Strophoids are auto-isogonal cubics

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Abstract. Strophoids are defined as plane circular cubics with a node and orthogonal node tangents. These rational curves are characterized by a series of properties. Of fundamental importance is their role as generalizations of Apollonian circles (Theorem 1). We also focus on quadratic transformations which keep strophoids invariant. At almost all properties a symmetric relation of points on the cubic is important.

Keywords: Strophoid, rational cubic curve, pedal curve, focal curve, equicevian curve, associated points, quadratic transformations, auto-isogonal curve.

1 Introduction

Definition 1. An algebraic curve in the Euclidean plane is called *circular* if it passes through the absolute circle-points. A circular curve of third degree is called *strophoid* if it has a double point (= node) with orthogonal tangents. An irreducible strophoid with an axis of symmetry is called *right* and otherwise *oblique* (see [9, p. 515] or [12, pp. 37–39]).

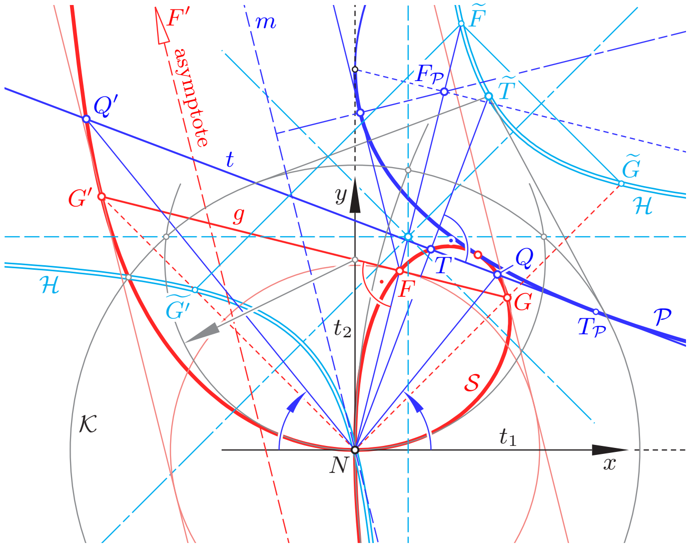

Figure 1: The irreducible strophoid $\mathcal{S}$ is the pedal curve of the parabola $\mathcal{P}$ and inverse w.r.t. the circle $\mathcal{K}$ to the equilateral hyperbola $\mathcal{H}$.

In a cartesian coordinate frame with the two tangents t_1, t_2 at the node N as x- and y-axes (see Figure 1), the equation of the *cubic* can be set as

$$\mathcal{S}\colon\ (x^2+y^2)(ax+by)-xy=0. \tag{1}$$

In fact, the origin N as a point of intersection between $\mathcal{S}$ and the x- or y-axis has the multiplicity 3, and for all other lines through N the multiplicity 2. On the other hand, the two complex conjugate points of intersection of $\mathcal{S}$ with the line at infinity are $(0:1:\pm i)$, when we use homogeneous coordinates $(x_0:x_1:x_2)$, which for finite points satisfy $(x,y)=(x_1/x_0,\, x_2/x_0)$. Hence, the cubic $\mathcal{S}$ is circular. The real point of $\mathcal{S}$ on the line at infinity is $F'=(0:-b:a)$.

It means no restriction to assume that the constant coefficients $a,b\in\mathbb{R}$ satisfy

$$a\geq b\geq 0. \tag{2}$$

The strophoid with the constants $(a,-b)$ is the reflection of $\mathcal{S}$ in the x-axis. The strophoid with constants (b,a) instead of (a,b) can be obtained by reflection of $\mathcal{S}$ in the line $x=y$.

In the case $a=b$ the strophoid $\mathcal{S}$ in (1) is symmetric with respect to ('w.r.t.' in brief) the line $x=y$. There are two types of *reducible* strophoids: For $a>b=0$ the curve (1) splits into the y-axis and a circle passing through the origin and with the center $(0,\,\frac{1}{2a})$. In the case $a=b=0$ the projective cubic consists of the two coordinate axes and the line at infinity (note Table 1).

Table 1: Types of affine strophoids

type	*coefficients*	*comments*
oblique strophoid	$a>b>0$	irreducible
right strophoid	$a=b$	
circle + diameter	$a>b=0$	reducible
orthogonal lines	$a=b=0$	

The line through N with inclination angle φ intersects $\mathcal{S}$ in the point

$$X=\left(\frac{\sin\varphi\ \cos^2\varphi}{a\,\cos\varphi+b\,\sin\varphi},\ \frac{\sin^2\varphi\ \cos\varphi}{a\,\cos\varphi+b\,\sin\varphi}\right). \tag{3}$$

This yields a parametrization of $\mathcal{S}$. The choice $\varphi=\pm 45°$ gives the points

G and G' with coordinates

$$G = \left(\frac{1}{2(a+b)},\ \frac{1}{2(a+b)}\right), \quad G' = \left(\frac{-1}{2(a-b)},\ \frac{1}{2(a-b)}\right). \tag{4}$$

In the symmetric case $a = b$ point G' is at infinity. The pedal point of the connecting line

$$[G, G']\colon\ 2bx + 2ay = 1$$

w.r.t. to the node N is the *focus* F of $\mathcal{S}$ with the coordinates

$$F = \left(\frac{b}{2(a^2+b^2)},\ \frac{a}{2(a^2+b^2)}\right).$$

It can be verified (compare [11, p. 66]) that the tangents to $\mathcal{S}$ at the absolute circle-points intersect at the focus F.

The line m, which connects N with the real point F' at infinity is called the *median* of $\mathcal{S}$. It is the reflection of the line $[N, F]$ in the node tangents t_1 or t_2 (Figure 1).

The parametrization (3) of $\mathcal{S}$ gives rise to the polar equation

$$\mathcal{S}\colon\ r(\varphi) = \frac{1}{\frac{a}{\sin\varphi} + \frac{b}{\cos\varphi}} \quad \text{for} \quad \varphi \in \left(0,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\ \pi\right). \tag{5}$$

Lemma 1. *Each irreducible strophoid $\mathcal{S}$ is inverse to an equilateral hyperbola $\mathcal{H}$ w.r.t. any circle centered at the node N of $\mathcal{S}$. The strophoid $\mathcal{S}$ is the pedal curve of a parabola $\mathcal{P}$ whose directrix passes through N and whose focus $F_{\mathcal{P}}$ is the reflection of N in the focus F of $\mathcal{S}$.*

Proof. (a) The inversion $T \mapsto \widetilde{T}$ in the circle $\mathcal{K}$ with radius R centered at N transforms the strophoid $\mathcal{S}$ with the polar equation (5) into the curve $\mathcal{H}$ with the polar equation

$$\mathcal{H}\colon\ r = \frac{aR^2}{\sin\varphi} + \frac{bR^2}{\cos\varphi} \quad \text{for} \quad \varphi \in \left(0,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\ \pi\right).$$

This is an *equilateral hyperbola* since it satisfies the cartesian equation

$$\mathcal{H}\colon\ (x - bR^2)(y - aR^2) = abR^4.$$

It passes through N, and its asymptotes are parallel to the tangents t_1, t_2 of $\mathcal{S}$ at N (see Figure 1). Due to properties of inversions, each asymptote of $\mathcal{H}$ is inverse to the circle of curvature of $\mathcal{S}$ at N for one of the two branches. The centers $(0, \frac{1}{2a})$ and $(\frac{1}{2b}, 0)$ of these circles are the intersections of the line $[G, G']$ with the tangents t_1 and t_2 (Figure 1).

The inverse images $\widetilde{G}, \widetilde{G'}, \widetilde{F} \in \mathcal{H}$ of the points $G, G', F \in \mathcal{S}$ form together with N a rectangle symmetric w.r.t. the axes of $\mathcal{H}$ (Figure 1).

(b) In order to prove the second part of Lemma 1, we select any point $T \in \mathcal{S}$ and determine the line t through T orthogonal to $[N, T]$ (Figure 1). If the line $[N, T]$ has the slope k then we obtain from (1) or (3)

$$T = \left(\frac{k}{(1+k^2)(a+bk)}, \frac{k^2}{(1+k^2)(a+bk)} \right) \tag{6}$$

and

$$t\colon\ x + ky = \frac{k}{a+bk}\,.$$

The homogeneous line coordinates of t,

$$(u_0, u_1, u_2) = \left(\frac{-k}{a+bk},\ 1,\ k \right);$$

satisfy the homogeneous quadratic equation

$$au_0u_1 + bu_0u_2 + u_1u_2 = 0. \tag{7}$$

This is the tangential equation of a parabola $\mathcal{P}$, since the polynomial on the left-hand side is irreducible and contains the zero $(u_0, u_1, u_2) = (1, 0, 0)$. Also the homogeneous line coordinates $(0 : 1 : 0)$ and $(0 : 0 : 1)$ of the y- and x-axis are zeros of this equation. We obtain the point-equation of $\mathcal{P}$ by inverting the symmetric coefficient matrix of (7):

$$\mathcal{P}\colon\ b^2x^2 + a^2y^2 - 2bx - 2ay - 2abxy + 1 = 0.$$

The parabola $\mathcal{P}$ contacts the line at infinity at the point $(0 : a : b)$. Hence, the axis of $\mathcal{P}$ is orthogonal to the median m. Since N is the intersection point of two tangents of $\mathcal{P}$, i.e., the coordinate axes, the median m is the directrix of $\mathcal{P}$. After some computations we obtain the parabola's focus $F_{\mathcal{P}}$ and the vertex $S_{\mathcal{P}}$ as

$$F_{\mathcal{P}} = \left(\frac{b}{a^2+b^2}, \frac{a}{a^2+b^2} \right), \quad S_{\mathcal{P}} = \left(\frac{b^3}{(a^2+b^2)^2}, \frac{a^3}{(a^2+b^2)2} \right).$$

The focus F of $\mathcal{S}$ is the midpoint between N and $F_{\mathcal{P}}$ and therefore a point of the parabola's tangent at the vertex $S_{\mathcal{P}}$. The parabola $\mathcal{P}$ is polar to the hyperbola $\mathcal{H}$ w.r.t. the circle $\mathcal{K}$ (Figure 1). □

2 Associated points

Definition 2. Given a strophoid $\mathcal{S}$, two real or complex conjugate points $Q, Q' \in \mathcal{S}$, both different from the node N, are called *associated* if and only if the lines $[Q, N]$ and $[Q', N]$ separate the two node tangents t_1 and t_2 harmonically. In the case of two real points (Q, Q'), the tangents t_1 and t_2 are the angle bisectors of $[Q, N]$ and $[Q', N]$.

The harmonic position of the lines $[Q, N]$ and $[Q', N]$ w.r.t. the tangents at the node can be used to define associated points on each cubic with the node N. In the following lemma we list some of the projective properties of associated points. Proofs are given in [1, 144–145].

Lemma 2. *1. Two points Q, Q' of a cubic $\mathcal{C}$ with a node N are associated if and only if the tangents to $\mathcal{C}$ at Q and Q' intersect at a point S which lies again on $\mathcal{C}$. This point S is associated to the remaining point S' of intersection between $\mathcal{C}$ and the line $[Q, Q']$.*
2. For each point $P \in \mathcal{C} \setminus \{N\}$, the connecting lines with a pair (Q, Q') of associated points are harmonic w.r.t. the line $[P, N]$ and the line where the contact points of the tangents drawn from P' to $\mathcal{C}$ are located.
3. For each quadrangle formed by pairs of associated points (P, P') and (Q, Q'), the diagonal points $R = [P, Q] \cap [P', Q']$ and $R' = [P, Q'] \cap [P', Q]$ lie again on $\mathcal{C}$, and they are associated, too. Hence, three collinear points $\{P, Q, R\} \subset \mathcal{C} \setminus \{N\}$ together with their associated points $\{P', Q', R'\}$ are the six vertices of a complete quadrilateral.

Though the following lemma remains valid also after a projective generalization, we restrict ourselves to the particular case of a strophoid $\mathcal{S}$:

Lemma 3. *If two different real or complex conjugate points Q and Q' of a strophoid $\mathcal{S}$ are associated then their connecting line $[Q, Q']$ is tangent to the parabola $\mathcal{P}$, which is the negative pedal curve of $\mathcal{S}$ w.r.t. its node N. Conversely, on each tangent t of $\mathcal{P}$ the remaining points of intersection between t and $\mathcal{S}$, besides the pedal point of t w.r.t. N, are associated.*

Proof. Let k be the slope of the line $[N, Q]$. Then, for the associated point Q' the connection with N has the slope $-k$, and by virtue of (6) the line $t = [Q, Q']$ has the homogeneous coordinates

$$(u_0, u_1, u_2) = \left(k^2,\ -bk^2(1 + k^2),\ -a(1 + k^2)\right),$$

which satisfy the quadratic equation (7) of the parabola $\mathcal{P}$. The third point of intersection between t and the strophoid is the pedal point

$$T = \left((a^2 + b^2k^4)(1 + k^2) : bk^4 : ak^2\right).$$

On the line t, the harmonic conjugate of T w.r.t. Q and Q' is the point of contact $T_{\mathcal{P}} = \big(2ab(1+k^2)^2 : a : bk^4\big)$ with the parabola $\mathcal{P}$, since the slopes k_T of $[N,T]$ and $k_{\mathcal{P}}$ of $[N,T_{\mathcal{P}}]$ satisfy the condition $k_T \cdot k_{\mathcal{P}} = -k^2$. Only in the case $k^3 = a/b$ the pedal point T coincides with Q, and we obtain a point of contact between the strophoid $\mathcal{S}$ and its negative pedal curve $\mathcal{P}$.

Conversely, let t be a tangent of $\mathcal{P}$ not passing through the node N, and let T be its pedal point w.r.t. N. If Q is a remaining point of intersection between t and $\mathcal{S}$, i.e., $Q \neq T$, then, by virtue of the first part of the proof, its connection with the associated point $Q' \in \mathcal{S}$ is tangent to $\mathcal{P}$. Therefore, $[Q,Q']$ is either coincident with t or normal to $[N,Q]$. However, in the latter case Q coincides with the pedal point of this line w.r.t. N, which has been excluded. $\square$

The connecting line of two real associated points Q and Q' contacts the parabola $\mathcal{P}$ within the arc which is terminated by the points of contact between $\mathcal{P}$ and the two node tangents t_1 and t_2. The terminating points lie on the polar of N w.r.t. $\mathcal{P}$. This polar is parallel to $g = [G,G']$ and passes through $F_{\mathcal{P}}$ (Figure 1).

One point of each pair (Q,Q') of associated points belongs to the finite loop of $\mathcal{S}$, while the other lies outside. The focus F is associated to the real point F' at infinity. The connections of F' with pairs of associated points are symmetric w.r.t. the median $m = [F',N]$; consequently, the midpoints of all finite pairs of associated points lie on m.

A further pair of associated points is (G,G') on the line g through F orthogonal to $[N,F]$. Also the absolute circle-points are associated, since the isotropic lines through N separate the node tangents t_1 and t_2 harmonically. In accordance to this, (a) the connecting line of the absolute circle-points, the line at infinity, is tangent to the parabola $\mathcal{P}$, and (b) the tangents at the absolute circle-points intersect at F. As a limit, the node N can be called self-associated.

The inversion between $\mathcal{S}$ and the hyperbola $\mathcal{H}$ transforms pairs (Q,Q') of associated points of $\mathcal{S}$ into pairs $(\widetilde{Q},\widetilde{Q}')$ of opposite points of $\mathcal{H}$. When $\mathcal{S}$ splits into a circle and its diameter line, associated points on the circle are symmetric w.r.t. the diameter.

Lemma 4. *A strophoid is uniquely defined by its node N and a pair (Q,Q') of associated points, provided that the three points are not collinear.*

Proof. We use the inversion w.r.t. any circle centered at N: The inverse points of Q and Q' define a diameter of an equilateral hyperbola $\mathcal{H}$ whose asymptotes are parallel to the angle bisectors of $[N,Q]$ and $[N,Q']$. Thus,

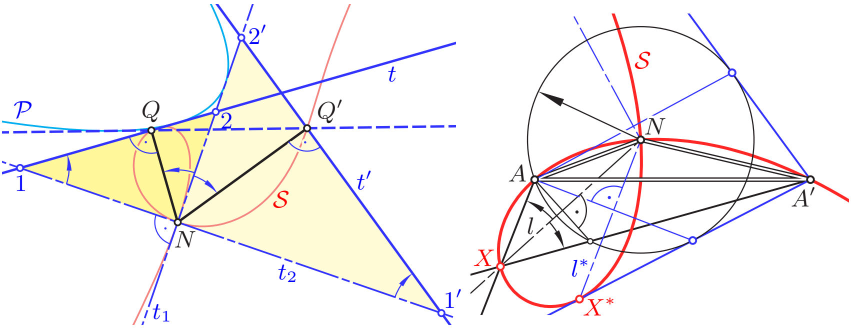

Figure 2: Left: A strophoid $\mathcal{S}$ is uniquely defined by its node N and a pair (Q, Q') of associated points (Lemma 4). Right: Constructing points $X \in \mathcal{S}$ and the tangents through A'.

the hyperbola and consequently its inverse, the wanted strophoid, are uniquely defined.

As an alternative, we can also look for a parabola $\mathcal{P}$ as the negative pedal curve of the strophoid. $\mathcal{P}$ is given by the following four tangents: the angle bisectors t_1, t_2 of the lines $[N, Q]$ and $[N, Q']$, the normal t to $[N, Q]$ through Q and the normal t' to $[N, Q']$ through Q'. No two of them are parallel, no three of them are concurrent. Therefore there exists exactly one parabola $\mathcal{P}$ which contacts them. Point N lies on the directrix of $\mathcal{P}$, since the tangents drawn through N to $\mathcal{P}$ are orthogonal.

By the same token, the parabola $\mathcal{P}$ is also tangent to the line $[Q, Q']$ (Lemma 3). We prove this in the following way (see Figure 2, left): Let the tangents t_1 and t_2 intersect t in the points $1, 2$ and t' in the points $1'$ and $2'$, respectively. Then the triangles $N12$ and $N1'2'$ are similar. The feet Q and Q' of the altitudes through N are corresponding in this similarity. Hence, the lines $t_1 = [1, 1']$, $t_2 = [2, 2']$ and $[Q, Q']$ together with t and t' must be tangents of a parabola, and this is $\mathcal{P}$. □

3 Strophoids as a geometric locus

Strophoids show up as the locus of points at various geometric problems in the Euclidean plane (see, e.g., [1, 10, 11]): Here we focus on one of the main properties and on a few consequences.

Theorem 1. *For given non-collinear points A, A' and N, the locus of points X such that the line $[X, N]$ bisects the angle between $[X, A]$ and $[X, A']$ is a strophoid $\mathcal{S}$ with the node N and with associated points A, A' (Figure 3). The strophoid $\mathcal{S}$ has this property w.r.t. all its pairs (A, A') of associated points, and this property holds also when A is at infinity. The respectively second angle bisectors are tangent to the parabola $\mathcal{P}$, which is*

the negative pedal curve of $\mathcal{S}$ w.r.t. the node N.
The strophoid $\mathcal{S}$ splits if and only if $\overline{NA} = \overline{NA'}$. In this case the requested locus of points X consists of the circumcircle of $AA'N$ and the diameter line orthogonal to $[A, A']$.

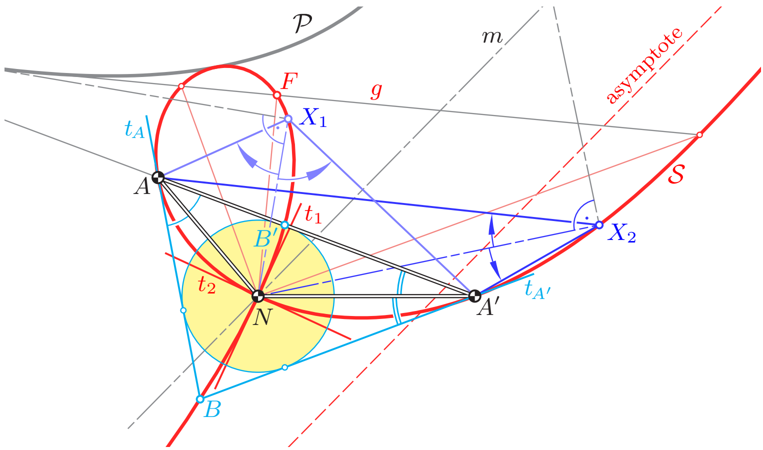

Figure 3: The strophoid is the locus of points X such that the line $[X, N]$ bisects an angle between the lines $[X, A]$ and $[X, A']$.

Proof. By virtue of Lemma 2, 2., the lines connecting any point $X \in \mathcal{S} \setminus \{N\}$ with all pairs of associated points belong to an involution with one fixed line passing through the node N. Since the absolute circle-points are associated too, the two fixed lines of the involution are orthogonal, i.e., the involution is the symmetry w.r.t. $[X, N]$. And this holds for all pairs (A, A') of associated points of $\mathcal{S}$.
Conversely, on each line l through N there is at most one point X with this property: We obtain this point by intersecting l with the line connecting A' with the reflection of A in l (Figure 2, right).[1] This reflection cannot coincide with A', except the case with N being equidistant to A and A'. Since the strophoid $\mathcal{S}$ has one remaining point of intersection with l, there are no other points than those of $\mathcal{S}$. □

Due to Theorem 1, the strophoid generalizes the *Apollonian circle*, which has the requested property in the case of collinear points $\{A, A', N\}$ (Figure 4, left). Then the complete locus of points X with $[X, A]$ and $[X, A']$ being symmetric w.r.t. $[X, N]$ includes also the line $[A, A']$. In the

[1]This defines between the line pencils with carriers A and A' a 2-1-correspondence which generates the strophoid $\mathcal{S}$ as the locus $\{X\}$ of intersection points.

case of N at infinity, but neither aligned with A and A' nor orthogonal to $[A, A']$, the analogue locus is an equilateral hyperbola (Figure 4, right).

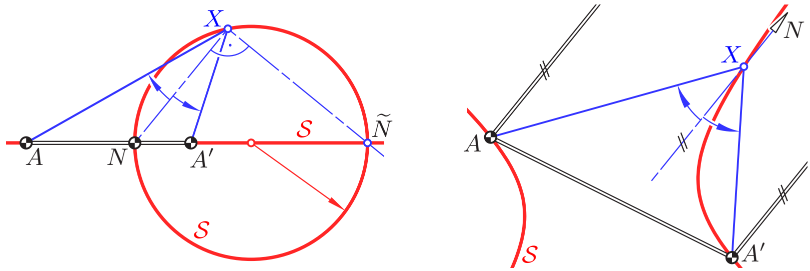

Figure 4: In the case of collinear points $\{A, A', N\}$ (left) the *Apollonian circle* together with $[A, A']$ is the locus of points X, as requested in Theorem 1, while for N at infinity (right) an equilateral hyperbola with diameter AA' is the result, provided that N is not orthogonal to $[A, A']$.

When X tends to $A \in \mathcal{S}$, then the line $[X, A]$ tends to the tangent t_A to $\mathcal{S}$ at point A. This reveals that $[A, N]$ bisects the angle between t_A and $[A, A']$. The analogue holds for the tangent $t_{A'}$ to $\mathcal{S}$ at A'. Hence, the tangents t_A and $t_{A'}$ together with $[A, A']$ determine a triangle $AA'B$, for which the node N is the center either of the incircle or of one of the excircles (Figure 3).

Figure 5 illustrates two consequences of Theorem 1 (see [7, p. 101] or [9, p 515, footnote 235], and [7, p. 120, footnote 417], respectively).

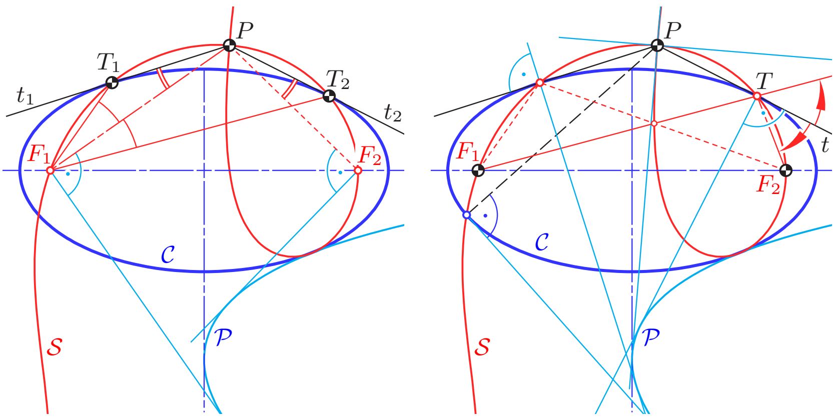

Figure 5: The strophoid and conics (Corollaries 1 and 2)

Corollary 1. *The foci of all conics in the pencil, which is defined by two given line elements* (T_1, t_1) *and* (T_2, t_2) *in admissable position, are located on a strophoid* $\mathcal{S}$ *with the node* $P := t_1 \cap t_2$ *and with* (T_1, T_2) *as a pair of associated points. For each included ellipse or hyperbola the two real foci as well as the two complex conjugate foci are associated points of the focal curve* $\mathcal{S}$*.*
The focal curve $\mathcal{S}$ *remains the same when* T_1 *and* T_2 *are replaced by any other pair of associated points of* $\mathcal{S}$*, while* P *is fixed.*

Proof. Let F be a focal point of any conic included in this pencil. Then, due to Desargues' involution theorem, the lines $[F, T_1]$ and $[F, T_2]$ are symmetric w.r.t. $[F, P]$ (Figure 5, left). Hence, by virtue of Theorem 1, F is a point of the strophoid with node P and associated (T_1, T_2) (see [1, Fig. 5] or [11, Fig. 6]).
If the conic is an ellipse or hyperbola with the real foci F_1 and F_2, then the lines $[F_1, P]$ and $[F_2, P]$ share with t_1 and t_2 the axes of symmetry (Figure 5, left). By virtue of Definition 2, the points F_1 and F_2 are associated. As a consequence of Lemma 2, 3., also the two imaginary foci are associated points of $\mathcal{S}$, since the absolute circle-points are associated as well. □

Corollary 2. *Let* $\mathcal{C}$ *be a conic, whose foci are associated points of the strophoid* $\mathcal{S}$ *mentioned in Corollary 1. For each conic* $\mathcal{C}'$ *confocal with* $\mathcal{C}$*, the points of tangency of the tangents from the node* P *to* $\mathcal{C}'$ *are also located on* $\mathcal{S}$*. Moreover,* $\mathcal{S}$ *is the locus of pedal points* N *of normals drawn from* P *to all conics* $\mathcal{C}'$*.*

Proof. Let T be a point of tangency (Figure 5, right). If F_1 and F_2 are the two real foci of $\mathcal{C}$ and $\mathcal{C}'$, then the lines $[T, F_1]$ and $[T, F_2]$ are symmetric w.r.t. $[T, P]$. Hence, by virtue of Theorem 1, T is a point of a strophoid with node P and associated (F_1, F_2) (see [11, Fig. 7]). By virtue of Lemma 4, this strophoid coincides with the strophoid $\mathcal{S}$ of Corollary 1. The negative pedal curve $\mathcal{P}$ of $\mathcal{S}$ w.r.t. P coincides with Chasles' parabola of P w.r.t. all confocal families (see [8, p. 342]).
Besides $\mathcal{C}'$, there is a second conic $\mathcal{C}''$ out of the confocal family passing through the point T of tangency. Since confocal conics form an orthogonal net, the line $[T, P]$ is orthogonal to $\mathcal{C}''$; hence, T is a pedal point of $\mathcal{C}''$ w.r.t. P. □

By the same token, the points of tangency of tangents from a generic fixed point P to all conics of a given pencil are located on a curve of degree 3. This can be proved by representing the equations of the conics as linear combinations. Then also the equations of the polar lines of P

are linear combinations, and the points of tangency are common solutions of corresponding pairs of equations.

A third consequence of Theorem 1 is the equicevian property, which is stated in the following Corollary. It has been discussed in more detail in [1, 2]. But already O. Bottema has noted in [5] that the equicevian property characterizes a circular cubic.

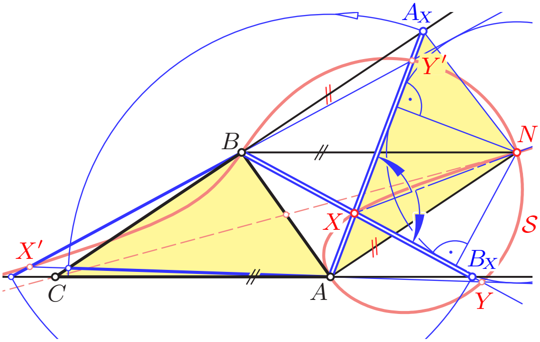

Figure 6: X is equicevian $\Leftrightarrow \overline{AA_X} = \overline{BB_X} \Leftrightarrow [X, N]$ bisects $\angle AXB$.

Corollary 3. *Let N be the node and A, B be associated points of a strophoid $\mathcal{S}$. We focus on the triangle ABC with C being the reflection of N in the midpoint of AB (Figure 2, right). The two cevians AA_X and BB_X of a point X have equal lengths if and only if X is a point of $\mathcal{S}$.*

Proof. The proof is based on the equivalence between the equality of distances $\overline{AA_X} = \overline{BB_X}$ and the symmetry of the lines $[X, A]$ and $[X, B]$ w.r.t. $[X, N]$.
Because of $A_X \in [B, C] \parallel [N, A]$ the triangles ANA_X, ANB and analogously also BNB_X have areas equal to that of ABC. Consequently, the distances $\overline{AA_X}$ and $\overline{BB_X}$ are equal iff the altitudes of N in the triangles NAA_X and NBB_X are equal. On the other hand, the lines $[A, A_X]$ and $[B, B_X]$ contact a circle with center N iff $[N, X]$ is an angle bisector of these lines (Figure 6). □

In Figure 6 three other points $X', Y, Y' \in \mathcal{S}$ are depicted for which the cevians through A and B have the same length as that of X.

4 Invariance against quadratic transformations

Theorem 2. *Let $\mathcal{S}$ be an irreducible strophoid with the node N, the focus F and with G, G' being the remaining points of intersection between $\mathcal{S}$ and the line g through F orthogonal to $[F, N]$. Then the inversions w.r.t.*

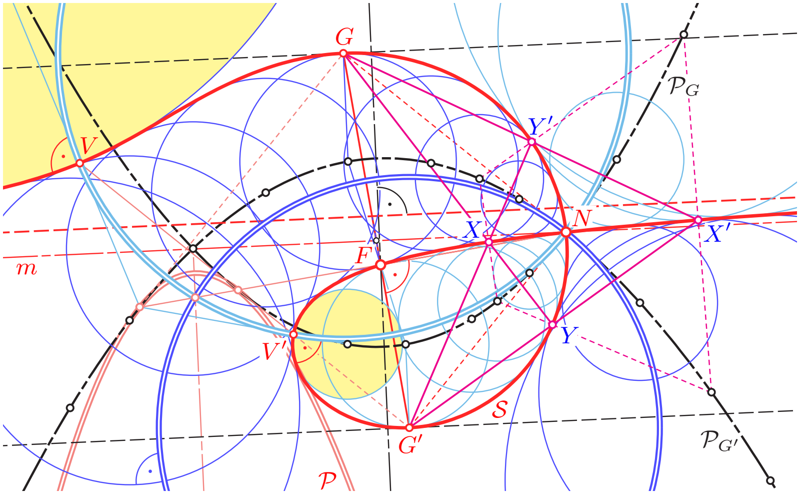

Figure 7: The strophoid $\mathcal{S}$ is self-inverse with respect to the circles $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ through N with centers G and G', respectively. Therefore $\mathcal{S}$ is the envelope of self-inverse circles with centers on confocal parabolas $\mathcal{P}_G$ and $\mathcal{P}_{G'}$ with the common focus F.

the two circles $\mathcal{K}_G$, $\mathcal{K}_{G'}$, centered at G and G', respectively, and passing through N and $F_{\mathcal{P}}$, map $\mathcal{S}$ onto itself.
Hence, $\mathcal{S}$ is the envelope of circles being orthogonal to $\mathcal{K}_G$ and centered on a parabola $\mathcal{P}_G$ with focus F and an axis orthogonal to the median m of $\mathcal{S}$. Similarily, $\mathcal{S}$ is also the envelope of circles being orthogonal to $\mathcal{K}_{G'}$ and centered on a parabola $\mathcal{P}_{G'}$, which is confocal to $\mathcal{P}_G$ (Figure 7).
In the case of a right strophoid, one of the circles $\mathcal{K}_G$ or $\mathcal{K}_{G'}$ and one of the parabolas $\mathcal{P}_G$ or $\mathcal{P}_{G'}$ degenerates to the axis of symmetry.

Proof. The inversion in the circle $\mathcal{K}$, as mentioned in Lemma 1, transforms the reflections in the axes of the hyperbola $\mathcal{H}$ onto the inversions in the circles $\mathcal{K}_G$ and $\mathcal{K}_{G'}$. All circles having a double contact with $\mathcal{H}$ are centered on one of the hyperbolas axes. The inversion in $\mathcal{K}$ maps them onto two families of circles being orthogonal either to $\mathcal{K}_G$ and $\mathcal{K}_{G'}$. In [10, p. 7] it is verified analytically that the centers of these circles lie on two confocal parabolas. The common axis of the two parabolas is orthogonal to the median m of $\mathcal{S}$. The directrices coincide with the tangents to $\mathcal{S}$ at G' and G. Both parabolas are tangent to the negative pedal curve $\mathcal{P}$ of $\mathcal{S}$ (see Figure 7). $\square$

The product of the reflections in the axes of the hyperbola $\mathcal{H}$ is the reflection in the hyperbola's center. The inversion in the circle $\mathcal{K}$ trans-

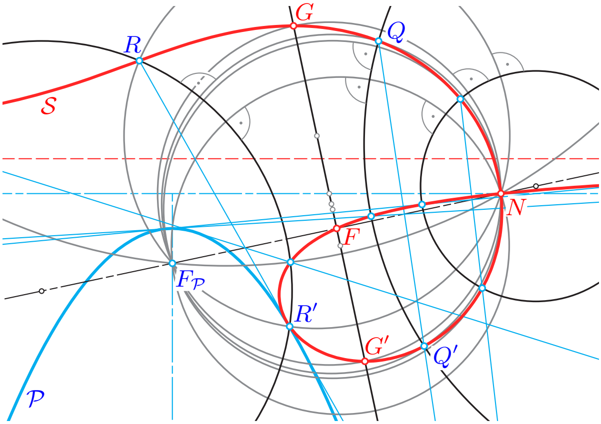

Figure 8: The circles through N and $F_{\mathcal{P}}$ intersect the strophoid $\mathcal{S}$ in pairs of associated points. Circles of the orthogonal pencil intersect $\mathcal{S}$ in two pairs of associated points.

forms this point reflection onto a Moebius-involution. Its restriction to $\mathcal{S}$ is the involution of associated points. Since the reflection in the hyperbola's center keeps diameters of $\mathcal{H}$ as well as all concentric circles fixed, we obtain

Corollary 4. *The remaining points of intersection between the strophoid $\mathcal{S}$ and circles centered on $[G, G']$ and passing through N and F_p are associated points of $\mathcal{S}$, or there is an osculation at N with one branch of $\mathcal{S}$. The circles being orthogonal to all circles through N and F_p share with $\mathcal{S}$ two pairs of associated points (Figure 8).*

Consequently, when X is a point of contact between $\mathcal{S}$ and a circle with double contact and centered on $\mathcal{K}_G$, then the second point of contact is $Y = [G, X] \cap [G', X']$ with X' associated to X (Figure 7). Circles of different families sharing one contact point have associated second points of contact. In one of the families there are two circles with a four-point contact with $\mathcal{S}$. Their contact points V and V' are associated as well.

The inversions in $\mathcal{K}_G$ and $\mathcal{K}_{G'}$ are the only ones which map $\mathcal{S}$ onto itself. This follows also from the fact that a four-point contact between a curve and a circle is invariant under inversions. Hence, the two points V, V' of $\mathcal{S}$ with stationary curvature must either be fixed or they commute under an inversion which preserves $\mathcal{S}$ (Figure 7).

Theorem 3. *Let $\mathcal{S}$ be an irreducible strophoid with node N, and let ABC be an inscribed triangle such that N is the center of a tritangent circle of*

ABC, i.e., center of the incircle or of an excircle. Then the isogonal transformation with respect to ABC maps $\mathcal{S}$ onto itself. The restriction of this isogonal transformation to $\mathcal{S}$ equals the involution of associated points (Figure 9).

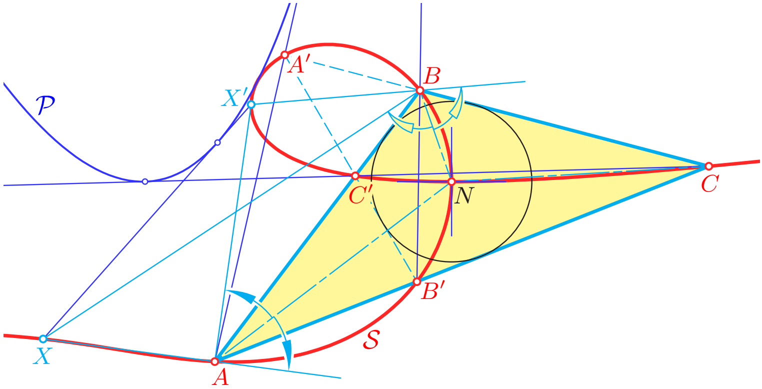

Figure 9: The isogonal transformation $X \mapsto X'$ w.r.t. the triangle ABC maps the strophoid $\mathcal{S}$ onto itself.

Proof. Given a strophoid $\mathcal{S}$ with node N, let $A, B \in \mathcal{S} \setminus \{N\}$ be two non-associated points. By reflecting $[A, B]$ in the lines $[A, N]$ and $[B, N]$ we obtain the side lines $[A, C]$ and $[B, C]$ of a triangle ABC, which has N as the center either of the incircle or of an excenter. The triangle has the property that for each vertex the associated point lies on the opposite side line (Figure 9).
We learned in Theorem 1, that for any two associated points $X, X' \in \mathcal{S}$ with $X, X' \neq A, B, C$ the connecting lines with A, B and C are symmetric with respect to the respective angle bisectors $[A, N]$, $[B, N]$ or $[C, N]$. Hence, X' is isogonal conjugate to X w.r.t. ABC. $\square$

Remark. The product of any two isogonal transformations of the type presented in Theorem 3 is a birational transformation which fixes all points of $\mathcal{S}$.

According to Bézout's theorem, two different strophoids with a common node N share at most three points besides the absolute circle-points, since N has an intersection multiplicity ≥ 4. The following theorem reveals that these three remaining points form a triangle of the type mentioned in Theorem 3.

Theorem 4. *Given a triangle NAB, each strophoid passing through A and B and with node N contains a third point C such that N is the center of a tritangent circle of ABC.*
All strophoids with node N and circumscribed to the triangle ABC belong to a linear system. These strophoids are orbits under the isogonal transformation w.r.t. ABC. Their foci lie on the circumcircle of ABC (Figure 10).

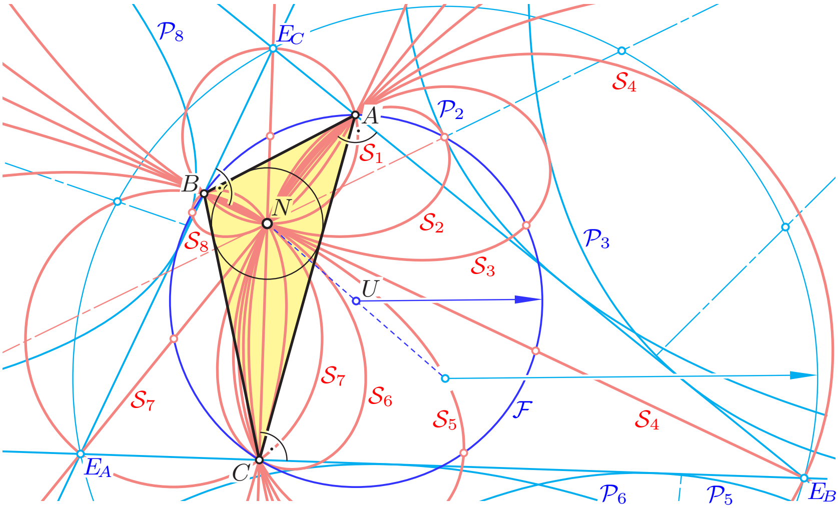

Figure 10: Strophoids with node N and circumscribed to the triangle ABC belong to a linear system. Their foci lie on the circumcircle $\mathcal{F}$ of ABC.

Proof. We rotate the coordinate frame, which was used in (1), and obtain the general equation of a strophoid with the node at the origin as

$$(x^2 + y^2)(cx + dy) + (ex + fy)(fx - ey) = 0 \tag{8}$$

with constant $c, d, e, f \in \mathbb{R}$ and $(e, f) \neq (0, 0)$. The linear factor in the cubic term gives the equation of the median, while the quadratic term describes the two orthogonal node tangents. The original parameters a, b used in (1) can be computed as

$$a = \frac{ec + fd}{\sqrt{e^2 + f^2}}, \quad b = \frac{ed - fc}{\sqrt{e^2 + f^2}}.$$

Any non-trivial linear combination of two such equations gives again an equation of this type, since the quadratic terms define a pencil of curves of

second degree, which consist of orthogonal lines. Hence, any two different strophoids with the node N at the origin and passing through A, B and C span a pencil of such strophoids.[2] Through each point different from N, A, B, and C passes exactly one strophoid of this linear set.
Conversely, for each strophoid $\mathcal{S}$ through ABC and with the node N the remaining point of intersection with $[A, B]$ must be associated to C. By virtue of Lemma 4, N and the associated pair (C, C') define a strophoid uniquely. Hence, $\mathcal{S}$ coincides with the strophoid which is included in the pencil and passes through C'.
The respective negative pedal curves are parabolas sharing three tangents, the lines through the three vertices A, B, C, orthogonal to the respective connections with the node N. These lines are the sides of the triangle $E_A E_B E_C$ formed by the centers of the remaining tritangent circles of ABC. By virtue of a wellknown theorem on parabolas (e.g., [8, p. 381]), the foci of these parabolas lie on the circumcircle of the triangle $E_A E_B E_C$. A dilation with center N and factor $\frac{1}{2}$ maps this circumcircle onto the Feuerbach circle $\mathcal{F}$ of $E_A E_B E_C$, which is the circumcircle of ABC and, by virtue of Lemma 1, the locus of the focal points of the strophoids.
Figure 10 shows some strophoides of the pencil, among them the three reducible ones, denoted as $\mathcal{S}_1$, $\mathcal{S}_4$, and $\mathcal{S}_7$. The strophoid $\mathcal{S}_2$ is right; $\mathcal{S}_6$ has its focus at C. For the strophoid $\mathcal{S}_8$ the points B and C are associated, with the respective tangents $[B, A]$ and $[C, A]$.
The inversion in a circle $\mathcal{K}$ centered in N maps the pencil of strophoids onto the pencil of equilateral hyperbolas passing through the vertices of the triangle $\widetilde{A}\widetilde{B}\widetilde{C}$ and its orthocenter N. $\square$

If the isogonal transformation w.r.t. any triangle ABC is given then there are four pencils of strophoids which serve as orbits, i.e., which contain with each point X also its isogonal conjugate X'. The nodes of these strophoids are the centers of the tritangent circles. It is a general property of the isogonal transformation that for any quadrangle consisting of two pairs (P, P') and (Q, Q') of isogonal conjugates the two diagonal points $R = [P, Q] \cap [P', Q']$ and $R' = [P, Q'] \cap [P', Q]$ are isogonal conjugates, too (note, e.g., [6, p. 47]). By virtue of Lemma 2, 3., all strophoids of the four pencils are closed under this operation.

Finally, it must be noted that the family of auto-isogonal cubics studied in [6] does not include strophoids. The cubics in [6] are defined by any point P (=pivot) as the locus of isogonal conjugates (X, X') which are collinear with P. These cubics are in general non-rational (see also [4, p. 1205] or [3]).

[2]The pencil contains also reducible strophoids, e.g., the circumcircle of ABN together with the diameter $[N, C]$.

5 Conclusion

Strophoids play an important role in plane geometry. The goal of this paper was to show how many properties can be derived from the fact that strophoids generalize Apollonian circles. From this particular property follows also that the strophoids of four pencils remain invariant under isogonal transformations with respect to a given triangle.

References

[1] S. Abu-Saymeh, M. Hajja, H. Stachel: *Equicevian Points and Cubics of a Triangle,* J. Geometry Graphics 18/2, 133–157 (2014)

[2] S. Abu-Saymeh, M. Hajja, H. Stachel: *Equicevian Points of a Triangle,* Amer. Math. Monthly **122**/10, 995–1000 (2015)

[3] R.C. Alperin, D.L. Johnson: *Pivot-isogonal cubics,* Beitr. Algebra Geom., DOI 10.1007/s13366-016-0284-8 (to appear)

[4] G. Berkhan, W.F. Meyer: *Neuere Dreiecksgeometrie,* In: *Encyklopädie der Mathematischen Wissenschaften* Bd.3, T.1, H.2, no. AB10, 1173–1276, B.G. Teubner, Leipzig, 1914–1931

[5] O. Bottema: *On some remarkable points of a triangle,* Nieuw Arch. Wiskd., III. Ser., 19, 46–57 (1971)

[6] H.M. Cundy, C.F. Parry: *Cubics Associated with a Triangle,* J. Geometry 53, 41–66 (1995)

[7] F. Dingeldey: *Kegelschnitte und Kegelschnittsysteme,* In: *Encyklopädie der Mathematischen Wissenschaften* Bd.3, T.2, H.1, no. C1, 1–160, B.G. Teubner, Leipzig, 1903

[8] G. Glaeser, B. Odehnal, H. Stachel: *The Universe of Conics,* Spektrum Akademischer Verlag, Heidelberg 2016 (ISBN 978-3-662-45449-7)

[9] G. Kohn, G. Loria: *Spezielle algebraische Kurven,* In *Encyklopädie der Mathematischen Wissenschaften* Bd.3, T.2, H.1, no. C5, 457–570, B.G. Teubner, Leipzig 1903–1915

[10] E. Schmidt: *Strophoiden,* available at `http://eckartschmidt.de/Stroid.pdf`, accessed 2015-12-15

[11] H. Stachel: *Strophoids, a Family of Cubic Curves with Remarkable Properties,* JIDEG (J. of Industrial Design and Engineering Graphics) 10/1, 65–72 (2015)

[12] H. Wieleitner: *Spezielle Ebene Kurven,* G.J. Göschen'sche Verlagsbuchhandlung, Leipzig 1908