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Two particular quadratic cones

Hellmuth Stachel

Vienna University of Technology
Wiedner Hauptstr. 8-10/104, A 1040 Wien
stachel@dmg.tuwien.ac.at

Abstract. The Euclidean geometry of quadratic cones is equivalent to the study of spherical conics. The *normal* (or orthogonal) quadratic cones have circular sections being orthogonal to vertex generators. These cones can be generated by congruent pencils of planes with intersecting axes. The corresponding conics are the spherical analogues of Thales circles.
Equilateral quadratic cones are characterized by a vanishing trace. The associated equilateral spherical conics have the property that the three vertices of a regular right-angled spherical triangle can simultaneously move along. Dualization yields cones which are the envelopes of triples of mutually orthogonal planes. If cones of this type are tangent to a regular quadric then their apices are located on a sphere. This reveals the movability of ellipsoids in circumscribed boxes.

Keywords: quadratic cone, spherical conic, normal cone, equilateral cone.

1 Introduction

A result of Linear Algebra says that for each quadratic cone (= cones of 2^{nd} degree) with real points other than the apex exists a coordinate frame such that the cone's equation has the standard form

$$c_1x^2 + c_2y^2 + c_3z^2 = 0 \quad \text{with} \quad c_1 > 0 > c_2 \geq c_3. \tag{1}$$

Irreducible quadratic cones intersect the unit sphere centered at the apex along two symmetric *spherical conics.* We call each connected component of its intersection with the unit sphere a *spherical ellipse.* Spherical ellipses share many properties with planar ellipses, e.g., the gardener's construction, or the optical property: all rays radiating from one focus pass through the other focus.

Each point P on the sphere has an antipode P^*. Therefore each spherical ellipse can also be seen as one branch of a spherical hyperbola. Furthermore, all spherical parabolas c are spherical ellipses with the major axis $2a = \frac{\pi}{2}$ [1, p. 444].

Among conic constructions which hold in the plane as well as on the sphere, we mention the construction of Proclus (or de la Hire) for points of ellipses. But also the centers of curvature at the vertices can be found for planar and spherical ellipses in a similar way. The latter construction can even be recognized as an analogue of that for planar hyperbolas.

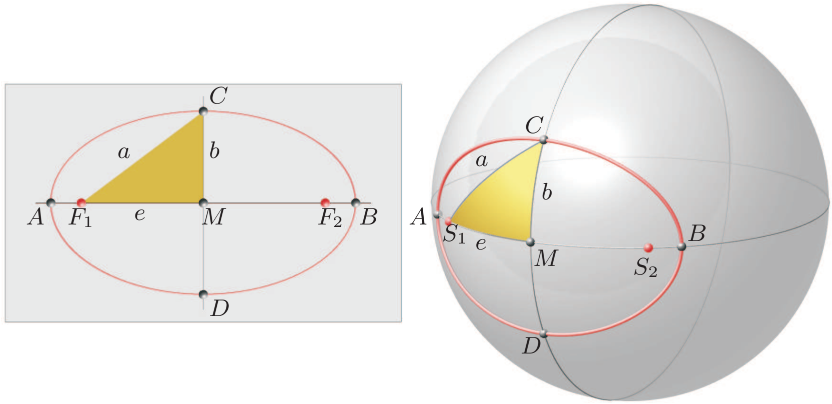

Figure 1: The semiaxes a, b and the eccentricity e of a spherical conic.

If a spherical conic on the unit sphere is given by the standard equation (1) then the semiaxes a, b and the eccentricity e satisfy (Fig. 1)

$$\tan^2 a = -\frac{c_1}{c_2}\,, \quad \tan^2 b = -\frac{c_1}{c_3}\,, \quad \cos a = \cos b \cos e.$$

We are going to study below the two particular cases:

- *normal cones or conics* are characterized by $c_1 + c_3 = c_2$.
- *equilateral cones or conics* have a vanishing trace, $c_1 + c_2 + c_3 = 0$.

2 Normal cones

The cone $c_1x^2 + c_2y^2 + c_3z^2 = 0$ with apex O at the origin is normal if and only if $\sin a = \tan b$. It is easy to confirm that exactly in this case the circular sections of the cone, which are parallel to one of the planes $x\sqrt{c_1 - c_2} \pm z\sqrt{c_2 - c_3} = 0$, are orthogonal to one of the generators in the plane $y = 0$ of symmetry.

Theorem 1. *On the sphere the set of points P with $\sphericalangle APB = \pi/2$, i.e., the spherical analogue of the Thales circle, is a normal conic c with A and B as vertices on the minor axis (Fig. 2).*

Proof. This can be proved using basic Descriptive Geometry: Let g, h be the lines which connect the given points A, B, respectively, with the center O of the sphere. We are looking for diameter lines p such that their connecting planes with g and h are orthogonal. If we specify g in vertical position then for any plane ε through g the normal line n through any point $H \in h$ to ε is horizontal, and n spans with h a plane orthogonal to ε. The line p of intersection of these two planes passes through the

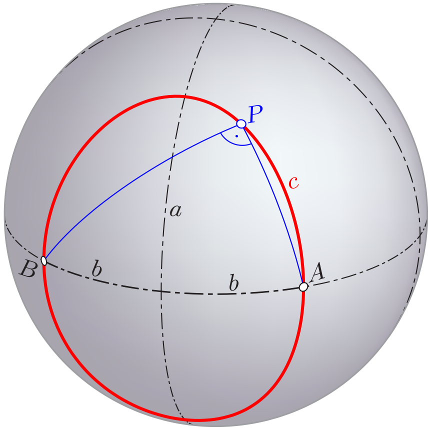

Figure 2: The spherical analogue of the Thales circle is a normal conic.

pedal point of H in ε. For $H \neq O$, this pedal point lies on a horizontal circle, due to Thales' theorem. Therefore the lines p belong to a cone, which contains a horizontal circle and a vertical generator g in a plane of symmetry. □

Remark. In other cases the spherical isoptic curves c for the segment PQ, i.e., the sets of points X with $\sphericalangle PXQ = \varphi$ or $\pi - \varphi$, where $\varphi \neq \frac{\pi}{2}$, are spherical quartics (see [1, Fig. 10.20, p. 465].

In the plane, the constance of the angle φ of circumference and the constant sum of interior angles in each triangle imply: Circles can be generated by an orientation-preserving congruence between two line pencils P, Q. On the sphere the analogue curves are no more isoptic.

Theorem 2. *An orientation-preserving congruence between the pencils P, Q of great circles generates a pair of antipodal normal conics with the spherical bisector of P and Q as the minor axis (Fig. 3).*

Proof. By virtue of standard results of Projective Geometry, the given congruence generates a pair of spherical conics c, and the great circle $1 = 2'$ connecting P and Q corresponds to the respective tangents $1'$ and 2 at Q and P. The angle bisectors $3, 4$ of $1, 2$ are mapped onto the angle bisectors $3', 4'$ of $1', 2'$. Their points of intersection $A \in 3, 3'$ and $B \in 4, 4'$ lie on the orthogonal bisector of P and Q, which is an axis of symmetry of the generated conic c. Hence, A, B and Q define c already uniquely, and the conic c is normal, by virtue of Theorem 1. □

Remark. After replacing one base point P by its antipode, we obtain a similar result in the case of an orientation-reversing congruence.

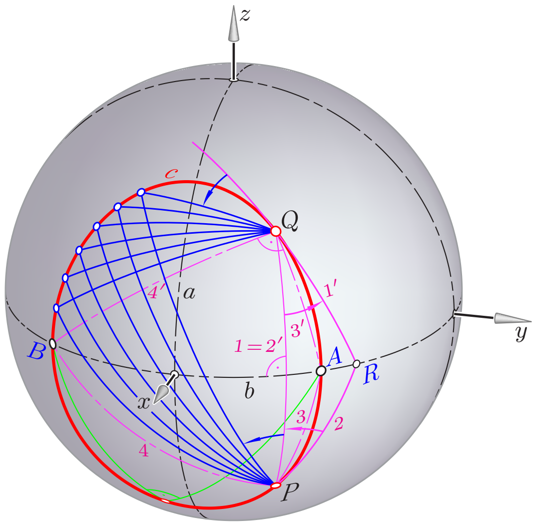

Figure 3: On the sphere an orientation-preserving congruence between two pencils of great circles generates a normal conic.

3 Equilateral cones

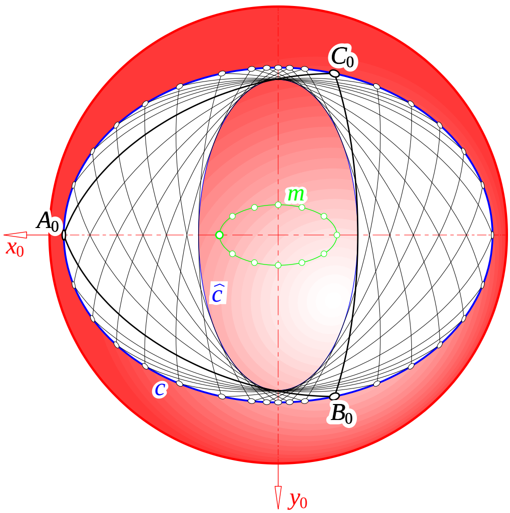

Figure 4: In an equilateral conic a spherical octant can move around.

The cone (1) is equilateral iff $c_1 + c_2 + c_3 = 0$, hence iff

$$\sin^2 b = \frac{\sin^2 a}{3\sin^2 a - 1}, \quad \text{where } \sqrt{\tfrac{2}{3}} < \sin a < 1.$$

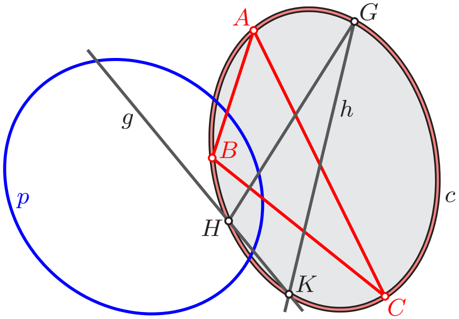

Figure 5: The conic p is inpolar to the conic c.

Theorem 3. *In each equilateral spherical conic c a regular right-angled triangle ABC can move around while all three vertices A, B, and C run along c.*

We prove this by using Projective Geometry.

Definition 1. A conic p is called *inpolar* (apolar) to the conic c if there exists a triangle ABC auto-polar w.r.t. p and inscribed in c (Fig. 5).

Together with ABC there is a always one-parameter set of triangles GHK which are auto-polar with respect to p and inscribed in c. All these triangles are circumscribed to the conic $\widehat{c}$ which is polar to c w.r.t. p.

Proof. A result of von Staudt says: If the two triangles ABC and GHK are auto-polar w.r.t. p the six vertices are located on a regular or singular curve of degree 2. This curve is already uniquely defined by the five points A, B, C, G, H. □

We show that in the case of Theorem 3 the polarity in p means orthogonality in the bundle, and the cone c is equilateral. For this purpose we need another theorem from analytic Projective Geometry [1, p. 420].

Theorem 4. *Given $p:\ \mathbf{x}^T\mathbf{P}\,\mathbf{x} = 0$ and $c:\ \mathbf{x}^T\mathbf{C}\,\mathbf{x} = 0$ with symmetric matrices $\mathbf{P}$ and $\mathbf{C}$, with c containing real points.*
Then, p is inpolar to c if and only if in the characteristic polynomial

$$\det(\sigma\mathbf{P} + \tau\mathbf{C}) = j_0\,\sigma^3 + j_1\,\sigma^2\tau + j_2\,\sigma\tau^2 + j_3\,\tau^3$$

the coefficient of $\sigma^2\tau$ vanishes, i.e., $j_1 = \det\mathbf{P}\cdot\mathrm{tr}(\mathbf{C}\mathbf{P}^{-1}) = 0$ [1, p. 420].

In our case we have $\mathbf{P} = \mathbf{I}_3$, hence $\mathrm{tr}(\mathbf{C}) = 0$.

The sides of the moving triangle envelope another spherical conic $\widehat{c}$. The cones connecting c and $\widehat{c}$ with the center are mutually orthogonal.

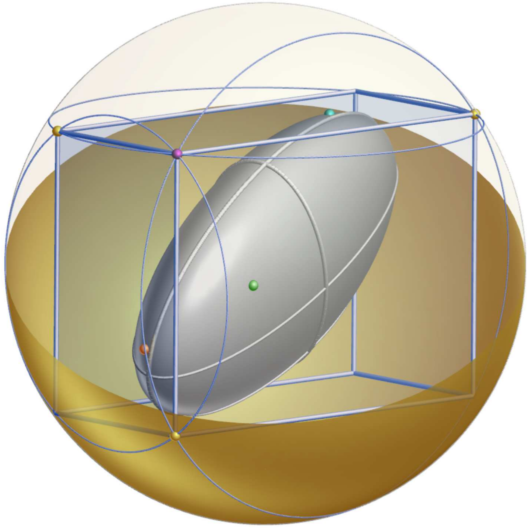

Figure 6: Boxes circumscribed to a tri-axial ellipsoid can move around.

A regular right-angled triangle circumscribed to $\widehat{c}$ can move such that all sides remain tangent to $\widehat{c}$. Such conics are called *dual-equilateral*.

Theorem 5. *The point* $S = (\xi, \eta, \zeta)$ *is the intersection of three mutually orthogonal tangent planes* τ_1*,* τ_2*,* τ_3 *of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *if*

$$\xi^2 + \eta^2 + \zeta^2 = a^2 + b^2 + c^2, \tag{2}$$

and then $\overline{M\tau_1}^2 + \overline{M\tau_2}^2 + \overline{M\tau_3}^2 = a^2 + b^2 + c^2$*.*

A parameter count reveals (Fig. 6): Boxes circumscribed to a tri-axial ellipsoid, hence inscribed in the *director sphere* (2), can move around [2].

4 Conclusion

Almost all presented theorems can be found in the classical geometry literature. Nevertheless, one can still enjoy their beauty and the elegance of reasoning. Furthermore, they are also a challenge to use modern media for their visualization.

References

[1] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics,* Springer Spektrum, Heidelberg 2016

[2] G. Glaeser, B. Odehnal, H. Stachel: *The Universe of Quadrics,* Springer Spektrum (in preparation)