Movement of conics on quadrics

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Abstract. Given any regular quadric, there is a three-parameter set of cutting planes, but the size of an ellipse or hyperbola depends only on its two semiaxes. This parameter count reveals that on each quadric $\mathcal{Q}$ there exist ellipses or hyperbolas with a one-parameter set of congruent copies on $\mathcal{Q}$, which can even be moved into each other. We present parametrizations for such movements on ellipsoids and hyperboloids. There is a close connection between these movements and the theory of confocal quadrics.

Keywords: confocal quadrics, conics on quadrics

1 Introduction

There are well-known examples of conics which can be moved on quadrics. Apart from the trivial case of circles on a sphere, paraboloids are surfaces of translation, even with a continuum of translational nets of parabolas. On quadrics of revolution, each planar section can be moved.

What's about general quadrics $\mathcal{Q}$? There is a three-parameter family of cutting planes, but the size of an ellipse or hyperbola depends only on its two semiaxes. The situation for parabolas is similar: Their size depends on one single length, its parameter, while on hyperboloids and paraboloids there exists a two-parameter family of planes which intersect along parabolas.

This parameter count reveals that on each quadric $\mathcal{Q}$ there exist conics with a one-parameter family of congruent copies on $\mathcal{Q}$. Below, we focus on central quadrics and provide parametrizations for the movement of appropriate ellipses and hyperbolas $\mathcal{Q}$. It turns out that there is a close connection with the theory of confocal quadrics.

2 Moving ellipses on an ellipsoid

On any regular quadric $\mathcal{Q}$, the intersections with parallel planes are homothetic. This means, in the case ellipses or hyperbolas, that they have parallel axes and the same ratio of semiaxes $a_e : b_e$. Moreover, their centers lie on the same diameter. This is a consequence of the polarity with respect to (w.r.t., in brief) $\mathcal{Q}$.

In the case of an ellipsoid $\mathcal{E}$, we obtain the biggest ellipse of this homothetic family in the plane through the center O. On the other hand, there is a point $P \in \mathcal{E}$ with a tangent plane τ_P parallel to the cutting planes, and the axes of the conics are parallel to the principal curvature directions at P. The conics are even homothetic to the Dupin indicatrix at P. This can be confirmed, e.g., by straight forward computation using the Taylor expansion of the quadratic polynomial at P.

According to the definition of the Dupin indicatrix, the ratio of the principal curvatures κ_1, κ_2 at P is reciprocal to the ratio of the squared semiaxes of the ellipses on $\mathcal{E}$ in planes parallel to τ_P, i.e.,

$$a_e : b_e = \sqrt{\kappa_1} : \sqrt{\kappa_2}, \quad \text{if} \quad \kappa_1 > \kappa_2. \tag{1}$$

The lines of curvature on quadrics are related to confocal quadrics. Therefore, we recall the relevant properties of confocal quadrics.

2.1 Confocal central quadrics

Let $\mathcal{E}$ be a triaxial ellipsoid with semiaxes a, b, and c. The one-parameter family of quadrics being confocal with $\mathcal{E}$ is given as

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1, \text{ where } k \in \mathbb{R} \setminus \{-a^2, -b^2, -c^2\} \tag{2}$$

serves as parameter. In the case $a > b > c > 0$ this family includes

$$\text{for} \left\{ \begin{array}{ll} -c^2 < k < \infty & \text{triaxial ellipsoids,} \\ -b^2 < k < -c^2 & \text{one-sheeted hyperboloids,} \\ -a^2 < k < -b^2 & \text{two-sheeted hyperboloids.} \end{array} \right. \tag{3}$$

Confocal quadrics intersect their common planes of symmetry along confocal conics. As limits for $k \to -c^2$ and $k \to -b^2$ we obtain 'flat' quadrics, i.e., the focal ellipse and the focal hyperbola.

The confocal family sends through each point $P = (\xi, \eta, \zeta)$ outside the coordinate planes exactly one ellipsoid, one one-sheeted hyperboloid and one two-sheeted hyperboloid. The corresponding parameters k define the three *elliptic coordinates* of P. We concentrate on points P of the ellipsoid $\mathcal{E}$ with $k = 0$, and denote the parameters of the two hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, by k_1 and k_2. Hence,

$$\mathcal{E}: \ \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1 \tag{4}$$

and, for $i = 1, 2$ and $-a^2 < k_2 < -b^2 < k_1 < -c^2 < 0$

$$\mathcal{H}_i: \ \frac{\xi^2}{a^2+k_i} + \frac{\eta^2}{b^2+k_i} + \frac{\zeta^2}{c^2+k_i} = 1. \tag{5}$$

For given Cartesian coordinates (ξ, η, ζ) of a point $P \in \mathcal{E}$, the parameters k_1 and k_2 of the hyperboloids through P are the two roots of the quadratic equation

$$k^2 + L\,k + M = 0 \tag{6}$$

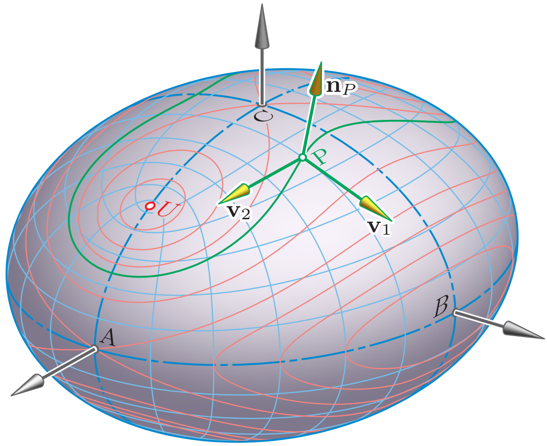

Figure 1: Curvature lines (blue), curves of constant ratio of principal curvatures $\kappa_1 : \kappa_2$ (red), and direction vectors $\mathbf{v}_1, \mathbf{v}_2$ of the principal curvature tangents at P.

with coefficients

$$\begin{aligned} L &= \frac{(b^2+c^2)\xi^2}{a^2} + \frac{(c^2+a^2)\eta^2}{b^2} + \frac{(a^2+b^2)\zeta^2}{c^2}, \\ M &= \frac{a^2b^2c^2}{h^2}, \text{ where } h = \overline{O\tau_P} \text{ and } \frac{1}{h^2} = \frac{\xi^2}{a^4} + \frac{\eta^2}{b^4} + \frac{\zeta^2}{c^4}. \end{aligned} \tag{7}$$

If, conversely, the tripel $(0, k_1, k_2)$ of elliptic coordinates is given, then the Cartesian coordinates (ξ, η, ζ) of the corresponding points satisfy

$$\begin{gathered} \xi^2 = \frac{a^2(a^2+k_1)(a^2+k_2)}{(a^2-b^2)(a^2-c^2)}, \quad \eta^2 = \frac{b^2(b^2+k_1)(b^2+k_2)}{(b^2-c^2)(b^2-a^2)}, \\ \zeta^2 = \frac{c^2(c^2+k_1)(c^2+k_2)}{(c^2-a^2)(c^2-b^2)}. \end{gathered} \tag{8}$$

There exist 8 such points, symmetric w.r.t. the coordinate planes.

The differences of any two of the equations in (4) and (5) yield

$$\begin{gathered} \frac{\xi^2}{a^2(a^2+k_i)} + \frac{\eta^2}{b^2(b^2+k_i)} + \frac{\zeta^2}{c^2(c^2+k_i)} = 0, \ i = 1, 2, \quad \text{and} \\ \frac{\xi^2}{(a^2+k_1)(a^2+k_2)} + \frac{\eta^2}{(b^2+k_1)(b^2+k_2)} + \frac{\zeta^2}{(c^2+k_1)(c^2+k_2)} = 0. \end{gathered} \tag{9}$$

This reveals, that confocal quadrics form a triply orthogonal system of surfaces. Due to a theorem of Dupin, the surfaces of a triply orthogonal system intersect each other along lines of curvature. Hence, the lines of curvature on ellipsoids and hyperboloids are of degree 4, except the principal sections in the coordinate planes (see Figure 1).

At each point P of the ellipsoid $\mathcal{E}$ the surface normal n_P to $\mathcal{E}$ at P has the direction vector

$$\mathbf{n}_P = \left(\frac{\xi}{a^2}\,,\ \frac{\eta}{b^2}\,,\ \frac{\zeta}{c^2}\right). \tag{10}$$

On the other hand, for point $P \in \mathcal{E}$ in general position, the two principal curvature tangents are the surface normals of the two hyperboloids $\mathcal{H}_1$ und $\mathcal{H}_2$ through P, therefore in direction of the vectors

$$\mathbf{v}_i := \left(\frac{\xi}{a^2+k_i}\,,\ \frac{\eta}{b^2+k_i}\,,\ \frac{\zeta}{c^2+k_i}\right). \tag{11}$$

2.2 Ellipses on ellipsoids

Now, we look for the biggest ellipse on $\mathcal{E}$ among the homothetic family in parallel planes.

Lemma 1. *The semiaxes of the ellipse in the diameter plane parallel to the tangent plane τ_P at the point $P \in \mathcal{E}$ with the elliptic coordinates $(0, k_1, k_2)$ are*

$$a_P = \sqrt{-k_2}, \quad b_P = \sqrt{-k_1}. \tag{12}$$

Proof. The diameter plane is spanned by the direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ given in (11). We look for $\lambda \in \mathbb{R}$ with $\lambda\mathbf{v}_i \in \mathcal{E}$, hence

$$\lambda^2\left[\frac{\xi^2}{(a^2+k_i)^2a^2} + \frac{\eta^2}{(b^2+k_i)^2b^2} + \frac{\zeta^2}{(c^2+k_i)^2c^2}\right] = 1.$$

This condition does not change if we subtract from the term in square brackets the left-hand side of the first equation in (9), divided by k_i. Thus, we obtain

$$\lambda^2\left[\frac{\xi^2}{(a^2+k_i)^2a^2} - \frac{\xi^2}{k_i(a^2+k_i)a^2} + \dots\right] = 1,$$

and, finally,

$$-\frac{\lambda^2}{k_i}\left[\frac{\xi^2}{(a^2+k_i)^2} + \frac{\eta^2}{(b^2+k_i)^2} + \frac{\zeta^2}{(c^2+k_i)^2}\right] = -\frac{\lambda^2}{k_i}\,\|\mathbf{v}_i\|^2 = 1,$$

hence, $a_P = |\lambda|\,\|\mathbf{v}_2\| = \sqrt{-k_2}$ and $b_P = |\lambda|\,\|\mathbf{v}_1\| = \sqrt{-k_1}$. These equations can already be found in [1, p. 517]. □

For the movement of a given ellipse e with semiaxes (a_e, b_e), Lemma 1 implies the necessary condition

$$a_e \le a_P = \sqrt{-k_2}, \text{ where } b < \sqrt{-k_2} < a\,. \tag{13}$$

Together with (1), we conclude

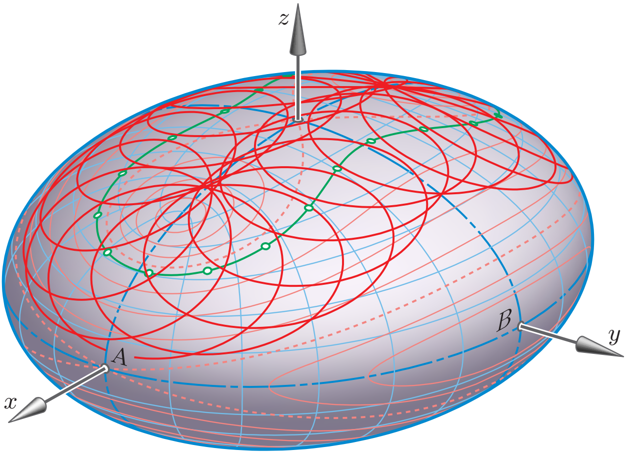

Figure 2: Moving an ellipse on an ellipsoid.

Theorem 1. *If an ellipse e with semiaxes (a_e, b_e) is moving on a triaxial ellipsoid $\mathcal{E}$, then the points $P \in \mathcal{E}$ with tangent planes τ_P parallel to the plane of e moves on a curve with proportional elliptic coordinates $k_2 : k_1 = -a_e^2 : -b_e^2$. This curve is also the locus of points with constant ratio of principal curvatures (Figure 1).*

All ellipses in planes parallel to τ_P have their principal vertices on an ellipse with the conjugate diameters OP and the major axis of the diametral section. Let $\mathbf{p}$ denote the position vector of P and $\mathbf{m} = \mu\,\mathbf{p}$ with $0 \le \mu = \sin x < 1$ that of the center M of any ellipse in this family. Then, its major semiaxis a_e equals $a_P \cos x = a_P\sqrt{1-\mu^2}$, which results in

$$\mu^2 = 1 - \frac{a_e^2}{a_P^2} = 1 - \frac{a_e^2}{t}\ . \tag{14}$$

When, during the movement of the ellipse e, the scalar μ vanishes, then its center M coincides with the center O of $\mathcal{E}$. The corresponding point P has the elliptic coordinate $k_2 = -a_e^2$. In order to continue the motion, point P has to jump to its antipode.

We set

$$v := \frac{k_2}{k_1} = \frac{a_e^2}{b_e^2} = \text{const.}, \text{ where } 1 < v < \frac{a^2}{c^2}\ , \tag{15}$$

and we use the parameter $t = -k_2$ for representing the motion. Then, t is restricted by the interval

$$\max\{b^2, vc^2, a_e^2\} \le t \le \min\{a^2, vb^2\}, \tag{16}$$

and $k_1 = t/v$. From (8) follows the parametrization $\mathbf{p}(t)$ by replacing (k_1, k_2) with $(t/v,\, t)$. This implies for the trajectory of the center M of e

$$\mathbf{m}(t) = \mu(t)\,\mathbf{p}(t) \text{ with } \mu(t) = \sqrt{1 - \frac{a_e^2}{t}}\,. \tag{17}$$

Now, we can express the movement of e in matrix form, in terms of position vectors $\mathbf{x}_m$ w.r.t. the moving space (attached to e) and $\mathbf{x}_f$ w.r.t. the fixed space (attached to $\mathcal{E}$), as

$$\mathbf{x}_f = \mathbf{m}(t) + \mathbf{M}(t)\,\mathbf{x}_m, \quad \text{where } \mathbf{M}(t) = \left[\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},\ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},\ \frac{\mathbf{n}_P}{\|\mathbf{n}_P\|}\right]. \tag{18}$$

The square brackets include the column vectors according to (11) and (10) in the orthogonal matrix $\mathbf{M}(t)$.

Note that this parametrization works only for point P in the octant $x, y, z > 0$. We get a closed movement after reflections in the planes of symmetry (see Figure 2). Algebraic properties of this movement are provided in [2].

3 Moving ellipses on a one-sheeted hyperboloid

Also on hyperboloids and paraboloids, the curves of intersection with parallel planes are homothetic. However, not in all cases the method, as used before for ellipsoids, can be applied since a point P either does not exist or lies at infinity. Moreover, paraboloids have no center O. Below, we analyse only the movements of ellipses on a one-sheeted hyperboloid $\mathcal{H}_1$. The case of moving parabolas is presented in [3].

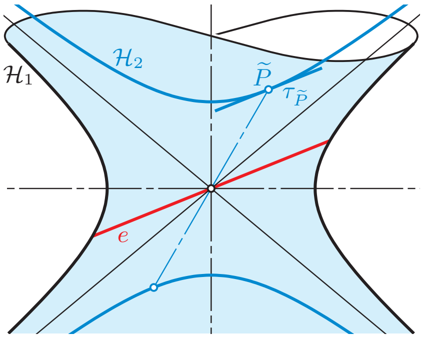

Figure 3: For ellipses e on a one-sheeted hyperboloid $\mathcal{H}_1$, there does not exist a point $P \in \mathcal{H}_1$ with the tangent plane τ_P parallel to the plane of e.

For ellipses $e \subset \mathcal{H}_1$, there is no point $P \in \mathcal{H}_1$ with a tangent plane τ_P parallel to e. However, we find an appropriate point $\widetilde{P}$ on the 'conjugate' two-sheeted hyperboloid $\mathcal{H}_2$ (Figure 3). The hyperboloid $\mathcal{H}_2$ shares the asymptotic cone with $\mathcal{H}_1$, and, therefore, the axes of the ellipse e are parallel to the principal curvature directions of $\mathcal{H}_2$ at $\widetilde{P}$. The two hyperboloids satisfy the respective equations

$$\mathcal{H}_1\colon\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad\text{and}\quad \mathcal{H}_2\colon\ -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

with $a>b$. The quadrics confocal with $\mathcal{H}_2$ are given by

$$-\frac{x^2}{a^2-k}-\frac{y^2}{b^2-k}+\frac{z^2}{c^2+k}=1.$$

Again, this family sends through each point $\widetilde{P}$ outside of the planes of symmetry three mutually orthogonal quadrics, one of each type. On the two-sheeted hyperboloid $\mathcal{H}_2$ with $k=0$, we use the respective parameters k_0 and k_1 of the ellipsoid and the one-sheeted hyperboloid as the elliptic coordinates of $\widetilde{P}$ with

$$k_0>a^2 \quad\text{and}\quad a^2>k_1>b^2.$$

Then, similar to Lemma 1, the ellipse $e\in\mathcal{H}_1$ in the diameter plane parallel to $\tau_{\widetilde{P}}$ has the semiaxes

$$a_{\widetilde{P}}=\sqrt{k_0} \quad\text{and}\quad b_{\widetilde{P}}=\sqrt{k_1}.$$

This is the smallest ellipse on $\mathcal{H}_1$ in the homothetic family.

Hence, if any given ellipse with semiaxes a_e and b_e should be moved on $\mathcal{H}_1$, the corresponding point $\widetilde{P}\in\mathcal{H}_2$ has to trace a curve with proportional elliptic coordinates

$$k_0:k_1=a_{\widetilde{P}}^2:b_{\widetilde{P}}^2=a_e^2:b_e^2$$

on $\mathcal{H}_2$. Similar to (8), we can parametrize the trajectory $\widetilde{\mathbf{p}}(t)=(\xi,\eta,\zeta)$ of $\widetilde{P}$ by $t:=k_0>a^2$, where

$$v:=\frac{k_0}{k_1}=\frac{a_e^2}{b_e^2}=\text{const.},$$

hence $k_1=t/v$ with $b^2\le k_1\le a^2$.

For each $\widetilde{P}$, the principal vertices of the ellipses in planes parallel to $\tau_{\widetilde{P}}$ are placed on a hyperbola, for which one principal vertex in the diameter plane and the point $\widetilde{P}$ define conjugate diameters. If $a_e=a_{\widetilde{P}}\cosh x$, then the position vectors $\mathbf{m}$ of the center of the ellipse e and $\widetilde{\mathbf{p}}$ of the point $\widetilde{P}$ are related by $\mathbf{m}=\sinh x\,\widetilde{\mathbf{p}}$. Thus, we obtain

$$\mathbf{m}=\mu\,\widetilde{\mathbf{p}} \quad\text{with}\quad \mu^2=\frac{a_e^2}{a_{\widetilde{P}}^2}-1\,. \tag{19}$$

This yields, similar to (18), a parametrization for the movement of the ellipse e on $\mathcal{H}_1$ (Figure 4). As a consequence of (19), on the trajectory of $\widetilde{P}$ only points with $a^2_{\widetilde{P}} = k_0 \leq a_e^2$ are admitted. Therefore, the parameter $t = k_0$ runs the interval

$$\max\{a^2,\, vb^2\} \leq t \leq \min\{a_e^2,\, va^2\}.$$

In the case $a_e^2 < va^2$, the same phenomenon appears as mentioned above. When the parameter t reaches a_e^2, then, for continuing the movement of the ellipse, the point $\widetilde{P}$ either has to jump to its antipode, or the scalar μ in (19) must get a negative sign.

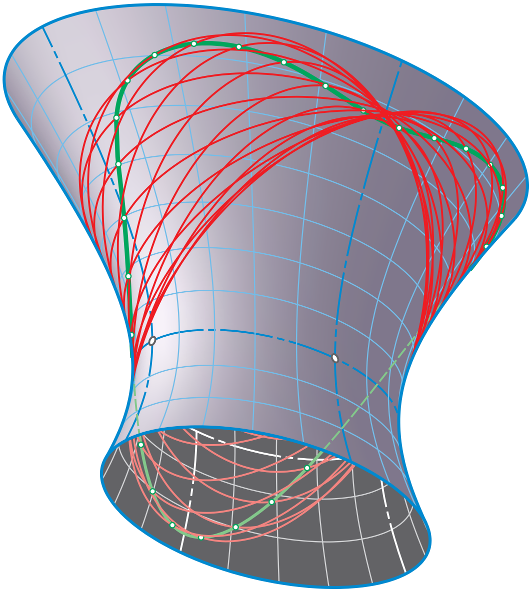

Figure 4: Movement of an ellipse on a one-sheeted hyperboloid.

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