the equations

$$\mp iX_0 + 2(a\pm ib)X_1 \pm 2i(a\pm ib)X_2 = 0.$$

They intersect at the point

$$F = \left(\frac{b}{2(a^2+b^2)}, \frac{a}{2(a^2+b^2)} \right), \qquad (4)$$

which is called the *focal point F* of S. It is easy to verify that F is again a point of S. The line g, which is orthogonal to NF and passes through F, intersects the cubic in the two points

$$G = \left(\frac{1}{2(a+b)}, \frac{1}{2(a+b)} \right), \quad G' = \left(\frac{-1}{2(a-b)}, \frac{1}{2(a-b)} \right),$$

which respectively lie on the angle bisectors $y = x$ or $y = -x$ of the orthogonal tangents t_1, t_2 to S at the node N.

The ratio $a : b$ of the coefficients in (1) can be seen as a shape-parameter of the strophoid S. By virtue of eqs. (3) and (4), $\psi = \arctan a/b$ is the angle between one tangent t_1 at the node N and both, either the asymptote or the connecting line NF. We can demand $0 < \psi \le \pi/4$. The lower limit defines the reducible case, the maximum gives a right strophoid.

Lines through the node N intersect the cubic in one remaining point which we can set up in polar coordinates by $(x, y) = (r \cos\ , r \sin\ )$. Then by (1) we obtain the polar equation

$$r = \frac{1}{\dfrac{a}{\sin\varphi} + \dfrac{b}{\cos\varphi}} \qquad (5)$$

and the parametrization

$$x = \frac{\sin\varphi \cos^2\varphi}{a\cos\varphi + b\sin\varphi}, \quad y = \frac{\sin^2\varphi \cos\varphi}{a\cos\varphi + b\sin\varphi}. \qquad (6)$$

When we apply the inversion, i.e., the reflection in the unit circle $\mathcal{K}$ (see Fig. 1), we obtain a curve $\mathcal{H}$ with the polar equation

$$r = \frac{a}{\sin\varphi} + \frac{b}{\cos\varphi} \qquad (7)$$

and the parametrization

$$x = a\cot\varphi + b, \quad y = a + b\tan\varphi. \qquad (8)$$

$\mathcal{H}$ satisfies the equation

$$(x-b)(y-a) = ab. \qquad (9)$$

This reveals that $\mathcal{H}$ is an equilateral hyperbola (Fig. 1). The center (a,b) of $\mathcal{H}$ is aligned with N and the focal point F.
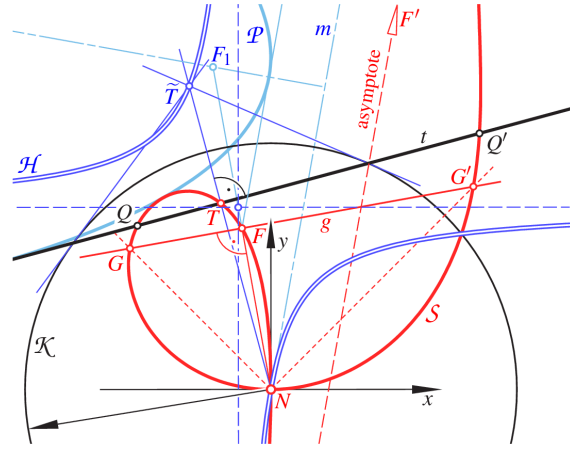

Fig. 1 The strophoid S is invers to an equilateral hyperbola $\mathcal{H}$ and the pedal curve of the parabola $\mathcal{P}$.

The polarity in the unit circle $\mathcal{K}$ transforms the points (4) of the hyperbola $\mathcal{H}$ onto lines with the equation

$$U_0 + U_1 x + U_2 y = \frac{-1}{a + b\tan\varphi} + x\cot\varphi + y = 0.$$

Their homogeneous line coordinates (U_0, U_1, U_2) obey the equation

$$aU_0U_1 + bU_0U_2 + U_1U_2 = 0, \qquad (10)$$

which is satisfied by the line at infinity $(U_0, U_1, U_2) = (1, 0, 0)$ as well as by the two tangents $t_1 = (0,1,0)$ and $t_2 = (0,0,1)$ of S at the node N. Therefore, eq. (6) is the tangential equation of a parabola $\mathcal{P}$. After inverting its symmetric coefficient matrix we obtain the homogeneous (point-) equation

$$(bx-ay)^2 - 2(bx+ay) + 1 = 0.$$

A dilatation with center N and factor 2 maps the focal point F of S onto the focal point F_1 of $\mathcal{P}$ (see Figs. 1 and 3). The directrix m of $\mathcal{P}$ passes through N and is parallel to the asymptote of the strophoid S. Proofs can be found in [1].

The product of the polarity and the inversion w.r.t. the unit circle $\mathcal{K}$ is the pedal transformation w.r.t. the origin N. The same is of course valid for all circles with center N. This confirms a well-known result (see, e.g., [15]).

Theorem 1: *The strophoid S is the pedal curve of a parabola $\mathcal{P}$ w.r.t. the node N, provided N is a point on the parabola's directrix m. On the other hand, the inversion in any circle $\mathcal{K}$ with center N maps S onto an equilateral hyperbola $\mathcal{H}$ (see Fig. 1).*
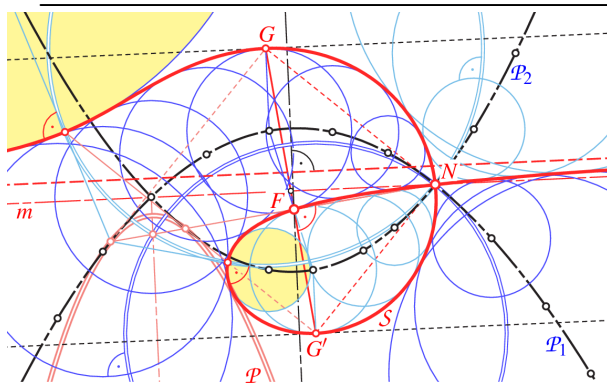
Fig. 2 Inversions in circles through *N* and with centers *G* or *G'* map the strophoid *S* onto itself. Therefore, *S* is the envelope of circles centered on two confocal parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

The inversion in $\mathcal{K}$ transforms the reflections in the axes of the hyperbola $\mathcal{H}$ onto inversions in circles passing through the node *N*. One circle has the center *G*, the other the center *G'*. The inversion in $\mathcal{K}$ maps the circles, which are centered on an axis and contact the hyperbola at two points, onto circles with double contact with the strophoid *S*. It can be proved (see [12]) that the centers of these circles (see Fig. 2) are located on two confocal parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$ with the focus *F*. The axis of these parabolas is orthogonal to the median line *m* of *S*. The associated directrices are tangent to *S* at *G* or *G'*. The two parabolas contact the negative pedal curve $\mathcal{P}$ in points of the line NF.

2.2 Associated Points of a Strophoid

For any point *T* of the strophoid *S*, *T ≠ N*, the line *t* through *T* and orthogonal to *TN* is tangent to the parabola $\mathcal{P}$, the ₋negative pedal curveø of *S*. There are two remaining, not necessarily real points *Q, Q'* of intersection between *t* and the strophoid *S*.

Definition 2: Tangents *t* of the negative pedal curve $\mathcal{P}$ of the strophoid *S* intersect the cubic *S* beside the pedal point *T* w.r.t. the node *N* in two real or conjugate complex points *Q* and *Q'*. We call them *associated points* of the cubic *S*.

Many properties of strophoids are related to pairs of associated points.

From each finite point *Q ∈ S \ {N}* two tangent lines to the parabola $\mathcal{P}$ can be drawn. One of them is normal to *NQ*. The other has, beside the pedal point w.r.t. *N*, exactly one remaining point *Q'* of intersection with *S*. Hence, for each point *Q ∈ S* there exists a unique associated point *Q'*; the relation between *Q* und *Q'* is one-to-one and symmetric.

Theorem 2: *Let S be a strophoid with the node N. For any pair (Q, Q') of associated points of S with Q,Q' ≠ N the connecting lines QN and Q'N with N are symmetric w.r.t. the tangents t_1 and t_2 at N (see Fig. 3).*

Proof: The line connecting the points (6) to the polar angles and ó has line coordinates

$$U_0 : U_1 : U_2 = \sin^2\varphi \, \cos^2\varphi : -b\sin^2\varphi : -a\cos^2\varphi ,$$

which satisfy eq. (10). On the other hand, its pedal point with respect to *N*, $(\cos^4\varphi / b, \; \sin^4\varphi / a)$, is in general different from *G* and *G'*.
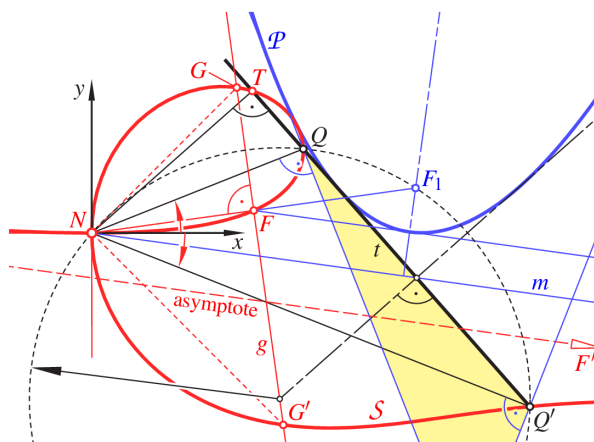

Fig. 3 The line *t* through *T ∈ S* and normal to *TN* intersects *S* beside *T* in a pair (*Q, Q'*) of associated points *S*.

The inversion $S \rightarrow \mathcal{H}$ mentioned in Theorem 1 sends associated points of the strophoid *S* onto diametral points of $\mathcal{H}$. Under the polarity $\mathcal{H} \rightarrow \mathcal{P}$ in the circle $\mathcal{K}$ they are mapped onto pairs of tangents which intersect on a line, which is orthogonal to *FN* and passes through the focal point F_1 of the parabola $\mathcal{P}$.

Remark 1: The lines through *Q* and *Q'*, respectively orthogonal to *QN* and *Q'N*, intersect at a point which lies together with *N*, *Q* and *Q'* on a circle (dotted in Fig. 3). This is the circumcircle of a trilateral (shaded in Fig. 3), which consists of three tangents of the parabola $\mathcal{P}$. Therefore the circumcircle contains the focal point F_1 of $\mathcal{P}$. Hence, its center lies on the perpendicular bisector of the segment NF_1. This bisector passes through *F* and is therefore identical with the line $g = GG'$. This gives rise to a ruler-and-compass construction of associated points (*Q, Q'*) on a given tangent *t* of the parabola $\mathcal{P}$ (see Fig. 3). When *Q* or *Q'* tends to the node *N*, the circle centered on *g* becomes the osculating circle of one of the two branches of *S* with the origin *N*.

Some properties of associated points on strophoids have a projective background. Therefore we deal in the coming subsection with a projective view on cubics of type 2.

2.3 Projective Properties of Cubics with a Node

Let $\mathcal{C}$ be a cubic with the node *N* (Fig. 4). Each line through *N* intersects $\mathcal{C}$ beside *N* in a single point. This defines a map of the pencil *N* of lines onto points of $\mathcal{C}$ which is one-to-one for lines which differ from the tangents t_1, t_2 at *N*, while both tangents t_1 and t_2 are sent to *N*.

The following definition refers to this correspondance.

Definition 3: Let $\mathcal{C}$ be a cubic with the node N. The involution in the pencil N with the two tangents t_1, t_2 at N as fixed lines induces an *involution* of type 1 on $\mathcal{C}$. Any involution in the pencil N which interchanges t_1 and t_2 induces an *involution* of type 2 on $\mathcal{C}$.
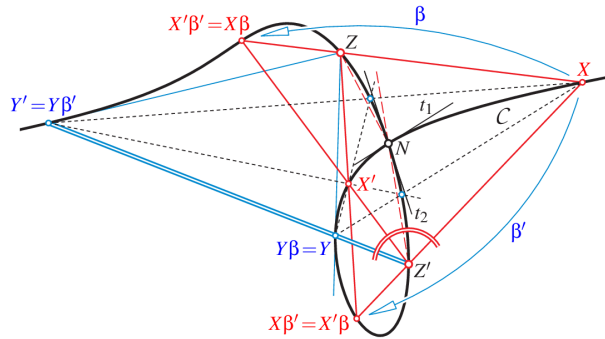

Fig. 4 On the cubic $\mathcal{C}$ with the node N we consider involutions , and ø= which commute pairwise.

Lemma 1:

1. *For any involution of type 2 on $\mathcal{C}$, all lines which connect corresponding points X, $X' \in \mathcal{C}$, $X \neq X'$, have a point $Z \in \mathcal{C} \setminus \{N\}$ in common, the center of . Also the tangent lines at the fixed points Y and Y' of pass through the center Z (see Fig. 4).*
2. *Each involution of type 2 commutes with the involution α of type 1. Therefore α maps each pair (X, X') of points corresponding under again onto such a pair, and vice versa. In particular, the fixed points Y and Y' of are corresponding under .*
3. *Each involution of type 2 defines another involution ø= = of type 2, and the involutions , and ø commute pairwise. The centers Z of and Z' of ø are corresponding under .*
4. *The lines connecting Z' with corresponding points X, $X' \in \mathcal{C} \setminus \{N, Z'\}$, constitute an involution in the pencil Z'. This involution keeps the line $Z'N$ fixed as well as the line through the fixed points of (Fig. 4).*
5. *For each quadrangle formed by pairs of points (X, X') and (Y, Y'), both corresponding under , the diagonal points $XY \cap X'Y'$ and $XY' \cap X'Y$ lie on $\mathcal{C}$, and they are corresponding under , as well.*

Proof: In the line pencil N and likewise on $\mathcal{C}$, two different involutions commute if and only if the fixed lines of one involution are corresponding under the other involution. Exactly in this case the composition ø= is an involution, too. For further details on the proof of items 1−5 we refer to the proof of Lemma 7 in [1].

Item 6 follows, since for any given pair (X, Y), $X, Y \in \mathcal{C} \setminus \{N\}$, there is an involution of the second kind with $X \rightarrow Y$, and, by item 2, at the same time with $X' \rightarrow Y'$. On the other hand, for any given point $Z \in \mathcal{C} \setminus \{N\}$ there is such an involution with the center Z.

2.4 Metrical Properties of Strophoids

Now we apply Lemma 1 to strophoids S. By virtue of Theorem 2, associated points on S are exactly corresponding under the involution of type 1.

Theorem 3: *Let S be any strophoid.*

1. *For S, the focal point F and the real point F' at infinity are associated.*
2. *The midpoint of associated points X, X' lies on the line m through N which is parallel to the asymptote.*
3. *The tangents of S at associated points X, X' meet each other at the point $T' \in \mathcal{C}$, which is associated to the pedal point T on the line $t = XX'$ w.r.t. N.*
4. *For any point $P \in S$, the lines PX and PX' are symmetric w.r.t. the line which connects P with the node N.*
5. *For any quadrangle formed by two pairs (X, X') and (Y, Y') of associated points of S the sides XY, YX', $X'Y'$ and $Y'X$ are tangent to a circle with center N. Therefore an alternate sum of their lengths vanishes.*

Proof: Let be the involution on S, which is induced by pairs of orthogonal lines through N. The absolute circle-points are the fixed points; therefore the focal point F of S is the center, and the remaining point F' at infinity is associated to F. By Lemma 1, item 4, the lines connecting F' with pairs (P, P') of associated points as well as those of are symmetric with respect to the median line $m = F'N$. The involution of lines $Z'X \rightarrow Z'X'$ mentioned in Lemma 1, item 4, includes also the two isotropic lines through Z'. As a consequence, the pairs $Z'X$ and $Z'X'$ are symmetric w.r.t. $Z'N$.

By virtue of items 1 and 2 of Lemma 1, the lines tangent to S at any pair (Q, Q') of associated points intersect each other at a point T' which is associated to the pedal point T of the line QQ' w.r.t. N. Due to the class 4 of S, beside the tangent of S at T', not more than two tangents of S can be drawn through T'.

Remark 2: Corresponding points X, X' of the involution mentioned above, bound a diameter of a circle which is centered on the line m and passes through the node N. The diameter line connecting X and X' passes through the focal point F. Beside the circles defined in Remark 1 (Fig. 3), also these circles with diameter X-X' can be used for a pointwise construction of the strophoid [12].

In Figs. 1 and 3 we find several pairs of associated points on S ó beside the absolute circle points and (F, F'): On the line g we have (G, G'); their tangents are parallel to m. The line t passing through T orthogonal to NT, contains the pair (Q, Q'). By virtue of Theorem 3, item 3, the tangents at these pairs intersect at a point of S. The tangential equation (6) of $\mathscr{P}$ shows that the two tangents t_1, t_2 of S at the node N are also tangent to $\mathscr{P}$. Hence, point N is self-associated.

Remark 3: By the same token, for circular cubics $\mathcal{C}$ with a node N the following statements are equivalent: (i) the tangents at N are orthogonal, (ii) the two absolute circle-points are associated, and (iii) the focal point F is a point of the cubic $\mathcal{C}$. In this sense, Definition 1 of strophoids

could be modified. Furthermore, the property claimed in Lemma 1, item 1, characterizes associated points on *S*.

We recall, that for three points *A, A'* and *N* on a line *l* the locus of points *X*, for which the line *XN* bisects the angle between the lines *XA* and *XA'*, is a circle centered on the line *l*. This is the so-called circle of Apollonius. But also points on the line *l* itself satisfy the required condition; hence, the complete geometric locus is a reducible cubic. The following generalization is closely related to item 4 of Theorem 3.

Theorem 4: *Let A, A' and N be three non-collinear points such that N does not lie on the perpendicular bisector of the segment AA'. Then the locus of points X with the property that one angle bisector of the two lines XA and XA' passes through N is a strophoid S with N as its node and (A, A') as a pair of associated points (see Fig. 5). The respective second angle bisectors envelope a parabola 𝒫, the negative pedal curve of S w.r.t. N.*
This is also valid when A' is a point at infinity; then A is the focal point of S.

Proof: There is a unique strophoid *S* with the node *N* and with (*A, A'*) as a pair of associated points. This follows from the property, that there is a unique parabola 𝒫 which contacts the following four lines: the angle bisectors of *NA* and *NA'* and the lines through *A* and *A'* which respectively are orthogonal to *NA* and *NA'*. The wanted strophoid is the pedal curve of 𝒫.

All points *X* of the strophoid *S* satisfy the required condition. However, there are no other points for the following reason: On each line *h* through *N* there is at most one point *Y* with this property, since the line *YA'* must pass through the mirror point of *A* w.r.t. *h*.
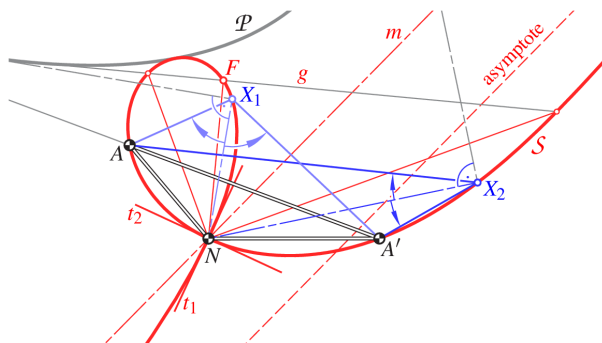

Fig. 5 The strophoid *S* is the locus of points *X* such that *XN* bisects the angle between *XA* and *XA'*, which holds for all pairs of associated points (*A, A'*) of *S*.

3. STROPHOIDS AS A GEOMETRIC LOCUS

3.1 Strophoids and conics

As a consequence of Theorem 4, strophoids play a role for conics in the following sense.

Theorem 5: *Let (A , a) and (A', a') be two given line elements with A∉a' and A'∉a such that a and a' are not symmetric w.r.t. the perpendicular bisector of the seg-*

ment *AA'* (Fig. 6). *Then the locus of focal points of all conics through the given line elements is a strophoid with A and A' as associated points and the point N = a ∩ a' as its node. The pairs of the real focal points as well as of the imaginary ones are associated, too. Their connecting lines, i.e., the axes of the conics, are tangent to a parabola.*
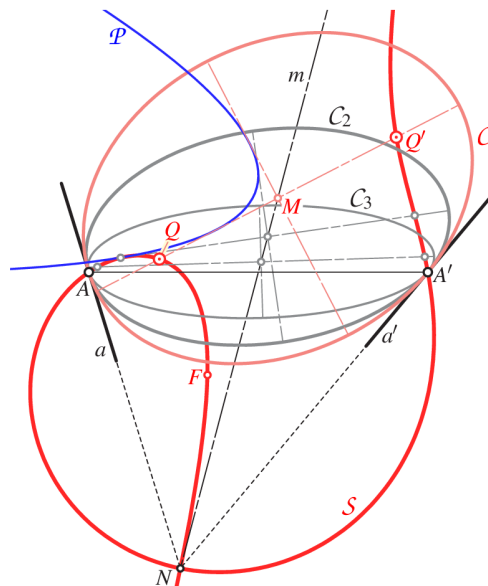

Fig. 6 The strophoid *S* is the focal curve of a pencil of conics sharing two line elements (*A, a*) and (*A', a'*).

Proof: With respect to any conic *C* of the pencil under consideration, *N* is the pole of the chord *AA'*. Focal points *Q* of *C* are characterized by the property that the line *QN* bisects an angle between *QA* and *QA'*. This can be concluded from Desarguesø involution theorem: The pencil of conics contains the pair of line pencils (*A, A'*) as well as the twofold counted pencil *N* as reducible curves. On the other hand, the tangents drawn from *Q* to the conic 𝒞 are isotropic. Thus, by virtue Theorem 4, the locus of focal points is a strophoid.

Theorem 6: *Let A, A' and N be three non-collinear points. Then for all conics 𝒞 with focal points A and A' the points T and T' of contact with tangents passing through N is the strophoid S with node N and A and A' as a pair of associated points (Fig. 7). S is also the locus of pedal points of normal lines drawn from point N to the confocal conics.*
For all conics 𝒞 of this confocal family, the normal lines at the points T of S as well as the polar lines TT' of N w.r.t. 𝒞 are tangent to a parabola 𝒫, the so-called Chasles parabola of N w.r.t. the confocal family.

Proof: The line *TN* is a bisector of *TA* and *TA'*. This characterizes the requested points of contact. The rest follows from Theorem 4. In the confocal family there are two conics passing through *T*. When the line *TN* is tangent to one of them, then it is orthogonal to the other. Hence *S* is also the locus of pedal points of normals drawn from *N* to the conics.
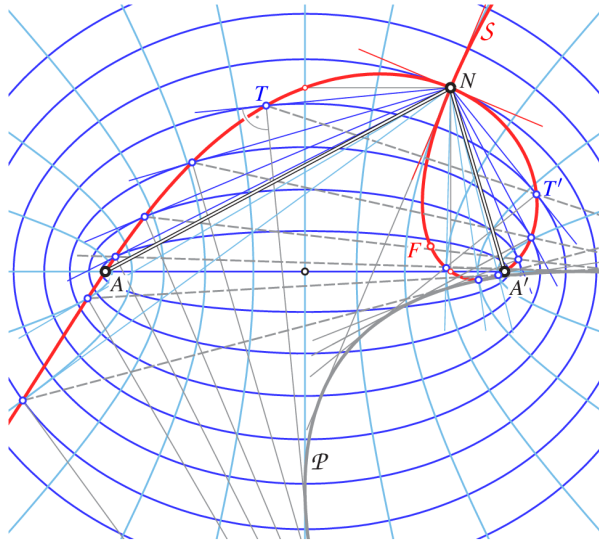
Fig. 7 For tangents drawn from a fixed point *N* to the conics of a confocal family, the points *T*, *T'* of contact lie on a strophoid *S*. At the same time is *S* is the locus of pedal points of normals drawn from *N* onto the confocal conics.

3.2 Perspective of Viviani's curve

Viviani's curve (or Viviani's window) $\mathcal{V}$ is defined as the curve of intersection between a sphere with radius *r* and a right cylinder with radius *r*/2, which contacts the sphere at a point *N*. We choose the cylinders's axis in vertical position.
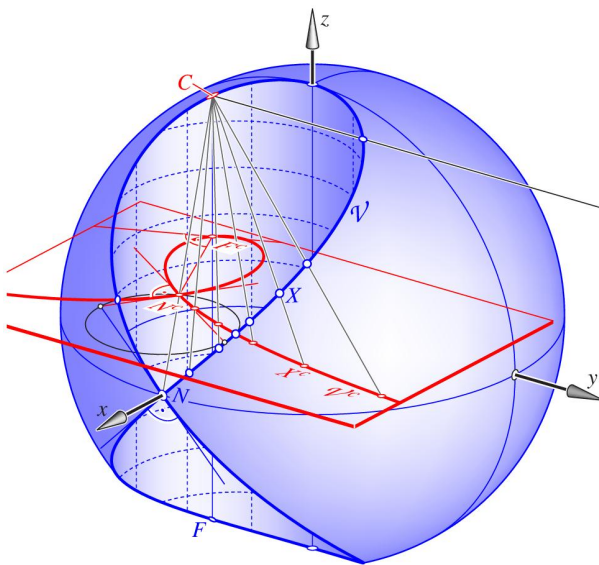
Fig. 8 Viviani's window can be projected onto a strophoid.

$\mathcal{V}$ is a curve of degree 4 with orthogonal tangents at *N*, and it passes through the absolute circle-points of the horizontal planes. We specify any horizontal plane as image plane of a perspective, whose center *C* lies on $\mathcal{V}$ (Fig. 8). Then the image of $\mathcal{V}$ is a circular cubic with a node. Since $\mathcal{V}$ is also located on a right cone with apex *N*

and 45° inclination, the orthogonal node tangents have also orthogonal images (compare [11]).

By the same token, Viviani's window is also located on a torus [8] (Fig. 9). Let *N* be the initial point for measuring the geographic longitude and latitude on the sphere; then $\mathcal{V}$ is the locus of points with equal longitude and latitude (Fig. 10). As a consequence, $\mathcal{V}$ is the locus of mirror points of *N* under reflection in diameter planes with 45°-inclination. Therefore, $\mathcal{V}$ is also a spherical trochoid: both polodes are circles with radius /4; in the initial pose the moving point coincides with the center of the fixed polode.
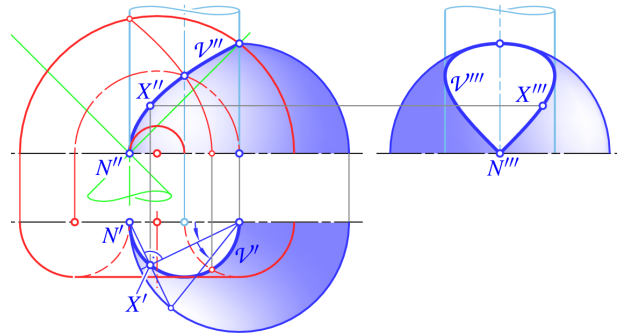

Fig. 9 Viviani's window $\mathcal{V}$ in top, front and side view. $\mathcal{V}$ is located on a sphere, a cylinder, a cone, and a torus.
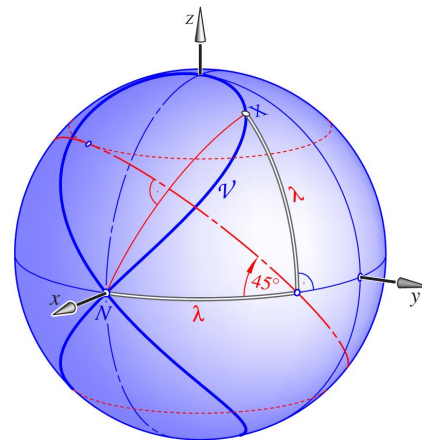

Fig. 10 For points of Viviani's window $\mathcal{V}$ the geographic longitude and latitude are equal.

3.3 Bricard's Flexible Octahedron, Type 3

According to R. Bricard (1899) there are three types of flexible octahedra (compare, e.g., [14] and the refer-ences therein; recent *n*-dimensional generalizations can be found in [6]). Those of type 3 have the property that they admit two flat poses. Such a pose can be obtained in the following way (Fig. 11): choose two dif-ferent circles with center *M* and two points *A, A'* outside the bigger one. Then, for each circle, intersect tangents drawn from *A* with those drawn from *A'* and select two opposite points. This gives finally two pairs (*B, B'*) and (*C, C'*) which together with *A* and *A'* build the three pairs of opposi̇ve vertices. The octahedron consists then of the eight triangles *ABC, ABC', AB'C, ..., A'B'C'*.

Since for *B* and *B'* as well as for *C* and *C'* one angle bisector of the connections with *A* and *A'* passes through *M*, by virtue of Theorem 4, all 6 points are located on a strophoid with the node *M*, and they are pairwise associated. As a consequence, also the lines *BC*, *BC'* and *B'C* and *B'C'* send an angle bisector through *M* (Fig. 12).
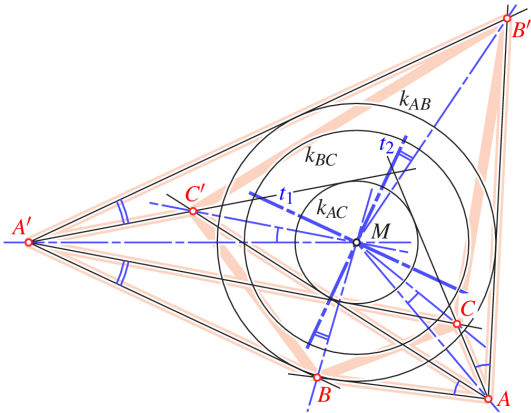

Fig. 11 Due to R. Bricard, the flexible octahedra of type 3 admit two flat poses with have local symmetries ó as depicted.

Theorem 6: *For any strophoid S, each triple of associated pairs (A, A'), (B, B') and (C, C') defines a flat pose of a flexible octahedron.*
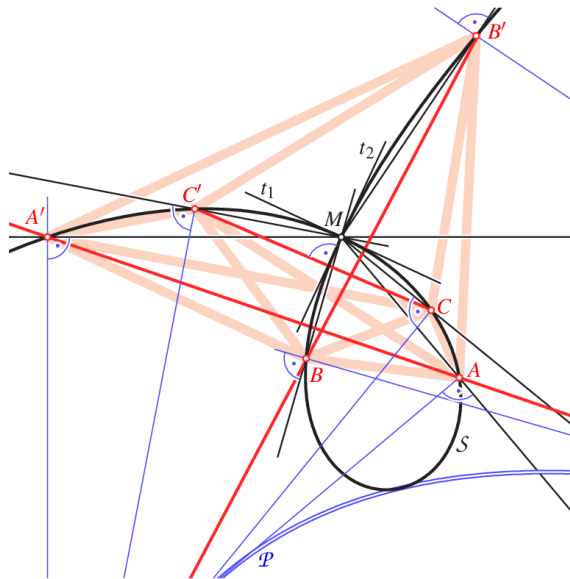

Fig. 12 For defining a flat pose of a type-3 flexible octahedron, the pairs of opposite vertices can be specified as associated points on a strophoid.

3.4 Euclidean Circle-Point Curve

A classical problem in plane kinematics is the question, for which moving points *X* there is instantaneously a 4-point contact between their trajectory and the circle of curvature. The requested locus is a cubic curve [4] which satisfies the equation

$$(x^2 + y^2)[(\kappa^* - 2\kappa) - (\kappa^{*\prime} - \kappa')^2 y] + 3xy = 0,$$

provided the pole *P* is the origin and the pole tangent the *x*-axis of the coordinate system. and * are the instant curvatures of the polodes, and ø and *ø are the derivatives with respect to the arc-length of the polodes. Upon comparison with eq. (1) we learn that the circle-point curve *S* is a strophoid. The same holds for the locus *S** of curvature centers which turns out to be the circle-point curve of the inverse motion (Fig. 13).

For a graphical construction of the circle-point curve it is standard (e.g., [16], p. 193) to use a map $X = (x, y) \to \underline{X} = (\underline{x}, \underline{y})$, which is defined as follows: For any point *X* determine the perpendicular bisector of the segment *XN*. This bisector intersects the pole tangent and the pole normal at two points. The image $\underline{X}$ of *X* forms with these two points and the pole *N* a rectangle. We compute

$$\underline{x} = \frac{x}{x^2 + y^2}, \quad \underline{y} = \frac{y}{x^2 + y^2},$$

which reveals that the strophoid *S* is mapped onto the line $\underline{S} = g$. We conclude

Theorem 7: *There is a cubic transformation $X \to \underline{X}$ which maps the strophoid S onto the line $\underline{S} = g$ being orthogonal to FN and passing through F.*

It should be noted that the analogue curves in spherical as well as in hyperbolic kinematics are closely related to strophoids (see [13, 7]).
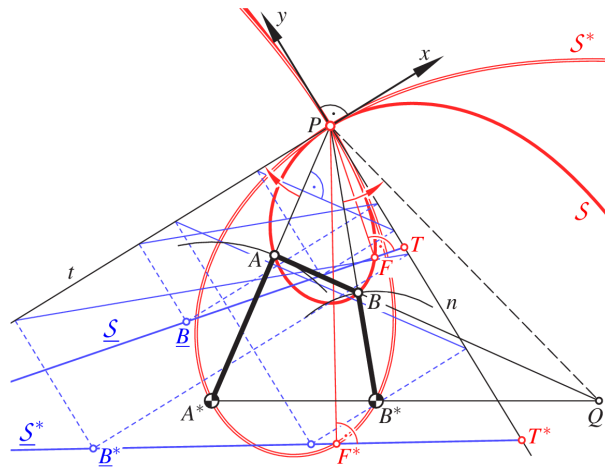

Fig. 13 In plane Euclidean kinematics, the circle-point curve *S* and the center-point curve *S** are strophoids.

3.5 Equicevian Points

Let *ABC* be any triangle and *P* a point outside the side-lines. On the lines connecting *P* with the vertices, the segments bounded by one vertex and by the point of intersection with the opposite sideline are called cevians of *P*. It is proved in [1] and [2] that the locus of points, for which the cevians with two fixed vertices have equal lengths, is a strophoid. It turns out that the focal points of the Steiner circumellipse of *ABC* are the only equicevian points, i.e., points for which all three cevians have the same length (Fig. 14).
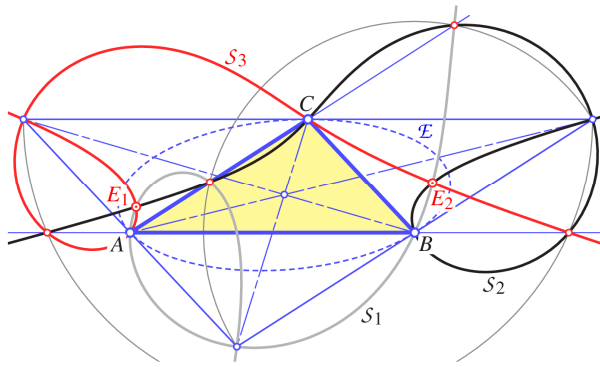
Fig. 14 The equicevian points of a triangle are the common points of the three equicevian cubics.

4. CONCLUSION

The aim of this paper was to emphasize that strophoids, i.e., circular cubics with orthogonal node tangents, play an important role at various problems in Euclidean geometry. Many of the presented theorems date already back to the 19th century, but are almost forgotten. However, these curves still deserve interest since with the aid of dynamic geometric software it is now possible to visualize many of their remarkable properties.

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