# Moving ellipses on quadrics

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**Abstract**. For each regular quadric in the Euclidean 3-space, there is a three-parameter set of cutting planes, but the size of an ellipse or hyperbola depends only on its two semiaxes. Therefore, on each quadric $\mathcal{Q}$ there exist ellipses or hyperbolas with a one-parameter set of congruent copies, which can even be moved into each other. For the case of ellipses, we present parametrizations of motions on ellipsoids, hyperboloids, and paraboloids. These motions are closely related to the theory of confocal quadrics.

*Keywords*: confocal quadrics, conics on quadrics

## 1 Introduction

There are well-known examples of conics which can be moved on quadrics. Apart from the trivial case of circles on a sphere, paraboloids are surfaces of translation, even with a continuum of translational nets of parabolas. On quadrics of revolution, each planar section can be rotated while it remains on the quadric.

What's about general quadrics $\mathcal{Q}$? There is a three-parameter family of planes which cut $\mathcal{Q}$ along a conic. However, the size of an ellipse or hyperbola depends only on its two semiaxes. This parameter count reveals that on each quadric $\mathcal{Q}$ there exist conics with a one-parameter family of congruent copies on $\mathcal{Q}$. Below, we focus on ellipses and provide parametrizations for the motion of appropriate ellipses on ellipsoids, hyperboloids, and paraboloids. The motions prove to be in close relation to the family of quadrics being confocal with $\mathcal{Q}$.

## 2 Moving ellipses on a triaxial ellipsoid

On each regular quadric $\mathcal{Q}$, two conics $e_1$ and $e_2$ in parallel planes are homothetic (Fig. 1). This means in the case ellipses, that they have parallel axes and the same ratio of semiaxes $a_e : b_e$. Moreover, their centers lie on the same diameter. This follows from the polarity with respect to (henceforth abbreviated as w.r.t.) $\mathcal{Q}$.

On an ellipsoid $\mathcal{E}$, we obtain the biggest ellipse within a homothetic family as the intersection with a plane through the ellipsoid's center $O$. On the other hand, there is a point $P \in \mathcal{E}$ with a tangent plane $\tau_P$ parallel to the cutting planes, and the axes of the homothetic conics are parallel to the principal curvature directions at $P$ (Fig. 1). The conics are even homothetic to the Dupin indicatrix at $P$. This can be confirmed, e.g., by straight forward computation using a Taylor expansion at $P$.

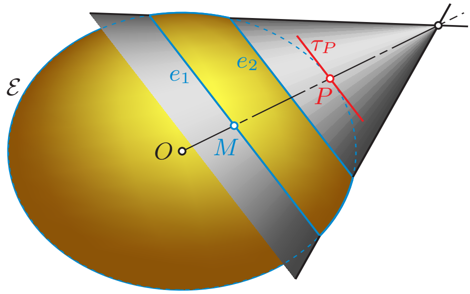


Figure 1: Homothetic sections $e_1, e_2$ of the ellipsoid $\mathcal{E}$ in parallel planes.

According to the definition of the Dupin indicatrix, the ratio of the principal curvatures $\kappa_1, \kappa_2$ at $P$ is reciprocal to the ratio of the squared semiaxes of the ellipses on $\mathcal{E}$ in planes parallel to $\tau_P$, i.e.,

$$a_e : b_e = \sqrt{\kappa_1} : \sqrt{\kappa_2}, \quad \text{if} \quad \kappa_1 > \kappa_2. \tag{1}$$

The lines of curvature on quadrics are related to confocal quadrics. Therefore, we recall some relevant properties of confocal quadrics.

### 2.1 Confocal central quadrics

Let $\mathcal{E}$ be a triaxial ellipsoid with semiaxes $a$, $b$, and $c$. The one-parameter family of quadrics being *confocal* with $\mathcal{E}$ is given as

$$F(x, y, z; k) := \frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} - 1 = 0, \tag{2}$$

where $k \in \mathbb{R} \setminus \{-a^2, -b^2, -c^2\}$ serves as a parameter. In the case $a > b > c > 0$, this family includes

$$\text{for} \begin{cases} -c^2 < k < \infty & \text{triaxial ellipsoids,} \\ -b^2 < k < -c^2 & \text{one-sheeted hyperboloids,} \\ -a^2 < k < -b^2 & \text{two-sheeted hyperboloids.} \end{cases} \tag{3}$$

Confocal quadrics intersect their common planes of symmetry along confocal conics. As limits for $k \to -c^2$ and $k \to -b^2$ we obtain ‘flat’ quadrics, i.e., the focal ellipse and the focal hyperbola.

The confocal family sends through each point $P = (\xi, \eta, \zeta)$ outside the coordinate planes, i.e., with $\xi\eta\zeta \neq 0$, exactly one ellipsoid, one one-sheeted hyperboloid, and one two-sheeted hyperboloid. The corresponding parameters $k$ define the three *elliptic coordinates* of $P$. We focus on points $P$ of the ellipsoid $\mathcal{E}$ with $k = 0$,

$$\mathcal{E}: \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1\,. \tag{4}$$

The two hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$ through $P$ with respective parameters $k_1$ and $k_2$, where

$$-a^2 < k_2 < -b^2 < k_1 < -c^2 < 0\,, \tag{5}$$

satisfy

$$\mathcal{H}_i:\ \frac{\xi^2}{a^2+k_i} + \frac{\eta^2}{b^2+k_i} + \frac{\zeta^2}{c^2+k_i} = 1, \quad i = 1, 2. \tag{6}$$

For given Cartesian coordinates $(\xi, \eta, \zeta)$ of a point $P$, we obtain the elliptic coordinates, i.e., the parameters of the quadrics through $P$, by solving $F(\xi, \eta, \zeta; k) = 0$ in (3) for $k$. This results in a cubic equation with three real roots. Conversely, if the tripel $(0, k_1, k_2)$ of elliptic coordinates is given, then the Cartesian coordinates $(\xi, \eta, \zeta)$ of the corresponding points $P \in \mathcal{E}$ satisfy

$$\begin{gathered} \xi^2 = \frac{a^2(a^2+k_1)(a^2+k_2)}{(a^2-b^2)(a^2-c^2)}\,, \quad \eta^2 = \frac{b^2(b^2+k_1)(b^2+k_2)}{(b^2-c^2)(b^2-a^2)}\,, \\ \zeta^2 = \frac{c^2(c^2+k_1)(c^2+k_2)}{(c^2-a^2)(c^2-b^2)}\,. \end{gathered} \tag{7}$$

There exist 8 such points, symmetric w.r.t. the coordinate planes.

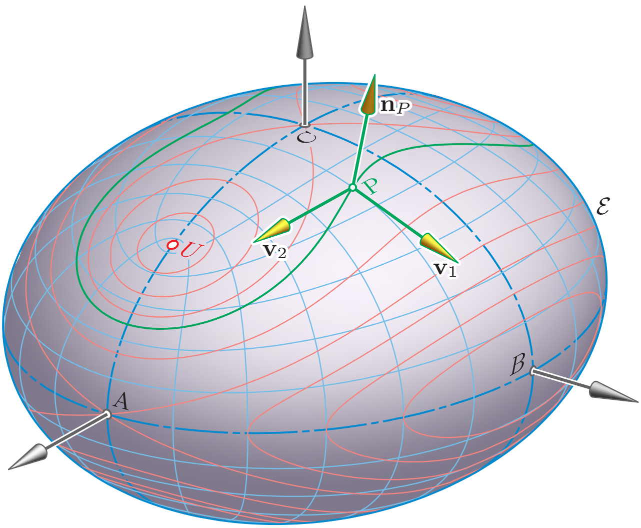


Figure 2: Ellipsoid $\mathcal{E}$ with lines of curvature (blue), curves of constant ratio of principal curvatures $\kappa_1 : \kappa_2$ (red), principal curvature directions $\mathbf{v}_1, \mathbf{v}_2$ at the point $P$, and one *umbilic point* $U$ with $\kappa_1 = \kappa_2$.

At each point $P$ of the ellipsoid $\mathcal{E}$, the surface normal $n_P$ to $\mathcal{E}$ has the direction vector

$$\mathbf{n}_P = \left(\frac{\xi}{a^2}\,,\ \frac{\eta}{b^2}\,,\ \frac{\zeta}{c^2}\right). \tag{8}$$

The surface normals of the two hyperboloids $\mathcal{H}_1$ und $\mathcal{H}_2$ through $P$ are in direction of the vectors

$$\mathbf{v}_i := \left(\frac{\xi}{a^2+k_i}\,,\ \frac{\eta}{b^2+k_i}\,,\ \frac{\zeta}{c^2+k_i}\right),\ i=1,2\,. \tag{9}$$

The differences of any two of the equations in (4) and (6) yield

$$\begin{gathered}\frac{\xi^2}{a^2(a^2+k_i)} + \frac{\eta^2}{b^2(b^2+k_i)} + \frac{\zeta^2}{c^2(c^2+k_i)} = 0,\ i=1,2\,, \quad \text{and}\\ \frac{\xi^2}{(a^2+k_1)(a^2+k_2)} + \frac{\eta^2}{(b^2+k_1)(b^2+k_2)} + \frac{\zeta^2}{(c^2+k_1)(c^2+k_2)} = 0.\end{gathered} \tag{10}$$

This is equivalent to vanishing dot products

$$\mathbf{n}_P\cdot\mathbf{v}_1 = \mathbf{n}_P\cdot\mathbf{v}_2 = \mathbf{v}_1\cdot\mathbf{v}_2 = 0.$$

Therefore, confocal quadrics form a triply orthogonal system of surfaces. Due to a theorem of Dupin, they intersect each other along lines of curvature. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from (9) define the principal curvature directions at $P$.

### 2.2 Ellipses on ellipsoids

Now, we look for the biggest ellipse on $\mathcal{E}$ within a homothetic family.

**Lemma 1.** *The semiaxes of the ellipse in the diameter plane parallel to the tangent plane $\tau_P$ at the point $P \in \mathcal{E}$ with elliptic coordinates* $(0, k_1, k_2)$ *are*

$$a_P = \sqrt{-k_2}, \quad b_P = \sqrt{-k_1}. \tag{11}$$

*Proof.* The diameter plane is spanned by the direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from (9). We look for $\lambda \in \mathbb{R}$ with $\lambda\mathbf{v}_i \in \mathcal{E}$, hence by (4)

$$\lambda^2\left[\frac{\xi^2}{(a^2+k_i)^2a^2} + \frac{\eta^2}{(b^2+k_i)^2b^2} + \frac{\zeta^2}{(c^2+k_i)^2c^2}\right] = 1.$$

This condition does not change if we subtract from the term in square brackets the left-hand side of the first equation in (10), divided by $k_i$. Thus, we obtain

$$\lambda^2\left[\frac{\xi^2}{(a^2+k_i)^2a^2} - \frac{\xi^2}{k_i(a^2+k_i)a^2} + \dots\right] = 1,$$

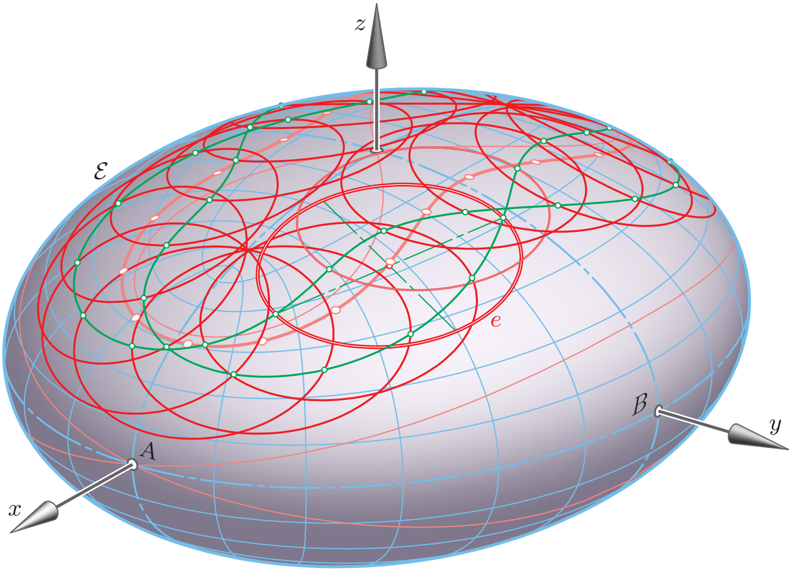


Figure 3: Moving the ellipse $e$ on the ellipsoid $\mathcal{E}$. The trajectories of the principal vertices of $e$ are displayed in green.

and, finally,

$$-\frac{\lambda^2}{k_i}\left[\frac{\xi^2}{(a^2+k_i)^2}+\frac{\eta^2}{(b^2+k_i)^2}+\frac{\zeta^2}{(c^2+k_i)^2}\right]=-\frac{\lambda^2}{k_i}\,\|\mathbf{v}_i\|^2=1,$$

hence, $a_P = |\lambda|\,\|\mathbf{v}_2\| = \sqrt{-k_2}$ and $b_P = |\lambda|\,\|\mathbf{v}_1\| = \sqrt{-k_1}$. These equations can already be found in [1, p. 517]. $\square$

For the motion of a given ellipse $e$ with semiaxes $(a_e, b_e)$, Lemma 1 implies the necessary condition

$$a_e \le a_P = \sqrt{-k_2},\ \text{where } b < \sqrt{-k_2} < a \tag{12}$$

by virtue of (5). We infer, under inclusion of (1),

**Theorem 1.** *If an ellipse $e$ with semiaxes $(a_e, b_e)$ is moving on a triaxial ellipsoid $\mathcal{E}$, then both points $P \in \mathcal{E}$ with tangent planes $\tau_P$ parallel to the plane of $e$ move on curves with proportional elliptic coordinates $k_2 : k_1 = -a_e^2 : -b_e^2$. Along these curves also the ratio of the principal curvatures remains constant (see Fig. 2).*

The ellipses of $\mathcal{E}$ in planes parallel to $\tau_P$ have their principal vertices in the plane spanned by the center $O$, point $P$, and by the principal curvature

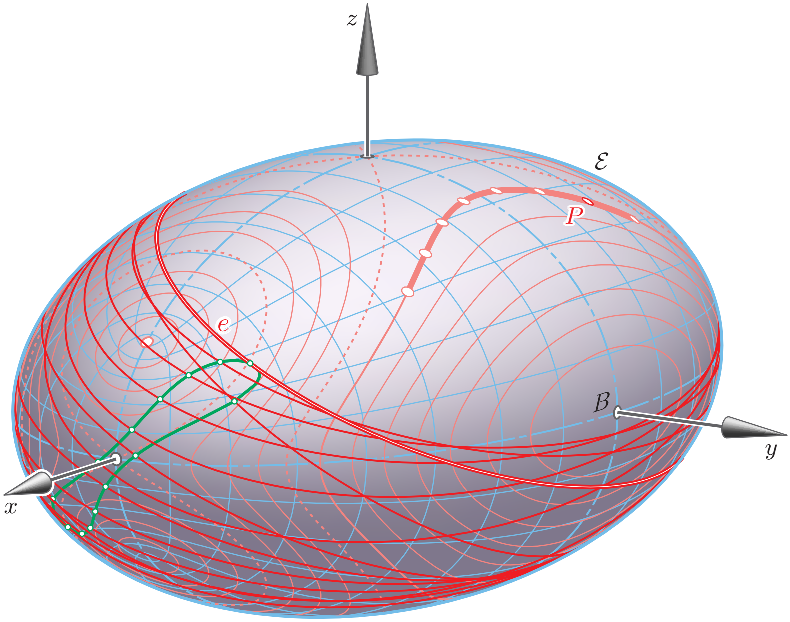


Figure 4: Motion of the ellipse $e$ on the ellipsoid $\mathcal{E}$ – displayed together with the trajectory of a principal vertex of $e$ (green) and that of the corresponding point $P \in \mathcal{E}$ (red) with the tangent plane $\tau_P$ parallel to $e$.

direction $\mathbf{v}_2$ from (9). Therefore, the principal vertices are located on an ellipse, for which $OP$ and the major axis with length $a_P$ in the plane through $O$ determine conjugate diameters. Let $\mathbf{p}$ denote the position vector of $P$ and $\mathbf{m} = \mu\,\mathbf{p}$ with $0 \le \mu = \sin x < 1$ that of the center $M$ of any ellipse in the homothetic family. Then, its major semiaxis $a_e$ equals $a_P \cos x = a_P\sqrt{1-\mu^2}$, which results in

$$\mu^2 = 1 - \frac{a_e^2}{a_P^2} = 1 + \frac{a_e^2}{k_2}\,. \tag{13}$$

When during the motion of the ellipse $e$, the scalar $\mu$ vanishes, then its center $M$ coincides with the center $O$ of $\mathcal{E}$. The corresponding point $P$ has the elliptic coordinate $k_2 = -a_e^2$. In order to continue the motion, point $P$ has to jump to its antipode (note the example in Fig. 4).

In order to parametrize the motion of the ellipse $e$ on the ellipsoid $\mathcal{E}$ (see Fig. 3), we set

$$v := \frac{k_2}{k_1} = \frac{a_e^2}{b_e^2} = \text{const.}, \quad \text{where } 1 < v < \frac{a^2}{c^2}\,, \tag{14}$$

and use the parameter $t = -k_2$ for representing the motion. Then, by virtue of (5), $t$ is restricted by the interval

$$\max\{b^2, vc^2, a_e^2\} \le t \le \min\{a^2, vb^2\}, \tag{15}$$

and $k_1 = t/v$. From (7) follows the parametrization $\mathbf{p}(t)$ by replacing $(k_1, k_2)$ with $(t/v,\, t)$. This implies for the trajectory of the center $M$ of $e$

$$\mathbf{m}(t) = \mu(t)\,\mathbf{p}(t) \text{ with } \mu(t) = \sqrt{1 - \frac{a_e^2}{t}}\ . \tag{16}$$

Now, we can express the motion of $e$ in matrix form, in terms of position vectors $\mathbf{x}_m$ w.r.t. the moving space (attached to $e$) and $\mathbf{x}_f$ w.r.t. the fixed space (attached to $\mathcal{E}$), as

$$\mathbf{x}_f = \mathbf{m}(t) + \mathbf{M}(t)\,\mathbf{x}_m, \ \text{ where } \mathbf{M}(t) = \left[\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},\ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},\ \frac{\mathbf{n}_P}{\|\mathbf{n}_P\|}\right]. \tag{17}$$

The three column vectors of the orthogonal matrix $\mathbf{M}(t)$ are given in (9) and (8).

Note that this parametrization is valid only for points $P$ in the octant $x, y, z > 0$. We get a closed motion after appropriate reflections in the planes of symmetry (see Figs. 3 and 4). By the same token, algebraic properties of this motion are reported in [2].

## 3 Moving ellipses on a one-sheeted hyperboloid

Also on hyperboloids and paraboloids, the conics in parallel planes are homothetic. However, not in all cases the method, as used before for ellipsoids, can be applied since a point $P$ either does not exist or lies at infinity. Moreover, paraboloids have no center $O$. Below, we analyse the motions of ellipses on a one-sheeted hyperboloid $\mathcal{H}_1$ and on an elliptic paraboloid $\mathcal{P}$ (see Section 4). The motion of an ellipse on a two-sheeted hyperboloid works similar to that of triaxial ellipsoids.[1]

For ellipses $e \subset \mathcal{H}_1$, there is no point $P \in \mathcal{H}_1$ with a tangent plane $\tau_P$ parallel to $e$. However, we find an appropriate point $\widetilde{P}$ on the 'conjugate' two-sheeted hyperboloid $\mathcal{H}_2$ (Fig. 5).

The hyperboloid $\mathcal{H}_2$ shares the asymptotic cone with $\mathcal{H}_1$, and therefore, the axes of the ellipse $e$ are parallel to the principal curvature directions of $\mathcal{H}_2$ at $\widetilde{P}$. The two hyperboloids satisfy the respective equations

$$\mathcal{H}_1\colon\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{and} \quad \mathcal{H}_2\colon\ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

[1]The motion of a parabola on a hyperboloid is discussed in [4, p. 355–357].

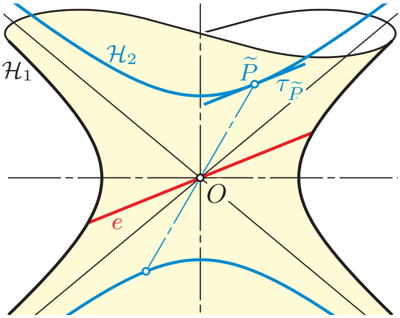


Figure 5: For ellipses $e$ on a one-sheeted hyperboloid $\mathcal{H}_1$, there does not exist a point $P \in \mathcal{H}_1$ with a tangent plane $\tau_P$ parallel to the plane of $e$.

with $a > b$. The quadrics confocal with $\mathcal{H}_2$ are given by

$$-\frac{x^2}{a^2 - k} - \frac{y^2}{b^2 - k} + \frac{z^2}{c^2 + k} = 1.$$

Again, this family sends through each point $\widetilde{P}$ outside of the planes of symmetry three mutually orthogonal quadrics, one of each type. On the two-sheeted hyperboloid $\mathcal{H}_2$ with $k = 0$, we use the parameters $k_0$ of the ellipsoid and $k_1$ of the one-sheeted hyperboloid as the elliptic coordinates of $\widetilde{P}$ with

$$b^2 < k_1 < a^2 < k_0.$$

Then, similar to Lemma 1, the ellipse $e \in \mathcal{H}_1$ in the diameter plane parallel to $\tau_{\widetilde{P}}$ has the semiaxes

$$a_{\widetilde{P}} = \sqrt{k_0} \quad \text{and} \quad b_{\widetilde{P}} = \sqrt{k_1}.$$

This is the smallest ellipse on $\mathcal{H}_1$ within the homothetic family.

If any ellipse with given semiaxes $a_e$ and $b_e$ is to be moved on $\mathcal{H}_1$, the corresponding point $\widetilde{P} \in \mathcal{H}_2$ has to trace a curve with proportional elliptic coordinates

$$k_0 : k_1 = a_{\widetilde{P}}^2 : b_{\widetilde{P}}^2 = a_e^2 : b_e^2.$$

Similar to (7), we can parametrize the trajectory $\widetilde{\mathbf{p}}(t) = (\xi, \eta, \zeta)$ of $\widetilde{P}$ by $t := k_0 > a^2$, where

$$v := \frac{k_0}{k_1} = \frac{a_e^2}{b_e^2} = \text{const.},$$

hence $k_1 = t/v$ with $b^2 < k_1 < a^2$.

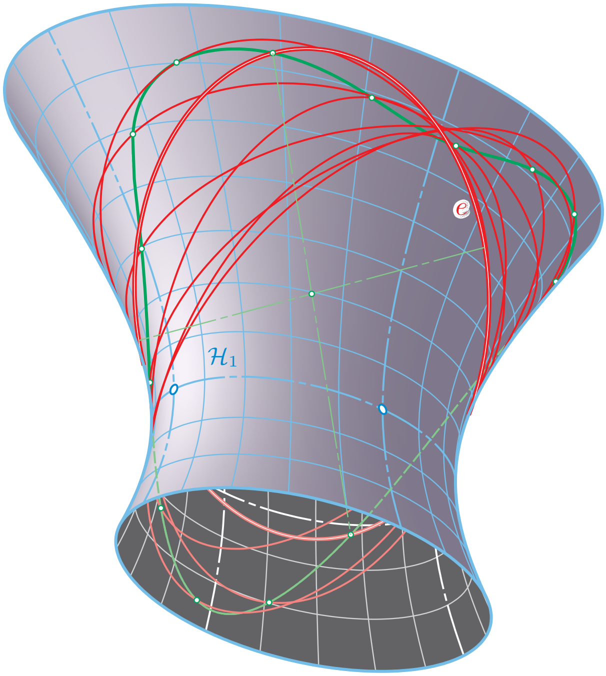


Figure 6: Movement of the ellipse $e$ on the one-sheeted hyperboloid $\mathcal{H}_1$. The two principal vertices of $e$ trace the same curve (green).

Now we have to find the center $M$ of the moving ellipse on the diameter line $[\widetilde{P}, O]$: For each $\widetilde{P}$, the principal vertices of the ellipses in planes parallel to $\tau_{\widetilde{P}}$ are placed on a hyperbola, for which the point $\widetilde{P}$ and one principal vertex in the plane through $O$ are the endpoints of conjugate diameters. If $a_e = a_{\widetilde{P}} \cosh x$, then the position vector $\mathbf{m}$ of the center $M$ of $e$ and $\widetilde{\mathbf{p}}$ of the point $\widetilde{P}$ are related by $\mathbf{m} = \sinh x\, \widetilde{\mathbf{p}}$. Thus, we obtain

$$\mathbf{m} = \mu\, \widetilde{\mathbf{p}} \quad \text{with} \quad \mu^2 = \frac{a_e^2}{a_{\widetilde{P}}^2} - 1\,. \tag{18}$$

This yields, similar to (17), a parametrization for the motion of the ellipse $e$ on $\mathcal{H}_1$ (Fig. 6). As a consequence of (18), on the trajectory of $\widetilde{P}$ only points with $a_{\widetilde{P}}^2 = k_0 \le a_e^2$ are admitted. Therefore, the parameter $t = k_0$ runs the interval

$$\max\{a^2,\, vb^2\} \le t \le \min\{a_e^2,\, va^2\}.$$

In the case $a_e^2 < va^2$, the same phenomenon appears as mentioned above. When the parameter $t$ reaches $a_e^2$, then, for continuing the motion of the ellipse, the point $\widetilde{P}$ either has to jump to its antipode, or the scalar $\mu$ in (18) must get a negative sign.

## 4 Moving ellipses on an elliptic paraboloid

The quadrics being confocal with an elliptic paraboloid can be represented as

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} - 2z - k = 0 \quad \text{for} \quad k \in \mathbb{R} \setminus \{-a^2, -b^2\}. \qquad (19)$$

In the case $a > b > 0$, this one-parameter family contains

$$\text{for} \begin{cases} -b^2 < k < \infty & \text{elliptic paraboloids,} \\ -a^2 < k < -b^2 & \text{hyperbolic paraboloids,} \\ k < -a^2 & \text{elliptic paraboloids.} \end{cases} \qquad (20)$$

For each $k$, the vertex of the corresponding paraboloid has the coordinates $(0, 0, -k/2)$. The point $(0, 0, b^2/2)$ is the common focal point of the principal sections in the plane $x = 0$, and $(0, 0, a^2/2)$ is the analogue for the sections with $y = 0$. The limits for $k \to -b^2$ or $k \to -a^2$ define the two *focal parabolas* (note [4, Fig. 7.5]).

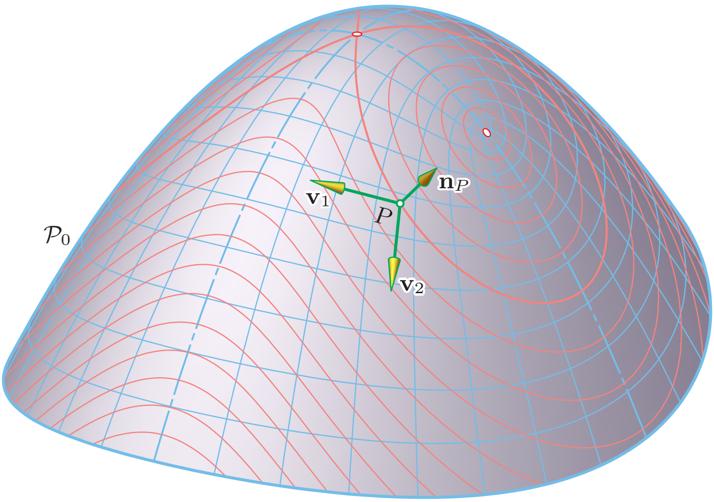


Figure 7: Elliptic paraboloid $\mathcal{P}_0$ with lines of curvature (blue), curves of constant ratio of principal curvatures $\kappa_1 : \kappa_2$ (red), and direction vectors $\mathbf{v}_1, \mathbf{v}_2$ of the principal curvature tangents at the point $P \in \mathcal{P}_0$.

The family of confocal parabolas sends through each point $P$ outside the planes of symmetry $x = 0$ and $y = 0$ three surfaces, one of each type.

Like before in the case of confocal central surfaces, we call the parameters of the three parabolas through $P$ the *elliptic coordinates* of $\mathcal{P}$. We focus on the elliptic paraboloid $\mathcal{P}_0$ with $k = 0$. Its points have the elliptic coordinates $(0, k_1, k_2)$, where

$$k_2 \le -a^2 \le k_1 \le -b^2.$$

Conversely, if any point $P \in \mathcal{P}_0$ is defined by the elliptic coordinates $(k_1, k_2)$, then its Cartesian coordinates $\xi, \eta, \zeta$ satisfy

$$\begin{gathered}\xi^2 = -\frac{a^2(a^2+k_1)(a^2+k_2)}{(a^2-b^2)}, \ \eta^2 = \frac{b^2(b^2+k_1)(b^2+k_2)}{(a^2-b^2)},\\ \zeta = -\frac{a^2+b^2+k_1+k_2}{2}.\end{gathered} \tag{21}$$

The normal vectors $\mathbf{n}_P$ of $\mathcal{P}_0$ and $\mathbf{v}_i$ of the paraboloid $\mathcal{P}_i$ with parameter $k_i$, $i = 1, 2$, at the point $P$ are (note Fig. 7)

$$\mathbf{n}_P = \begin{pmatrix} \frac{\xi}{a^2} \\ \frac{\eta}{b^2} \\ 1 \end{pmatrix}, \quad \mathbf{v}_i = \begin{pmatrix} \frac{\xi}{a^2+k_i} \\ \frac{\eta}{b^2+k_i} \\ 1 \end{pmatrix}. \tag{22}$$

Also confocal paraboloids form a triply orthogonal system of surfaces, and consequently, they intersect each other along lines of curvature. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in (22) define the principal curvature directions at $P$.

**Lemma 2.** *Given a regular quadric $\mathcal{Q}_0$, let $P \in \mathcal{Q}_0$ be a point in general position with the tangent plane $\tau_P$ to $\mathcal{Q}_0$. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the remaining two confocal quadrics through $P$, the pole of $\tau_P$ w.r.t. $\mathcal{Q}_2$ is the center of curvature of the orthogonal section of $\mathcal{Q}_0$ at $P$ through the principal curvature tangent $t_P$ orthogonal to $\mathcal{Q}_2$.*

*Proof.* We can verify this by straight forward computation: Based on the parametrizations of $\mathcal{Q}_0$ by elliptic coordinates $(k_1, k_2)$, as given in (7) for central quadrics and in (21) for paraboloids, we compute the first and second fundamental form and the center of curvature (= Meusnier point) for the orthogonal section of $\mathcal{Q}_0$ through $t_P$ (see, e.g., [3]).

A synthetic proof runs as follows: Let $c$ be the line of intersection between the confocal quadrics $\mathcal{Q}_0$ and $\mathcal{Q}_1$. Then, $c$ is a line of curvature for both. The developable $\mathcal{T}$ which contacts $\mathcal{Q}_0$ along $c$ has generators orthogonal to $c$. Also the surface normals to $\mathcal{Q}_0$ along $c$ form a developable $\mathcal{N}$. Its cuspidal points are the centers of curvature of the orthogonal sections of $\mathcal{Q}_0$ through the tangents to $c$ (note [4, p. 418ff]).

At the point $P \in c$, the tangent $t_P$ to $c$, the surface normal $n_P$ to $\mathcal{Q}_0$, and the generator $g_P$ of $\mathcal{T}$ are mutually orthogonal. Any two of them define the principal curvature directions at $P$ for one of the three confocal quadrics. For example, the lines $g_P$ and $n_P$ are conjugate tangents of $\mathcal{Q}_2$, and therefore, even polar w.r.t. $\mathcal{Q}_2$.

The polarity w.r.t. $\mathcal{Q}_2$ transforms the developable $\mathcal{T}$ through $g_P$ into a developable $\mathcal{T}'$ through $n_P$, while tangent planes $\tau_X$ of $\mathcal{T}$ and $\mathcal{Q}_0$ at points $X \in c$ are sent to points $X'$ of the cuspidal edge $c_{\mathcal{T}'}$ of $\mathcal{T}'$. The poles of each plane w.r.t. the quadrics of a confocal family lie on a line orthogonal to the given plane (see, e.g., [4, p. 292]). Therefore, the $\mathcal{Q}_2$-pole $X'$ of $\tau_X$ lies on the normal $n_X$ of $\mathcal{Q}_0$ at $X$. Consequently, the cuspidal edge $c_{\mathcal{T}'}$ of $\mathcal{T}'$ is a curve on the developable $\mathcal{N}$. The polarity w.r.t. $\mathcal{Q}_2$ takes the generator $g_X \subset \mathcal{T}$ to the tangent $g'_X$ to $c_{\mathcal{T}'}$ at $X'$, which is also a tangent of $\mathcal{N}$.

Now we prove, that the cuspidal edge $c_{\mathcal{T}'}$ of $\mathcal{T}'$ passes through the cuspidal point $C_{\mathcal{N}}$ of $n_P \subset \mathcal{N}$:

The tangent plane $\tau_P$ to $\mathcal{T}$ at $P$ is the limit $X \to P$ of a plane connecting the generator $g_P$ with any point of $g_X$. By virtue of the polarity w.r.t. $\mathcal{Q}_2$ with $\mathcal{T} \to \mathcal{T}'$, the cuspidal point $P' \in c_{\mathcal{T}'}$ on $n_P$ is the limit $X \to P$ of the point of intersection between $n_P$ and any plane through $g'_X$. As noted before, the tangent plane $[n_X, t_X]$ along $n_X$ to $\mathcal{N}$ is such a plane, since it passes through $g'_X$. However, the limit $X \to P$ of the point of intersection $n_P \cap [n_X, t_X]$ yields also the cuspidal point $C_{\mathcal{N}}$ of $n_P$ w.r.t. the developable $\mathcal{N}$. This means, that $C_{\mathcal{N}}$ equals the pole $P'$ of $\tau_P$ w.r.t. $\mathcal{Q}_2$. $\square$

We apply Lemma 2 to the elliptic paraboloid $\mathcal{P}_0$. The tangent plane $\tau_P$ to $\mathcal{P}_0$ at $P = (\xi, \eta, \zeta)$ has the equation

$$\tau_P\colon\ \frac{\xi}{a^2}\,x + \frac{\eta}{b^2}\,y + z = \zeta.$$

Its pole w.r.t. the paraboloid $\mathcal{P}_i$ with parameter $k_i$ is

$$C_i = \begin{pmatrix} \dfrac{a^2 + k_i}{a^2}\ \xi \\ \dfrac{b^2 + k_i}{b^2}\ \eta \\ \zeta + k_i \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + k_i \begin{pmatrix} \dfrac{\xi}{a^2} \\ \dfrac{\eta}{b^2} \\ 1 \end{pmatrix}. \tag{23}$$

This confirms that the principal curvatures of $\mathcal{P}_0$ at $P$ are

$$\kappa_i = 1/\,\overline{PC_i} = \frac{1}{-k_i \|\mathbf{n}_P\|},\ \text{where } \kappa_1 > \kappa_2\,. \tag{24}$$

Now we have to place a given ellipse $e$ with semiaxes $a_e$ and $b_e$, where $a_e^2 : b_e^2 = k_2 : k_1$, in a plane parallel to $\tau_P$ in the correct way on $\mathcal{P}_0$. This means, the center $M$ of $e$ lies on the diameter $d_P$ of the paraboloid $\mathcal{P}_0$ and the major axis is parallel to the principal curvature tangent $t_P$ in direction $\mathbf{v}_2$, i.e., orthogonal to the paraboloid $\mathcal{P}_2$ through $P$ (Fig. 7).

The major axis lies in the plane $\varepsilon$ spanned by $t_P$ and $d_P$. This plane intersects $\mathcal{P}_0$ along a parabola $p$. Due to Meusnier's theorem, we obtain the center of curvature $P^*$ of $p$ at $P$ as the pedal point of $C_2$ from (23) in $\varepsilon$. Let $\rho = \overline{PP^*}$ denote the radius of curvature at $P$ (Fig. 8). Then the chord $S_1S_2$ of $p$ parallel to $t_P$ through the midpoint of $PP^*$ has its midpoint $S_0$ on the diameter $d_P$ and the length $2\rho$.

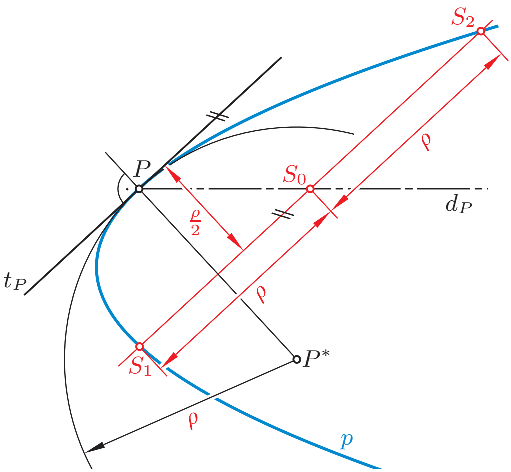


Figure 8: For a given parabola $p$ with point $P \in p$ and corresponding center of curvature $P^*$, this is a construction of the endpoints $S_1, S_2$ on a particular chord of $p$.

This follows with the help of a *shear*, i.e., a perspective affine transformation in $\varepsilon$ with $t_P$ as axis and the ideal point of $t_P$ as its center. This shear transforms $p$ into a parabola $p'$ which osculates $p$ at $P$. We can define a shear such that $P$ becomes the vertex of $p'$. Then, the midpoint of $PP^*$ if the focal point of $p'$, and for $p'$ the chord parallel to $t_P$ through the focal point has the length $2\rho$. Under the inverse shear, the chord is just translated parallel to $t_P$.

For the parabola $p$, the squared length of chords parallel to $t_P$ is proportional to the distance between $P$ and the midpoint of the chord. According to Fig. 8, in our case the factor of proportionality is known as $\overline{S_1S_2}^2/\overline{PP_0}$. Consequently, the respective position vectors $\mathbf{p}$, $\mathbf{s}_0$, and $\mathbf{m}$

of $P$, $S_0$, and the center $M$ of the wanted ellipse $e$ are related by

$$\mathbf{m} = \mathbf{p} + \frac{a_e^2}{\rho^2}(\mathbf{s}_0 - \mathbf{p}). \tag{25}$$

Now, we can parametrize the motion of a given ellipse $e$ on $\mathcal{P}_0$ in the following way. By (24), the given semiaxes define the locus of points $P \in \mathcal{P}_0$ with proportional elliptic coordinates

$$v := \frac{k_2}{k_1} = \frac{a_e^2}{b_e^2}, \text{ where } v > 1.$$

In the same way as before, we use $t := -k_2$ as the motion parameter. Then the pair of elliptic coordinates $k_1 = t/v$ and $k_2 = t$ yields the trajectory $\mathbf{p}(t)$ of the point $P \in \mathcal{P}_0$ by (21). For each admissible $t$, we compute the Meusnier point $C_2$ by (23) and then its pedal point $C^*$ in the plane $\varepsilon$, as described above. Finally, due to (25), we can find the correct position of the ellipse $e \subset \mathcal{P}_0$ in a plane parallel to $\tau_P$.

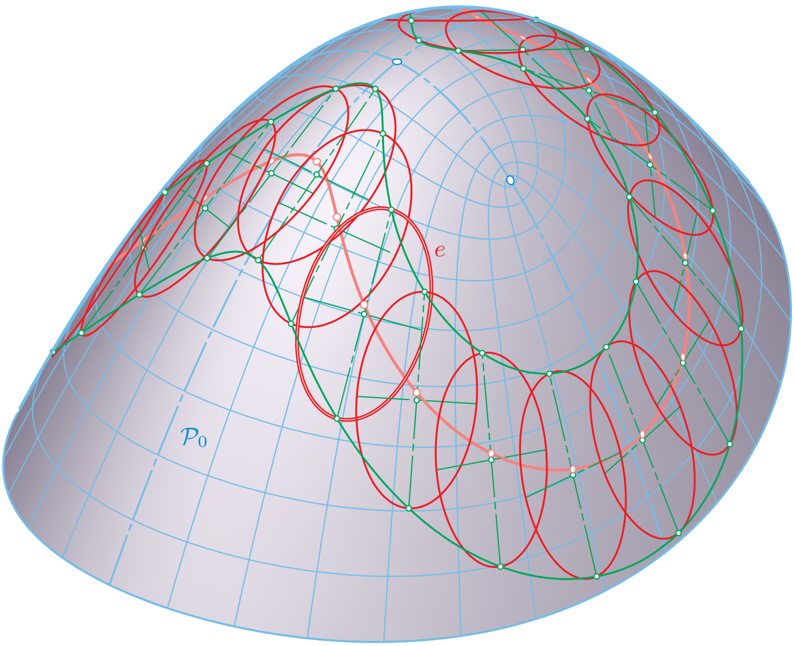


Figure 9: Ellipse $e$ moving on the elliptic paraboloid $\mathcal{P}_0$ – displayed together with the trajectories of the principal vertices of $e$ (green) and the related curve of constant ratio of principal curvatures (red).

We summarize:

**Theorem 2.** *On regular quadrics $\mathcal{Q}$, all ellipses $e$ other than circles can be moved, except on a one-sheeted hyperboloid the gorge ellipse and on a triaxial ellipsoid the ellipse with the longest and the shortest diameter as axes. During these motions, the points $P \in \mathcal{Q}$ with a tangent plane parallel to the plane of $e$ trace curves along which the ratio of elliptic coordinates remains constant.*

## References


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