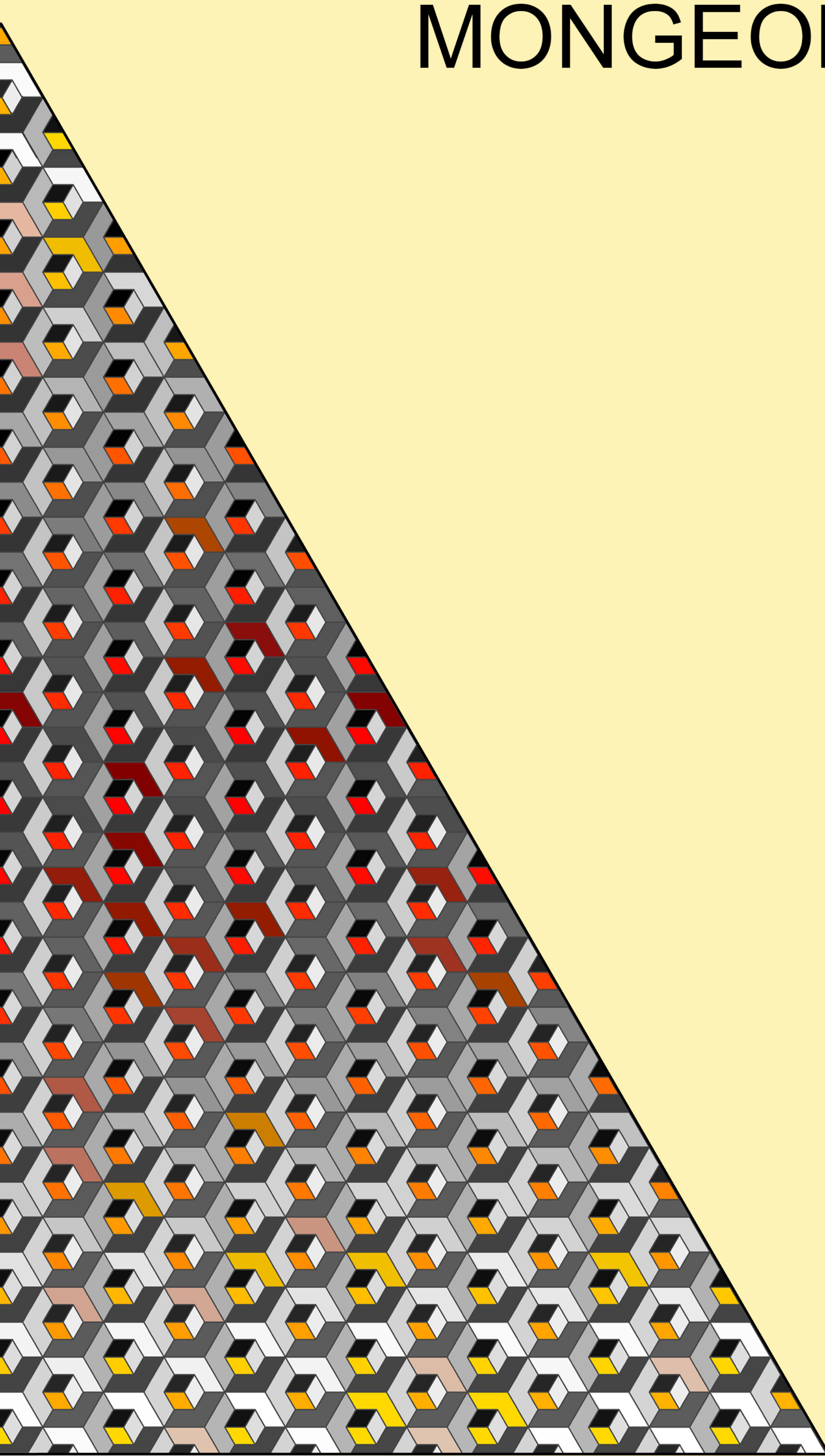
6th International Conference on Geometry and Graphics

MONGEOMETRIJA 2018



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RECALLING IVORY'S THEOREM

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Abstract

Ivory's Theorem states that in each curvilinear quadrangle of a confocal net of conics the two diagonals have the same lengths. This theorem is valid not only in the Euclidean plane, but also in planar hyperbolic, spherical and pseudo-Euclidean (or Minkowski) geometry, and similar statements hold in all dimensions. Recent publications on this theorem and its generalizations on surfaces are the reason to focus again on this topic and to show a few algebraic consequences.

Keywords: Ivory's Theorem, confocal conics, incircular net, Poncelet grid.

1 INTRODUCTION

Ivory's Theorem [11] states that in each curvilinear quadrangle *PP'Q'Q* of a confocal[51] family of conics the two diagonals have the same lengths *d(PQ') = d(P'Q)* (Fig. 1). This theorem is valid not only in the Euclidean plane, but also in hyperbolic, spherical and pseudo-Euclidean (or Minkowski) geometry. Similar statements are valid in all dimensions (see, e.g., [4, 8, 10, 11, 12, 14, 15]). A converse of the Euclidean version is proved in [13].

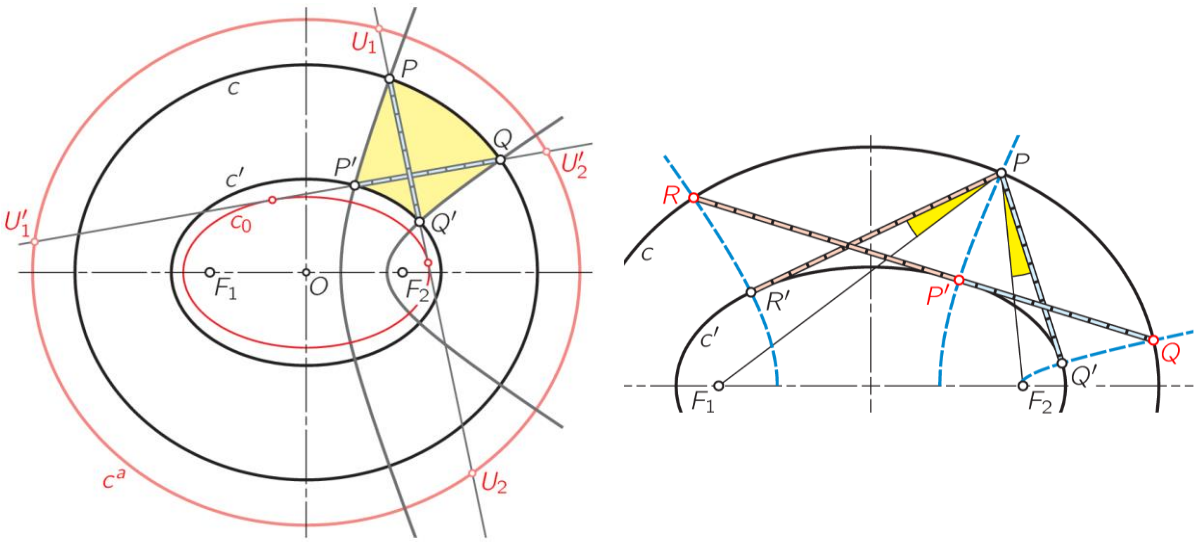

Fig. 1. Ivory's Theorem in the Euclidean and hyperbolic plane, d(PQ') = d(P'Q), d(PR') = d(P'R)

Due to recent publications [1, 12] we focus again on this topic and show a few consequences. While [12] presents a differential-geometric approach, which is also valid for Liouville and Stäckel nets on surfaces, we emphasize algebraic aspects.

Two confocal central conics of the same type in the Euclidean plane can be represented as

$$c:\ (1-\lambda^2)x^2 + (1-\mu^2)y^2 = C, \qquad c':\ \frac{1-\lambda^2}{\lambda^2}x^2 + \frac{1-\mu^2}{\mu^2}y^2 = C$$

where $\lambda, \mu \in \mathbb{R} \setminus \{0,1\}$ and $C = \mathrm{const}$. The affine transformation $(x, y) \mapsto (x', y') = (\lambda x, \mu y)$ maps *c* onto *c'*. Confocal parabolas can be represented as

[51] Conics with a center (*central* conics, in brief) are called *confocal* when they share the two focal points. Two parabolas are confocal if they share the focal point and the axis.

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$$c:\ -4px+(1-\mu^2)y^2=C, \qquad c':\ -4p(x-p)+\frac{1-\mu^2}{\mu^2}y^2=C,$$

where $p \neq 0$ and $\mu \neq 0,1$. Here the affine transformation $(x, y) \mapsto (x', y') = (x+p, \mu y)$ takes *c* onto *c'*. In both cases a straight forward computation shows that for any two points $P = (x_1, y_1) \in c$ and $Q' = (x_2', y_2') \in c'$ their distance equals that between the image $P' = (x_1', y_1')$ of *P* and the preimage $Q = (x_2, y_2)$ of *Q'*.[52]

In the case of confocal conics with the center *O* we can confirm similarily that

$$d(OP)^2 + d(OQ')^2 = d(OP')^2 + d(OQ)^2,$$

and for the dot product of vectors

$$vec(OP) \cdot vec(OQ') = vec(OP') \cdot vec(OQ).$$

More general, for any coaxial conic *d* holds: *P and Q' are conjugate with respect to* ('w.r.t.' in brief) *d if and only if P' and Q are conjugate w.r.t. the conic d.* This means in the case *d* = *c'* : If $Q,R \in c$ lie on the tangent to c' at *P'* then $Q', R' \in c'$ are the points of contact of the tangents drawn from $P \in c$ to *c'* (Fig. 1, right). By the same token, these tangents have common angle bisectors with the lines connecting *P* with the focal points (see, e.g., [9, p. 42]).

If a conic of a given confocal family serves as absolute conic c^a in the Cayley-Klein model of a hyperbolic geometry (Fig. 1, left) then the conics *c* and *c'* are also confocal in the hyperbolic sense, i.e., they share the (complex conjugate) common tangents with the absolute conic c^a. Therefore (e.g., according to [15]) there are equal cross ratios $(PQ'U_1U_2) = (P'QU_1'U_2')$ with the respectively collinear absolute points $U_1, \ldots, U_2'$. In the limiting case $U_1 = U_2$ we obtain a result, which is cited in [2, p. 153], but probably has been known earlier:

Lemma 1. *In each Ivory quadrangle the two diagonal lines are tangent to the same conic c_0 of the confocal family (Fig. 1, left).*

2 SOME CONSEQUENCES

Ivory's Theorem and Lemma 1 can be used to reprove a theorem recently published.

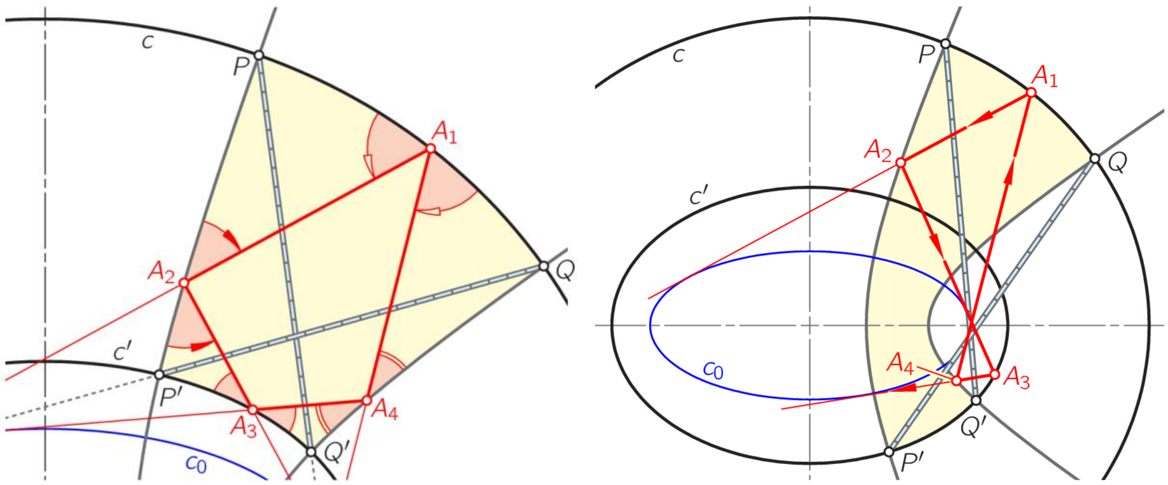

Fig. 2. All billiards consisting of tangents to c_0 are closing 'within' PP'Q'Q

Theorem 2. (Izmestiev, Tabachnikov [12]) *Given a confocal family of conics, in each Ivory quadrangle PP'Q'Q with diagonals tangent to c_0 all billiards with sides being tangent to c_0 are closing and have equal lengths (Fig. 2).*

Proof: Let *c*, h_1, *c'*, h_2 be the conics carrying the curved sides of the Ivory quadrangle *PP'Q'Q* (Fig. 3). We choose a point $A_1 \in c$ within the curved side *PQ* and denote with h_3 the second confocal conic

[52] To be precise, Ivory's Theorem (Euclidean version) is valid only for curvilinear quadrangles *PP'Q'Q* where opposite sides (‚*arcs semblables*' according to [6]) are corresponding under an affine transformation which fixes the axes of the confocal conics. In this case we speak of an *Ivory quadrangle*.

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passing through A_1. If D_1 is a point of intersection between h_3 and the diagonal PQ' then there is a sub-quadrangle $PA_1D_1A_2$ with $A_2 \in h_1$ and one diagonal tangent to c_0. Hence, by virtue of Lemma 1, also the line A_1A_2 contacts c_0.

The second confocal conic c'' through A_2 intersects $P'Q$ at D_2, and thus we obtain a second sub-quadrangle $P'A_2D_2A_3$ with diagonals tangent to c_0. Similarly we find a point $D_3 \in PQ'$ and furtheron $A_4 \in h_2$. Finally, the quadrangle $A_1D_4A_4Q$ reveals that the line A_1A_4 is tangent to c_0, too.

We infer from Ivory's Theorem that

$$d(A_1A_2) + d(A_2A_3) + d(A_3A_4) + d(A_4A_1) = d(PD_1) + d(P'D_2) + d(D_3Q') + d(D_4Q)$$
$$= (d(PQ') - d(D_1D_3)) + (d(P'Q) + d(D_2D_4)) = 2d(PQ').$$

On the other hand, the focal properties yield equal angles at A_1, ..., A_4, as indicated in Fig. 2, left. This confirms that $A_1 ... A_4$ is a closing billiard within the Ivory quadrangle $PP'Q'Q$.

Figure 2 shows on the right hand side that for Ivory quadrangles, which cross an axis, the billiard $A_1 ... A_4$ can look quite different. This calls to mind that the construction of the billiard for given A_1, as listed above, is ambiguous, since the point of intersection between a line and a conic is not unique. Only the respective fourth vertex of a curved Ivory quadrangle is unique because of the affine transformations mapping one side on the opposite side. □

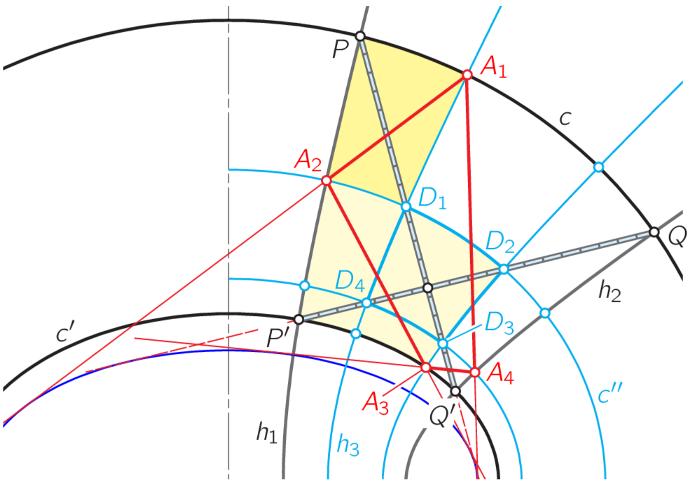

Fig. 3. The sides of the billiard and the diagonals of the bounding quadrangle PP'Q'Q

In [12] the authors reprove in a differential-geometric way (note Section 3) the following theorem (see Fig. 4).

Theorem 3. *If the tangents drawn from any two points A_1, B_1 of a conic c_1 to a confocal conic c_0 form a quadrilateral then each other pair of opposite vertices (A_i, B_i), i = 2,3, belongs to the same conic c_i of the confocal family. The quadrilateral is 'incircular', i.e., it has an incircle.*

This theorem has already been published by Chasles [6, p. 841] and later by Böhm in [3, p. 221]. The same theorem was studied recently in [1]. A projective version of this statement is given below in Theorem 4 and proves at the same time the properties of Poncelet grids (see, e.g., [9, p. 412]).

The second part of Theorem 3, which is also discussed in [5], can be concluded from Ivory's Theorem, as shown in Fig. 4: The respectively second confocal conics through A_1, B_1, A_2, and B_2 define four Ivory quadrangles. The diagonals passing through the common vertex S must be aligned, since by Lemma 1 they are tangent to c_0. Thus we can immediately figure out that in the quadrangle $A_1A_2B_1B_2$ the sums of lengths of opposite sides equal *d(PQ')*.

In the sequel, the term 'conic' stands for regular conics, seen as set of their tangent lines, as well as for pairs of line pencils and for single line pencils with multiplicity two. Expressed in terms of homogeneous line coordinates, the corresponding quadratic forms have rank 3, 2 or 1, respectively. Moreover, we use the term *range* for a pencil of dual conics, i.e., a pencil in line coordinates. The term *net* denotes a 2-parametric linear system of dual curves of degree 2. Obviously, conics and ranges

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included in a net can be seen as points and lines of a projective plane. Any two ranges in a net must have a *conic* in common (compare with [7, Théorèmes I – IV]).

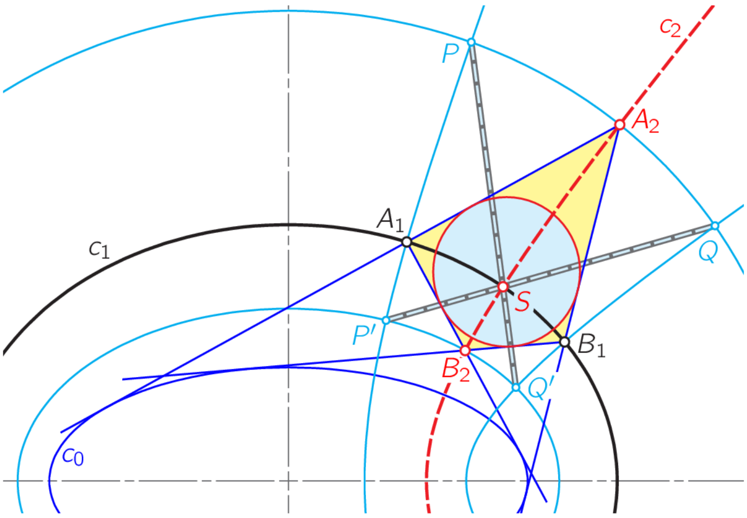

Fig. 4. An incircular quadrangle $A_1A_2B_1B_2$ of tangents (Theorem 3)

Theorem 4. *Let c_0 be any conic and A_1, B_1 two points such that the tangents t_1, ..., t_4 drawn from A_1 and B_1 to c_0 form a quadrilateral. Its remaining pairs of opposite vertices are denoted* by (A_i, B_i) for i = 2,3 *(Fig. 5).*

(i) *For any conic c_1 passing through A_1 and B_1 there exist conics c_i through A_i and B_i such that c_i belongs to the range $\mathcal{R}_c$ spanned by c_0 and c_1.*

(ii) *If $\mathcal{R}_c$ includes pairs of line pencils with carriers (E_j, F_j), j = 1,..., then there exist conics d_j tangent to t_1,...,t_4 and passing through E_j and F_j.*

(iii) *The tangents at A_i and B_i to c_i for i = 1,2,3, as well as the tangents at E_j and F_j to d_j for j =1,... meet at a common point T.*

(iv) This result holds also true in the limiting case $t_1 = t_2$, where the chord A_1B_1 of c_1 contacts c_0 at B_2 and coincides with two of the four tangents t_1, ..., t_4. Then all conics d_j touch c_0 at B_2 and are tangent to t_3 and t_4.

Figure 5 illustrates Theorem 4 in the particular case where c_0 and c_1 span a confocal range $\mathcal{R}_c$. Then the real focal points and the absolute circle points serve as pairs of points (E_j, F_j), as mentioned in (ii) and (iii). The latter correspond to the incircle d_2 of the quadrilateral t_1...t_4. This circle has the center T.

Proof. The conics being tangent to t_1, ..., t_4 define a range $\mathcal{R}_t$, which includes for i = 1,2,3 the pairs of line pencils (A_i, B_i) as well as the initial conic c_0. On the other hand, c_0 and c_1 span a range $\mathcal{R}_c$, which contains the pairs of line pencils (E_j, F_j). Since both ranges share the conic c_0, they span a net $\mathcal{N}$ of conics.

The pair (A_1, B_1) of line pencils spans together with c_1 the range of conics sharing the points A_1, B_1 and the tangents there, which meet at point T. This range, which also belongs to $\mathcal{N}$, contains the rank-1-conic with carrier T. Now each pair of line pencils (A_i, B_i), i = 2,3, spans with the pencil T again a range within $\mathcal{N}$. This range shares with the range $\mathcal{R}_c$ a conic c_i passing through A_i and B_i with respective tangent lines through T. A similar argument holds for the pair of line pencils (E_j, F_j) which proves the existence of a conic d_j through E_j and F_j with tangent lines passing through T, which also belongs to the range $\mathcal{R}_c$.

All these conclusions remain valid in the case (iv), when $\mathcal{R}_t$ consists of conics which touch c_0 at B_2 and are tangent to t_3 and t_4. □

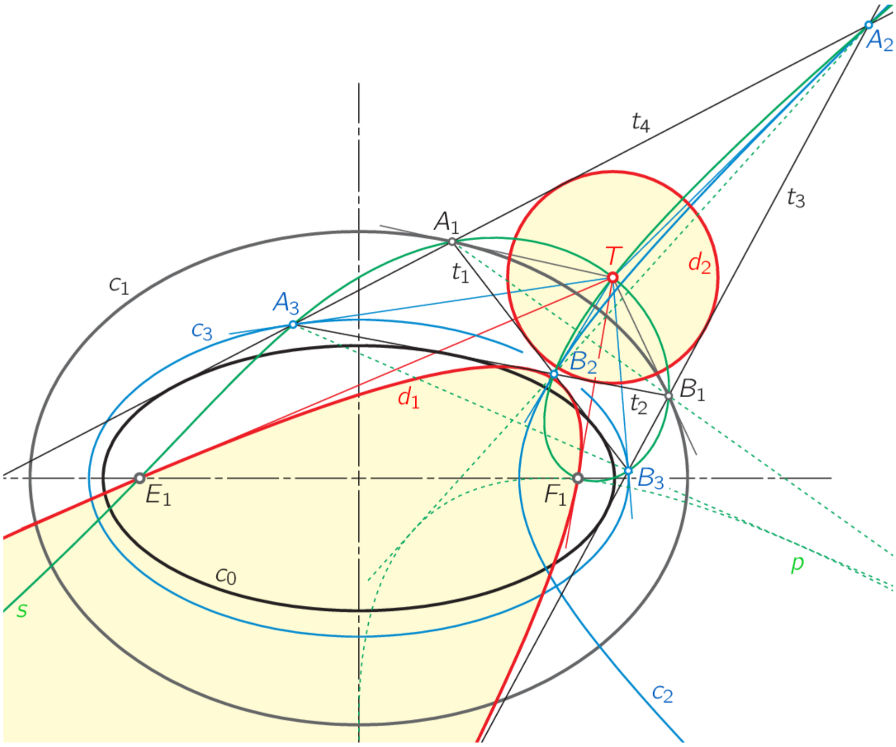

Fig. 5. Illustrating Theorem 4

Remarks. 1) With each net of conics two particular algebraic curves are associated (see [8, Sect. 77]). One is the *Hessian curve* of degree three. It is the envelope of all lines *l* with an undeterminate pole w.r.t. a (singular) conic included in the net. This means that line *l* either connects the carriers of the two line pencils of a rank-2-conic or it passes through the carrier of a rank-1-conic. On the other hand, the locus of the carriers of included line pencils is called *Cayley's curve* of the net. In general it has degree six.

2) In the particular case depicted in Fig. 5 the Hessian consists of the line pencil *T* and of *Chasles's parabola p* of *T* w.r.t. the confocal range $\mathcal{R}_c$. Cayley's curve contains the strophoid *s* comprising the points of contact for all tangents drawn from *T* to any conic in $\mathcal{R}_c$ (compare with [9, p. 342, Fig. 7.60]).

3) Nets of conics, which include a rank-1-conic, are characterized by a reducible Hessian curve. If only one rank-1-conic is included the Hessian splits into a regular conic and this singular line pencil.

In the particular case of Theorem 4 depicted in Fig. 5 with confocal c_0, c_1 the absolute points of the Euclidean plane determine one pair (E_2, F_2) of line pencils included in the range $\mathcal{R}_c \subset \mathcal{N}$. An analogous example works in the projective models of the elliptic or hyperbolic plane. Then instead of the absolute points (E_2, F_2) the absolute conic c^a is included in $\mathcal{R}_c$. We note that c^a spans together with the pencil *T* a range of twice touching conics. Hence, d_2 is a conic which contacts c^a at two points with tangents passing through *T*. This characterizes d_2 as a non-Euclidean circle with center *T*. In the hyperbolic case the points of contact with c^a can be real, complex conjugate or coinciding. Accordingly, the circle d_2 has either a center, or it is a hyper- or horocircle.

Figure 6 shows the spherical model of elliptic geometry and a spherical Poncelet grid starting with a closed billiard in c_1 with 9 edges tangent to c_0 (note [16]). The extended sides of the billiard form a grid of nine great circles. Any two pairs of adjacent great circles form a spherical quadrangle with an incircle, which gives the depicted incircular net [1, 5]. Similarily to the Euclidean case (Fig. 4), the existence of an incircle could also be concluded from equal sums of opposite side lengths.

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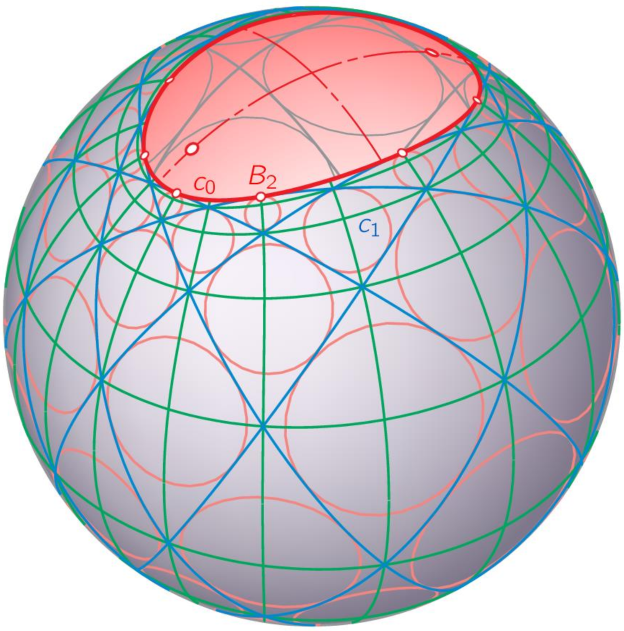

Fig. 6. An incircular net of great circles tangent to c_0 on the sphere

Among the circles depicted in Fig. 6 there are also circles according to the limiting case $B_2 \in c_0$, as mentioned in Theorem 4, (iv). Point B_2 of contact belongs together with the opposite point A_2 to the second confocal conic passing through B_2. We summarize this particular case of Theorem 4.

Corollary 5. *Given a conic c_0 with the tangent t at the point $B_2 \in c_0$. Let t intersect any confocal conic c_1 at the points A_1, B_1, and suppose that the second tangents drawn from A_1 and B_1 to c_0 intersect at the point A_2. Then for all conics c_1 the locus of points A_2 is a confocal conic c_2 passing through B_2.*

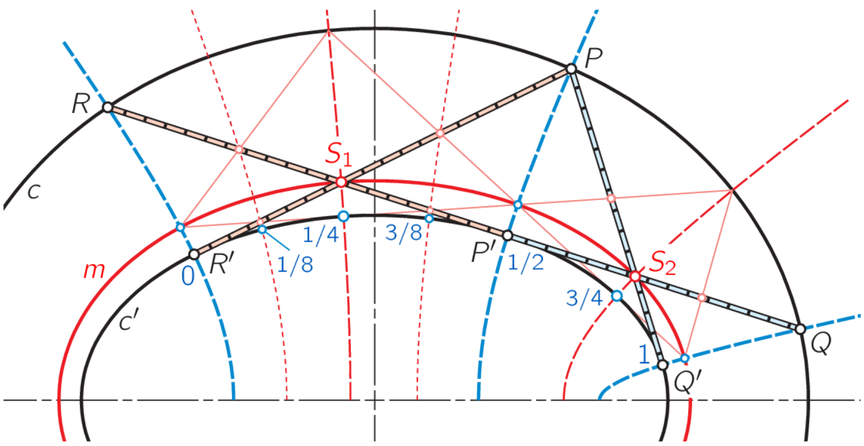

Fig. 7. The crossing points S_1, S_2 lie on the same conic m

Another consequence of this subcase (iv) is illustrated in Fig. 7. Since *P* and *P′* belong to the same conic of the family, the opposite vertices S_1, S_2 in the degenerated quadrangle of tangents belong to the same conic *m*, too. Due to properties of Poncelet grids the conic *m* does not change while *P* varies along *c*. The same can be concluded by the use of canonical coordinates, as explained in the coming section.

3 CANONICAL COORDINATES

The elegant differential-geometric proofs in [12] are based on the Arnold-Liouville theorem from the theory of completely integrable system. According to this, there exist cyclic canonical coordinates on

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c_0 with the following property[53]: If any point X in the exterior of c_0 is parametrized by the canonical coordinates (u,v) of the tangency points of the tangent lines from X to c_0 then the lines $u \pm v$ = const. are located on conics of the confocal family. In [12] c_0 is an ellipse; all curves $u - v$ = const. are on ellipses and $u + v$ = const. are on hyperbolas. There is a period $p > 0$ such that the parameters u and $u + p$ define the same point on c_0.

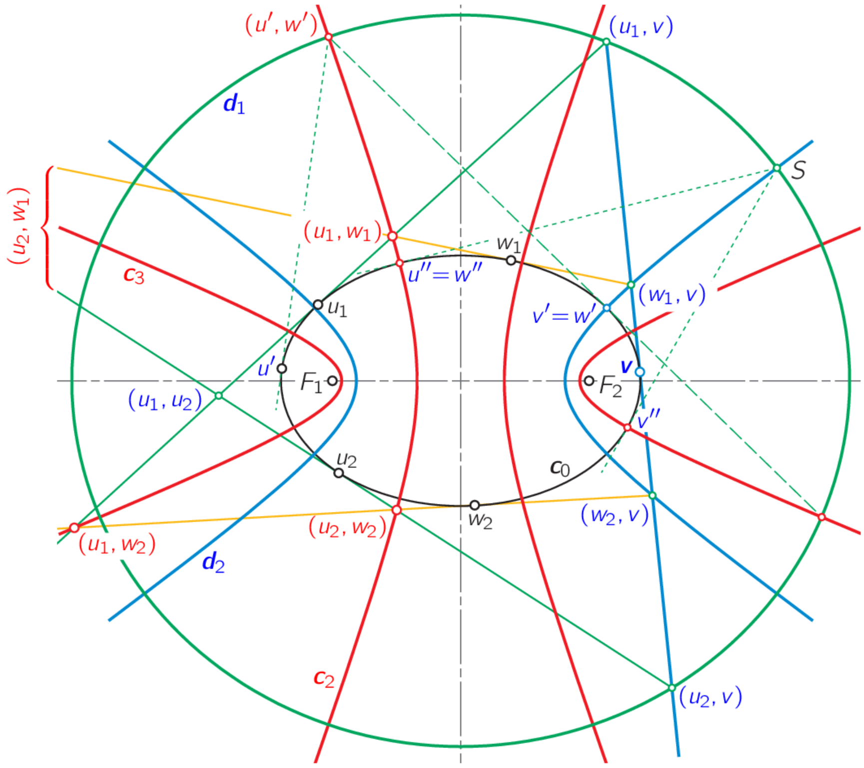

Fig. 7. On the composition of the correspondances (u,v) *w.r.t.* d_1 *and* (v,w) *w.r.t.* d_2 *(Theorem 6)*

Algebraically, each conic d, confocal with c_0, defines a symmetric 2-2-correspondance on c_0 such that the tangents to c_0 at corresponding points intersect on d, or in other words, values (u,v) corresponding w.r.t. d define an unique point of d.

Obviously we can compose the correspondences w.r.t. two conics confocal with c_0. Suppose, (u,v) are corresponding w.r.t. the conic d_1 and (v,w) w.r.t. the conic d_2. In Fig. 8 d_1 is an ellipse and d_2 is a hyperbola. Then $u - v = C_1$ and $v + w = C_2$, and consequently $u + w = C_1 + C_2$ = const. The algebraic version is given below and shown in Fig. 8. The points on c_0 in this figure are denoted by their parameters, while points in the exterior are labelled by pairs of parameters.

Theorem 6. *Let three confocal conics* c_0, d_1, d_2 *be given with canonical coordinates on* c_0. *Suppose that* (u,v) *are corresponding coordinates w.r.t.* d_1, *i.e., the tangents to* c_0 *at the related points intersect on* d_1, *and* (v,w) *are corresponding w.r.t.* d_2.*Then the points defined by parameters* (u,w) *are located on two conics* c_2, c_3 *of the confocal range.*

Proof. Let u_1 and u_2 be corresponding to the coordinate v w.r.t. d_1, and w_1 and w_2 be corresponding to v w.r.t. d_2. According to Corollary 5, for any point $V \in c$ with coordinate v the two corresponding u-values (u_1, u_2) w.r.t. d_1 define a point on the second confocal conic passing through V, which is the same for the conic d_2. Hence, the points A_1 = (u_1,u_2) and B_1 = (w_1,w_2) satisfy the conditions of Theorem 4. Consequently, opposite vertices of the quadrilateral of tangents at u_1, ..., w_2 belong to the same conic c_1 (not displayed in Fig. 8). This means that the points (u_1,w_1) and (u_2,w_2) belong to a conic c_2 and (u_1,w_2) and (u_2,w_1) to another conic c_3.

How is this compatible with the local point of view in [12] ? Why are there two conics? Globally, the parameters (u,v) of points on d_1 are symmetric; this yields two values u_1, u_2 corresponding to v with u_1

[53] In Fig. 7 is illustrated how on c' such coordinates could be constructed by iterated subdivision, when R' and Q' get the respective coordinates 0 and 1.

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$-v = C_1 = v - u_2$. Analogously, the two w-values for the same conic d_2 satisfy $w_1 + v = C_2$ and $w_2 + v = 2p - C_2$. Thus we obtain $u_1 + w_1 = C_1 + C_2$ and $u_2 + w_2 = 2p - (C_1 + C_2)$, and on the other hand $u_2 + w_1 = C_2 - C_1$ and $u_1 + w_2 = 2p - (C_2 - C_1)$. This reveals also that these two conics remain constant while the parameter v varies. □

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