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A Spatial Version of the Theorem of the Angle of Circumference

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Abstract. The presented spatial version of the theorem of the angle of circumference in three-dimensional Euclidean space deals with pairs of planes (ε, φ) passing through two skew straight lines e and f, respectively, such that the angle α enclosed by ε and φ is constant. It turns out that the set of intersection lines $r = \varepsilon \cap \varphi$ is a quartic ruled surface Φ with $e \cup f$ being its double curve. We analyse the properties of Φ and discuss the special cases showing up for special values of some shape parameters such as the slope of e and f (with respect to a fixed plane) or the angle α.

Key Words: ruled surface, angle of circumference, quartic ruled surface, Thaloid, isoptic surface

MSC 2010: 51N20, 51N35, 51M30, 14J16

1. Introduction

The theorem of the angle of circumference states that a straight line segment (bounded by two points E and F) in the Euclidean plane is seen at a constant angle α from any point of a pair of circular arcs passing through E and F. Especially, if the visual angle α is a right angle, the pair of circles becomes one circle with diameter EF, usually referred to as the Thales circle.

It would be natural to generalize the theorem of the angle of circumference in Euclidean three-space by asking for all points that see a straight line segment bounded by two points E and F under a constant angle α. The locus of all such points is an algebraic surface of degree four. It is obvious that the latter surface has a rotational symmetry with respect to the straight line $[E, F]$. This isoptic surface can be obtained by rotating the pair of circular arcs through E and F: it is therefore a torus (see Figure 1). By the same token, also isoptic curves of conics as well as those of pairs of points on the sphere are well-known (see [1]).

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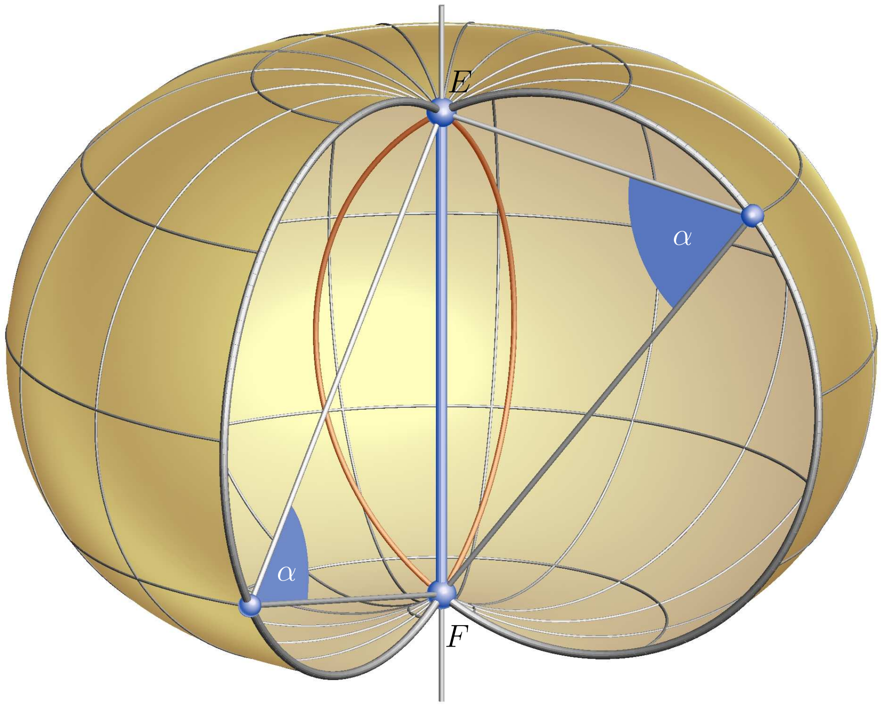

Figure 1: A possible generalization of the theorem of the angle of circumference in three-dimensional Euclidean space.

In this paper, we study a line geometric generalization: We ask for the set of intersection lines r of planes ε and φ from two pencils with $\alpha = \sphericalangle\varepsilon\varphi =$ const.. Each line r can be considered as *one-dimensional eye* seeing a pair of straight lines under a constant angle.

In Section 2, we derive the equation of the ruled surface Φ carrying all lines r that see a pair (e, f) of skew straight lines under constant angle α. From the equation of Φ we can deduce some properties of the surface which shall be the contents of Section 3. Finally, in Section 4 we look at special cases of Φ that arise when the axes e and f reach a special relative position or the angle α attains special values.

2. Equation of the ruled surface

It is favorable to represent points in Euclidean three-space $\mathbb{R}^3$ by Cartesian coordinates (x, y, z). It means no restriction to assume that the axes e and f of the two pencils of planes are given by

$$e = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ k \end{pmatrix}, \quad f = \begin{pmatrix} -d \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \\ -k \end{pmatrix}, \quad t, u \in \mathbb{R}. \tag{1}$$

Here and in the following, $\overline{ef} = |2d| \in \mathbb{R}$ is the distance between the straight lines e, f and $k \in \mathbb{R}$ is their slope with respect to the plane $z = 0$ (see Figure 2).

Since $\mathbf{g} = (0, 1, k)$ and $\mathbf{h} = (0, 1, -k)$ are direction vectors of the lines e and f, the normal vectors $\mathbf{n}_\varepsilon$ and $\mathbf{n}_\varphi$ of the planes ε through e and φ through f are linear combinations of

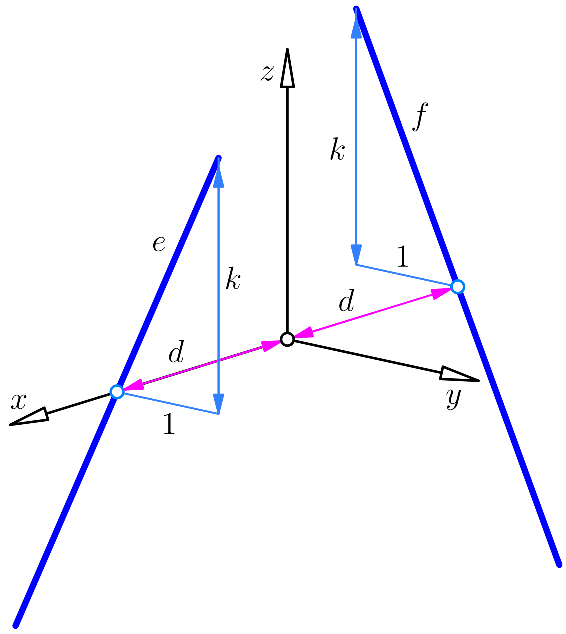

Figure 2: Choice of a Cartesian coordinate system and the meaning of d and k.

$\mathbf{g}_1 = (0, -k, 1)$, $\mathbf{g}_2 = (1, 0, 0)$ or $\mathbf{h}_1 = (0, k, 1)$, $\mathbf{h}_2 = \mathbf{g}_2$, respectively. With $\lambda, \mu \in \mathbb{R}$ we let

$$\mathbf{n}_\varepsilon = \mathbf{g}_1 + \lambda\mathbf{g}_2 = \begin{pmatrix} \lambda \\ -k \\ 1 \end{pmatrix}, \qquad \mathbf{n}_\varphi = \mathbf{h}_1 + \mu\mathbf{h}_2 = \begin{pmatrix} \mu \\ k \\ 1 \end{pmatrix}. \tag{2}$$

Now, we can write down the condition $\sphericalangle(\varepsilon, \varphi) = \sphericalangle(\mathbf{n}_\varepsilon, \mathbf{n}_\varphi) = \alpha$ by evaluating

$$\langle \mathbf{n}_\varepsilon, \mathbf{n}_\varphi \rangle^2 = A^2 \langle \mathbf{n}_\varepsilon, \mathbf{n}_\varepsilon \rangle \langle \mathbf{n}_\varphi, \mathbf{n}_\varphi \rangle$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the canonical scalar product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $A := \cos\alpha$. This gives

$$(1 - k^2 + \lambda\mu)^2 = A^2(1 + k^2 + \lambda^2)(1 + k^2 + \mu^2). \tag{3}$$

The planes from either pencil have normal vectors given in (2), and thus, they have the equations

$$\begin{aligned} \varepsilon: \quad \lambda x - ky + z &= \ \ d\lambda, \\ \varphi: \quad \mu x + ky + z &= -d\mu. \end{aligned} \tag{4}$$

If both λ and μ can vary freely in $\mathbb{R}$, the planes ε and φ intersect in the lines of a hyperbolic linear line congruence with axes e and f, parametrized by

$$\mathbf{r}(t, \lambda, \mu) = \frac{d}{k} \begin{pmatrix} k \\ -\mu \\ -\mu k \end{pmatrix} + t \begin{pmatrix} -2k \\ \mu - \lambda \\ k(\lambda + \mu) \end{pmatrix} \tag{5}$$

where $t \in \mathbb{R}$ is the parameter on the lines in the congruence. The ruled surface Φ we are aiming at is precisely that subset of the congruence (5) where λ and μ are subject to (3).

The equation of the ruled surface Φ in terms of Cartesian coordinates is obtained from the parametrization (5) by eliminating all parameters t, λ, μ: Assume $\mathbf{r} = (r_x, r_y, r_z)$. Then,

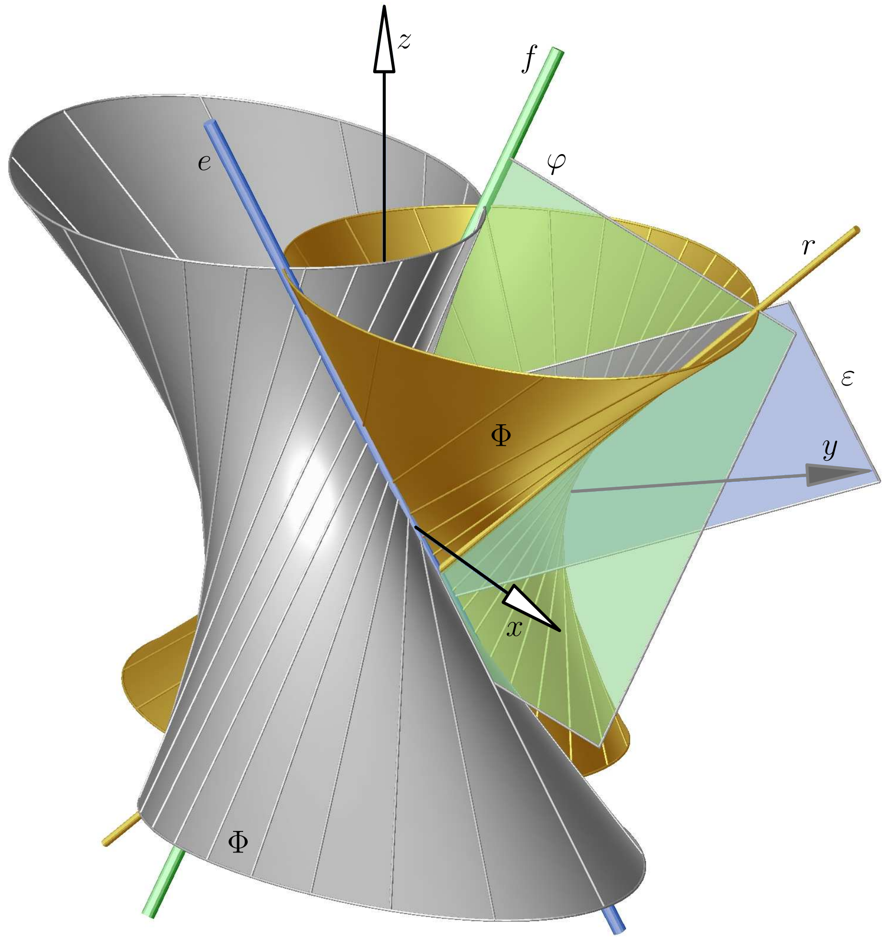

Figure 3: The quartic *isoptic* ruled surface of a pair of skew straight lines e and f.

we eliminate t from $x - r_x$, $y - r_y$, and $z - r_z$ by computing the resultants

$$r_1 := \text{res}\,(x - r_x,\ z - r_z,\ t),$$
$$r_2 := \text{res}\,(y - r_y,\ z - r_z,\ t).$$

In the next step, we eliminate λ from both, r_1 and r_2 using (3) which results in two further polynomials $r'_1 \in \mathbb{R}[x, z, \mu]$ and $r'_2 \in \mathbb{R}[y, z, \mu]$. It would make no difference if we eliminate μ first. Finally, the resultant of r'_1 and r'_2 with respect to μ contains a non-trivial factor which is the equation of Φ. (The trivial factors of the latter resultant are detected by substituting (5) and verifying that they do not vanish.)

So, we obtain the following equation of Φ:

$$\begin{gathered}\sigma_1\sigma_2(x^2 - d^2)^2 - B^2(z^2 - k^2y^2)^2 + 2\sigma_3(d^2z^2 + k^2x^2y^2) + 2\sigma_4(d^2k^2y^2 + x^2z^2) \\ -8A^2dk(1 + k^2)xyz = 0\end{gathered} \tag{6}$$

with the abbreviations

$$\sigma_{1,2} := Ak^2 \pm k^2 + A \mp 1, \quad \sigma_{3,4} := A^2k^2 \mp k^2 + A^2 \pm 1,$$

and $B^2 = 1 - A^2$ (or $B = \sin\alpha$). Summarizing, we can state:

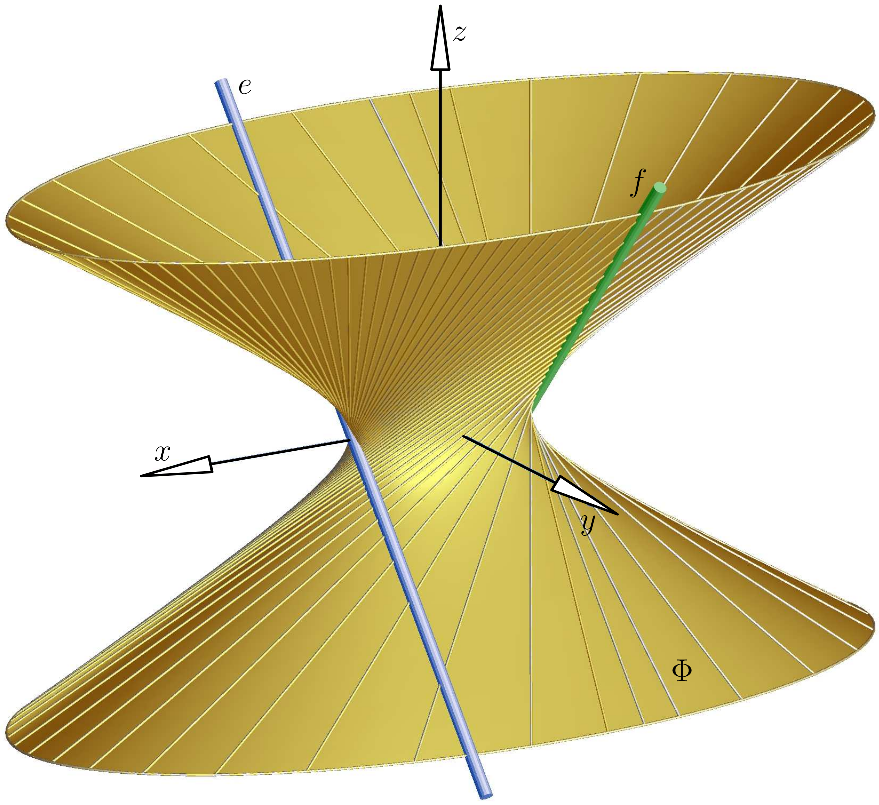

Figure 4: A one-sheeted hyperboloid appears if $A = 0$, $d, k \in \mathbb{R} \setminus \{0\}$.

Theorem 1. *The isoptic ruled surface* Φ *as the set of intersection lines* r *of planes* ε, φ *from two pencils with axes* e, f *and* $\sphericalangle(\varepsilon, \varphi) = \alpha = \text{const.}$ *is the algebraic ruled surface with the equation* (6). *In general, it is of degree four.*

Figure 3 shows an example of a ruled surface Φ together with the axes e and f of the pencils of planes.

In the case $A = 0$ which is equivalent to $\alpha = \frac{\pi}{2}$, there exists a generation of Φ by means of a projective mapping κ from the pencil of planes about e to the pencil of planes about f. The projectivity κ assigns to each plane ε (through e) precisely one plane φ (through f) such that $\varepsilon \perp \varphi$. Thus, the lines $\varepsilon \cap \kappa(\varepsilon)$ form a regulus, *i.e.*, one family of straight lines on a (regular) ruled quadric. Inserting $A = 0$ into (6) returns the equation of the regular ruled quadric which is in any case a hyperboloid (with multiplicity two),

$$\big((1 - k^2)x^2 - k^2y^2 + z^2 + d^2(k^2 - 1)\big)^2 = 0, \tag{7}$$

an example of which is shown in Figure 4.

3. Properties of Φ

From the construction of Φ it is clear that the lines e and f are part of the surface. Moreover, the union of these lines is the double curve of Φ. Hence, Φ is of Sturm type 1 (cf. [2]). Surfaces of this type are elliptic.

Each plane ε in the pencil about e intersects Φ along e with multiplicity 2. Since each such plane ε contains at least one generator, the remaining part of $\varepsilon \cap \Phi$ has to be a straight

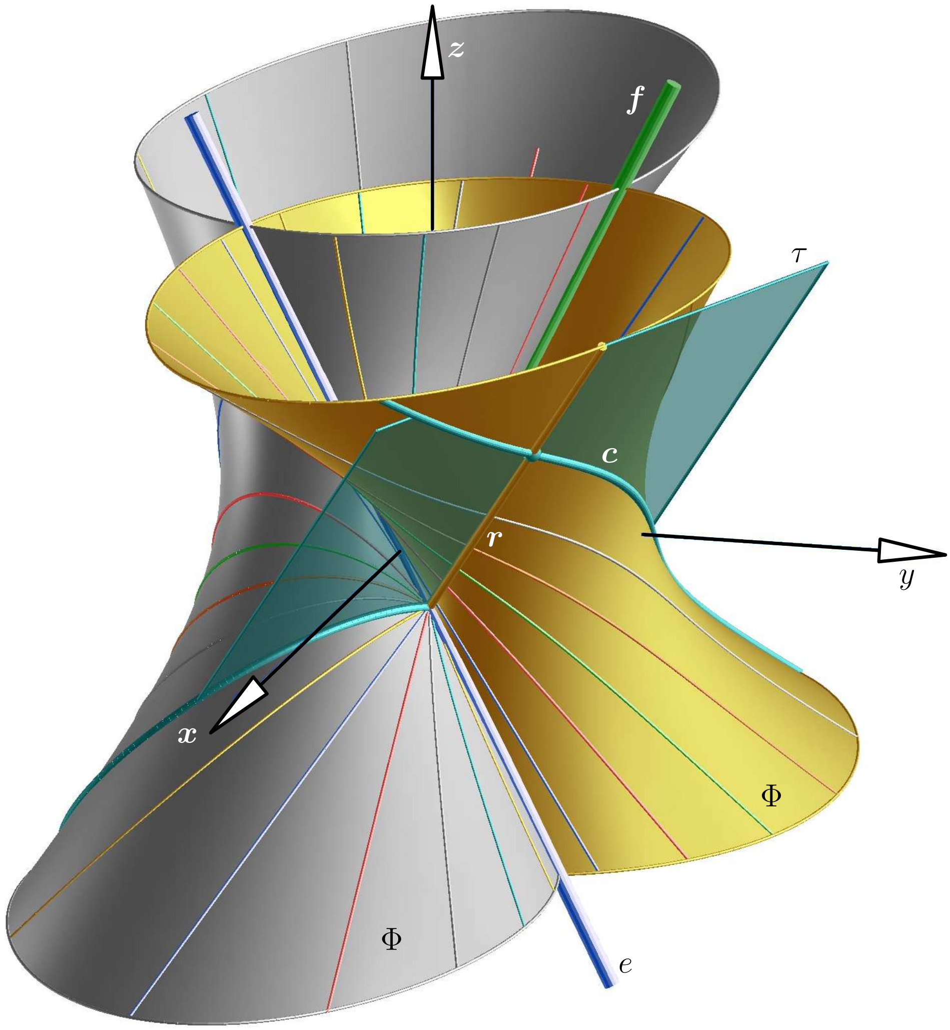

Figure 5: The quartic ruled surface Φ carries a two-parameter family of cubic curves which show up as the intersection of Φ with its tangent planes. Here: $\tau \cap \Phi = r \cup c$ where $r \subset \tau$ is a ruling, τ is a tangent plane through r, and c is the cubic curve.

line s too. The line s is a further generator of Φ. A similar statement can be made about the planes φ through f.

The tangent planes of Φ meet Φ along (planar) cubic curves which are either rational or elliptic (see Figure 5). The quartic ruled surface Φ carries no regular conic: Any plane ε through a pair of intersecting generators g_1, g_2 shares one of the axes, say e, with Φ. Therefore, the remaining part of $\Phi \cap \varepsilon \setminus \{e, g_1, g_2\}$ has to be a straight line l and $l \cup e$ is a singular conic (cf. [2]).

If we perform the projective closure of the Euclidean three-space, then we can look for Φ's intersection Φ_∞ with the ideal plane (see Figure 6). The ideal points E_∞ and F_∞ of the straight lines e and f are the only double points of the elliptic quartic Φ_∞. An equation of Φ_∞ can be obtained from (6) by removing all terms of degree three and less:

$$\Phi_\infty\colon\ (\sigma_1\sigma_2 x^2 + 2k^2\sigma_3 y^2 + 2\sigma_4 z^2)x^2 - B^2(k^2y^2 - z^2)^2 = 0. \tag{8}$$

Then, we interpret $x : y : z$ as homogeneous coordinates of points in the plane at infinity and note that in Φ_∞'s equation d does not show up.

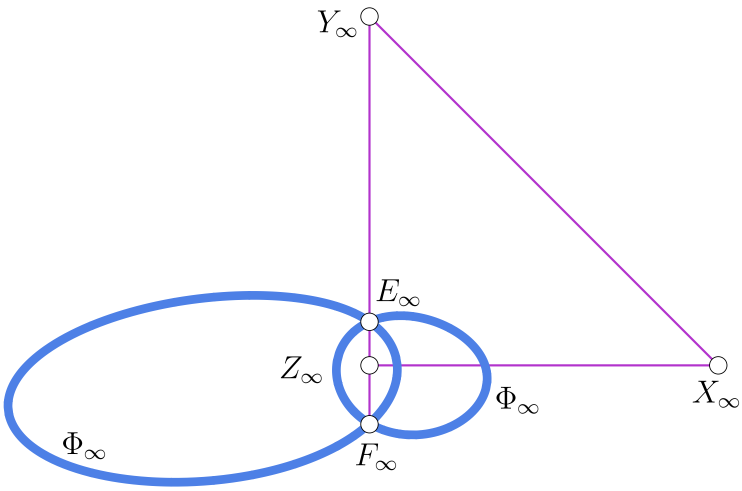

Figure 6: The intersection of Φ with the plane at infinity is an elliptic quartic curve Φ_∞ with the two double points E_∞ and F_∞ which are the ideal points of Φ's double curve $e \cup f$.

4. Special cases

We can expect exceptional appearances of the quartic ruled surface Φ if we choose special values for A, d, or k. Although all these values are originally assumed to be real and especially

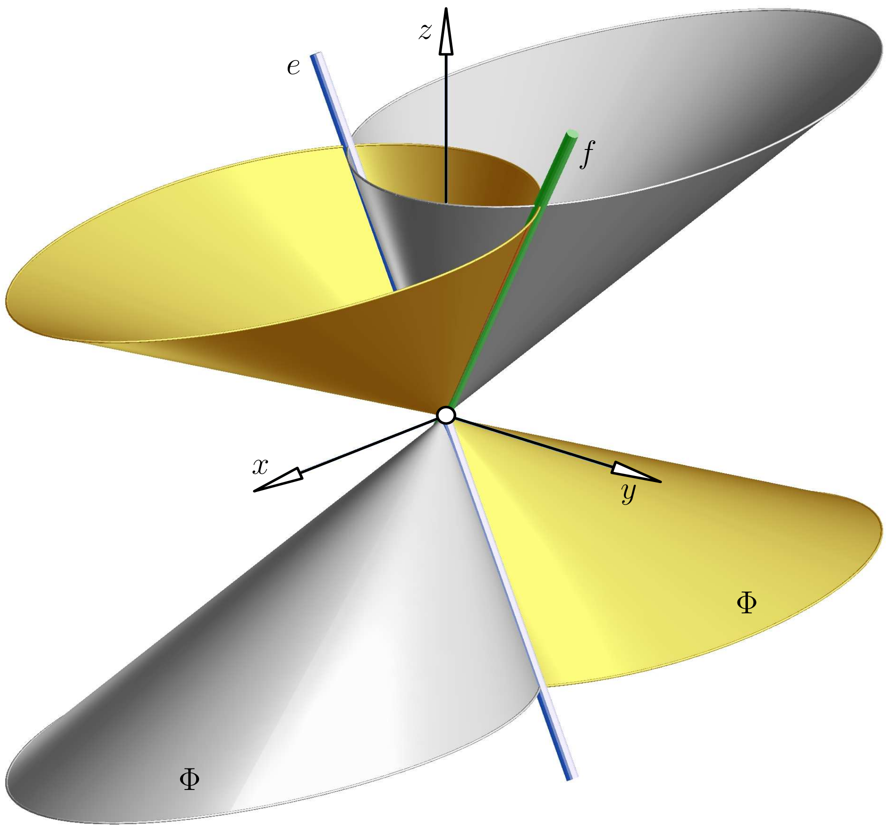

Figure 7: Intersecting lines e and f produce a quartic cone which is also the asymptotic cone of the generic (non-degenerate) quartic ruled surface Φ.

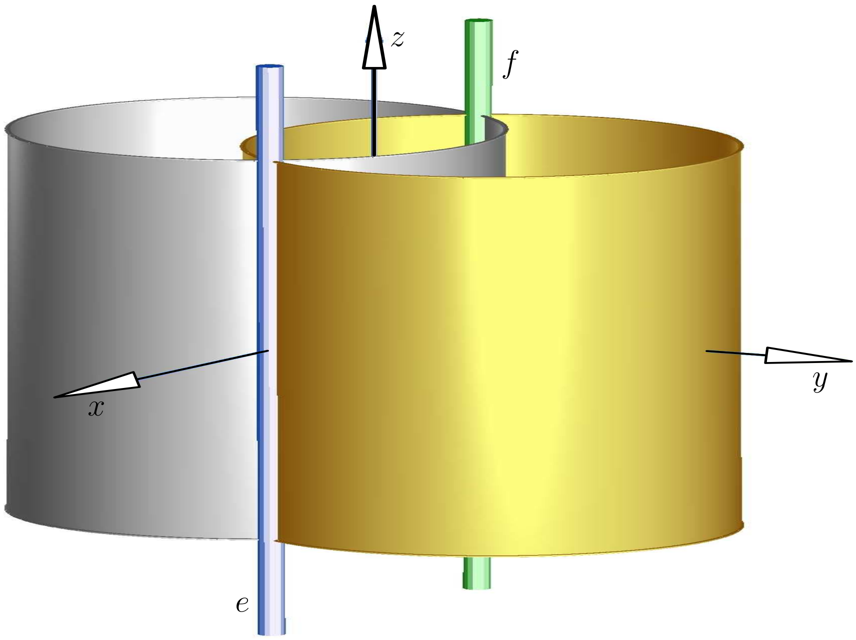

Figure 8: A pair of cylinders of revolution as the isoptic ruled surface of parallel lines e and f.

$|A| \leq 1$, we need not restrict ourselves to real values. In the following, we shall discuss some of these special choices that lead to sometimes unexpected surfaces Φ which, eventually, are then no longer ruled surfaces with real rulings.

4.1. One-sheeted hyperboloids

4.1.1. Intersecting lines e and f.

In the very beginning, we made the natural assumption $2d = \overline{ef} \neq 0$, *i.e.*, the lines e and f are skew. If we allow $d = 0$, then (6) simplifies to (8) which comes as no surprise, since (8) is independent of d. Therefore, (8) can also be viewed as the equation of a quartic cone Γ emanating from $(0, 0, 0)$. Obviously, e and f are generators with multiplicity two. The cone Γ is the asymptotic cone of Φ an example of which is displayed in Figure 7.

Further, if $k = 0$ (together with $d = 0$ this actually means $e = f$), then Γ degenerates and becomes the pair of isotropic planes $x^2 + z^2 = 0$ with multiplicity two.

If we allow $A = 0$, the cone Γ becomes the quadratic cone

$$(k^2 - 1)x^2 + k^2y^2 - z^2 = 0 \tag{9}$$

with multiplicity two. The quadratic cone (9) is a *normal cone* (cf. [1, p. 463–467]) and it is the asymptotic cone of the double hyperboloid (7) being the special form of Φ if $A = 0$.

An interesting case occurs if $k = \pm\mathrm{i}$ (besides $d = 0$), *i.e.*, the axes e and f of the pencils of planes are isotropic lines. In this case, Φ splits into two singular quadrics:

$$\begin{aligned} 2x^2 + (1 - A)y^2 + (1 - A)z^2 &= 0, \\ 2x^2 + (1 + A)y^2 + (1 + A)z^2 &= 0. \end{aligned} \tag{10}$$

One of these becomes a plane with multiplicity two if either $A = +1$ or $A = -1$ while the other one becomes an isotropic cone $x^2 + y^2 + z^2 = 0$.

Table 1: Special shapes of Φ caused by $A = 0, 1,\ \alpha = 0, \frac{\pi}{2},\ d = 0,\ k = 0, \infty, \mathrm{i}\,$; the integer μ denotes the multiplicities of the components.

$A = 0$		
$k = 0$	$k = \mathrm{i}$	$k = \infty$
$(d^2 - x^2 - z^2)^2 = 0$ right cylinder, $\mu = 2$	$(2d^2 - 2x^2 - y^2 - z^2)^2 = 0$ ellipsoid, $\mu = 2$	$(d^2 - x^2 - y^2)^2 = 0$ right cylinder, $\mu = 2$
$d = 0$	$d = 0$	$d = 0$
$(x^2 + z^2)^2 = 0$ compl. conj. planes, $\mu = 2$	$(2x^2 + y^2 + z^2)^2 = 0$ cone, no real point $\neq (0,0,0)$, $\mu = 2$	$(x^2 + y^2)^2 = 0$ compl. conj. planes, $\mu = 2$
$d = \mathrm{i}$	$d = \mathrm{i}$	$d = \mathrm{i}$
$(1 + x^2 + z^2)^2 = 0$ right cylinder, no real points, $\mu = 2$	$(2x^2 + y^2 + z^2 + 2)^2 = 0$ ellipsoid, no real point, $\mu = 2$	$(1 + x^2 + y^2)^2 = 0$ right cylinder, no real point, $\mu = 2$
$A = 1$		
$k = 0$	$k = \mathrm{i}$	$k = \infty$
$z^2 = 0$ plane, $\mu = 2$	$(x^2 - d^2)(d^2 - x^2 - y^2 - z^2) = 0$ sphere $\cup$ tangent planes	$y^2 = 0$ plane, $\mu = 2$
$d = 0$	$d = 0$	$d = 0$
empty set	$x^2(x^2 + y^2 + z^2) = 0$ real double plane $\cup$ $\cup$ isotropic cone	empty set
$d = \mathrm{i}$	$d = \mathrm{i}$	$d = \mathrm{i}$
$z^2 = 0$ plane, $\mu = 2$	$(x^2 + 1)(1 + x^2 + y^2 + z^2)^2 = 0$ sphere, no real point $\cup$ $\cup$ compl. tang. planes	$y^2 = 0$ plane, $\mu = 2$

The case $|A| < 1$ turns both of the quadrics into cones without any real points besides the common vertex $(0,0,0)$.

$|A| > 1$ corresponds to purely imaginary angles

$$\alpha \equiv \mathrm{i} \cdot \ln(A + \sqrt{A^2 - 1}) \pmod{2\pi}.$$

Nevertheless, inserting $|A| > 1$ into (10) makes either the first or the second quadric a cone with real points while the other still has only one real point, namely the vertex $(0,0,0)$.

4.1.2. Parallel lines e and f.

The case of parallel axes e and f is clearly an extrusion of the planar figure of the theorem of the angle of circumference. Thus, the ruled surface Φ (6) will split into two cylinders Δ_1

and Δ_2 of revolution erected on those circular arcs in the $[x, y]$-plane which are the locus of all points seeing the line segment EF (with $E, F = (\pm d, 0, 0)$) under constant angle α (cf. Figure 8).

From (6) we find the equation of the degenerate quartic by replacing k with $1/K$ and subsequently setting $K = 0$. (Otherwise, we would have to set $k = \infty$.) This results in the expected pair of cylinders of revolution

$$\Delta_{1,2}: \ x^2 + y^2 \pm \frac{2dA}{\sqrt{1 - A^2}} y - d^2 = 0.$$

In order to find real surfaces Δ_i, the values for A are restricted to $|A| \leq 0$. The Thaloid $x^2 + y^2 = d^2$ (cylinder of revolution) through e and f cannot be obtained directly from the cylinders' equations, since then $A = 1$.

4.1.3. Other quadrics

The axes e and f of the pencils of planes can be chosen as isotropic lines. Therefore, we let $k = \mathrm{i}$. (The choice $k = -\mathrm{i}$ produces the same result.) Again, we find that (6) degenerates and splits into quadratic polynomials:

$$\begin{aligned} Q_1: &\ \ 2x^2 + (1 - A)y^2 + (1 - A)z^2 = 2d^2, \\ Q_2: &\ \ 2x^2 + (1 + A)y^2 + (1 + A)z^2 = 2d^2. \end{aligned} \tag{11}$$

The case $d = 0$ was discussed earlier, so we have $d \neq 0$ in the following. Independent of the choice of A and regardless of the regularity, both quadrics Q_1 and Q_2 have the x-axis for their common axis of revolution.

In the very special case $A = \pm 1$, the pair of quadrics (11) contains precisely the singular quadric $x^2 - d^2 = (x - d)(x + d) = 0$ (a pair of (real) parallel planes) and the Euclidean sphere $x^2 + y^2 + z^2 = d^2$ with radius d centered at $(0, 0, 0)$ touching the planes at $(\pm d, 0, 0)$.

If $|A| > 1$, the pair (Q_1, Q_2) of quadrics consists of a two-sheeted hyperboloid of revolution and an ellipsoid of revolution.

Finally, we obtain two ellipsoids of revolution if $|A| < 1$.

Table 1 on page 155 summarizes the special and degenerate cases of Φ depending on special choices of A, k, and d.

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