Plücker's Conoid Revisited

Hellmuth Stachel

Abstrakt
Plückerov konoid (cylindroid) $\mathcal{C}$ je priamková plocha tretieho stupňa s jednou vlastnou dvojnásobnou priamkou. Táto plocha zohráva kľúčovú úlohu v geometrickej literatúre, pretože všetky jej úpätnice sú rovinné krivky. Je geometrickým miestom dvojíc mimobežných priamok, pre ktoré je daný kolmý hyperbolický paraboloid bisektorom. V priestorovej kinematike je $\mathcal{C}$ geometrickým miestom okamžitých polôh osí relatívneho skrutkového pohybu dvoch otáčajúcich sa kolies s pevnými mimobežnými osami. Napokon, štyri koncyklické generátory $\mathcal{C}$ sú spoločnými dotyčnicami nekonečného počtu guľových plôch, a v článku študujeme ich obalovú kanálovú plochu.

Kľúčové slová: Plückerov konoid, cylindroid, bisektor, jednodielny rotačný hyperboloid, kolmý hyperbolický paraboloid

Abstract
Plücker's conoid (cylindroid) $\mathcal{C}$ is a ruled surface of degree three with a finite double line. This surface plays a major role in the geometric literature since all its pedal curves are planar. It is the locus of pairs of skew lines for which a given orthogonal hyperbolic paraboloid is the bisector. In spatial kinematics, $\mathcal{C}$ is the locus of instantaneous screw axes of the relative motion for two rotating wheels with fixed skew axes. Finally, four concyclic generators of $\mathcal{C}$ are common tangents of infinitely many spheres, and we study their enveloping canal surface.

Keywords: Plücker's conoid, cylindroid, bisector, one-sheeted hyperboloid of revolution, orthogonal hyperbolic paraboloid

1 Introduction

Plücker's conoid $\mathcal{C}$, which is also known under the name *cylindroid*, is a ruled surface of degree three with a finite double line and a director line at infinity (see Fig. 1). Using cylinder coordinates (r, φ, z), the conoid can be given by the equation

$$\mathcal{C}\colon\ z = c\,\sin 2\varphi \tag{1}$$

with a constant $c \in \mathbb{R}_{>0}$. All generators of $\mathcal{C}$ are parallel to the $[x, y]$-plane. The z-axis is the double line d of $\mathcal{C}$ and an axis of symmetry. The conoid passes through the x- and y-axis. These two lines c_1, c_2, called *central generators* of $\mathcal{C}$, are axes of symmetry, as well. The Plücker conoid $\mathcal{C}$ is the trajectory of the x-axis under a motion composed from a rotation about the z-axis and a harmonic oscillation with double frequency along the z-axis [13, p. 37] (Fig. 2).

The substitution $x = r\cos\varphi$ and $y = \sin\varphi$ in (1) yields the Cartesian equation

$$(x^2 + y^2)\,z - 2c\,xy = 0, \tag{2}$$

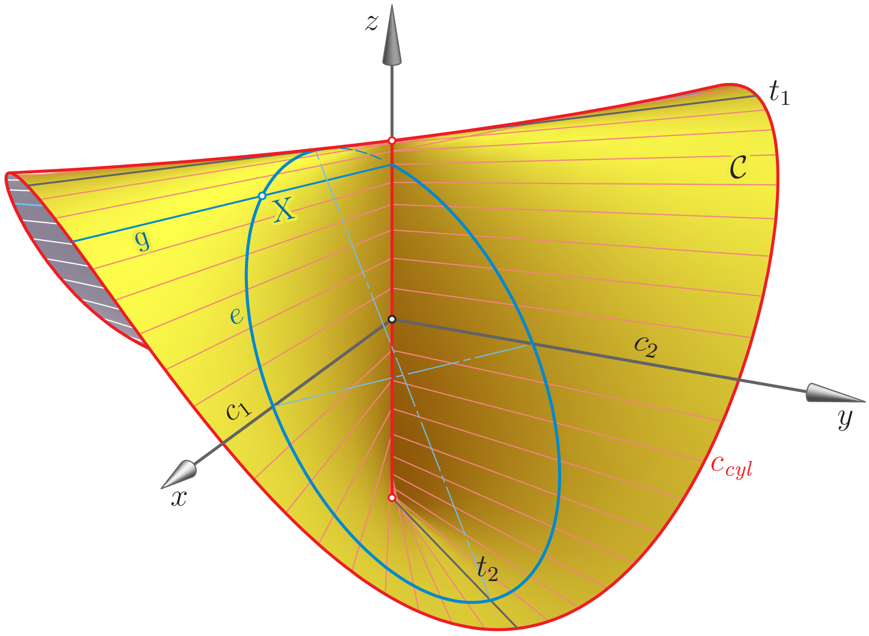

Fig. 1. Plücker's conoid $\mathcal{C}$ with central generators c_1 and c_2, torsal generators t_1 and t_2, the generator g through X, and the ellipse e in the tangent plane τ_X to $\mathcal{C}$ at point X.

which reveals that reflections in the planes $x \pm y = 0$ map $\mathcal{C}$ onto itself. The origin O is called the *center* of $\mathcal{C}$.

The right cylinder $x^2 + y^2 = R^2$ intersects the Plücker conoid $\mathcal{C}$ along a curve c_{cyl} of degree 4[1] (see Fig. 1), which in the cylinder's development appears as the Sine-curve with amplitude c and wavelength $R\pi$ (Fig. 2). The generators of $\mathcal{C}$ connect opposite points c_{cyl}.[2] The conoid is bounded by the planes $z = \pm c$, which contact $\mathcal{C}$ along the torsal generators t_1 and t_2. Their distance $2c$ is called the *width* of $\mathcal{C}$.

For the sake of simplicity, we assume that the $[x, y]$-plane and all generators of $\mathcal{C}$ are horizontal and the z-axis is vertical. In this sense, the *top view* stands for the image after vertical projection into the $[x, y]$-plane; a prime will be used to indicate the top views of geometric objects.

The top view reveals that the intersection of Plücker's conoid $\mathcal{C}$ with any right cylinder $\mathcal{Z}$ through the double line d gives a curve e which in the cylinder's development shows up as one period of a Sine curve (Fig. 3). Therefore, e is an ellipse with principal vertices on the torsal generators. There exists a two-parameter set of ellipses e on $\mathcal{C}$. They all have the same excentricity c, as it equals the z-coordinates' difference of a principal vertex and the center of e [6, p. 208].

The secondary vertices of e lie on the central generators c_1 and c_2. Ellipses $e \subset \mathcal{C}$ with the same minor semiaxis are congruent, and their planes have the same slope. All these ellipses are poses of one ellipse when it performs the 3D-continuation of an elliptic motion (see [14, p. 45]) under

[1] The remaining part of the curve of intersection consists of the lines at infinity of the two complex conjugate planes $x \pm iy = 0$.

[2] See models #96 – #100 of the collection of mathematical models at the Institute of Discrete Mathematics and Geometry, Vienna University of Technology, https://www.geometrie.tuwien.ac.at/modelle/models_show.php?mode=2&n=100&id=0, retrieved Sept. 2022. All these models originate from Schilling's collection as presented in [9].

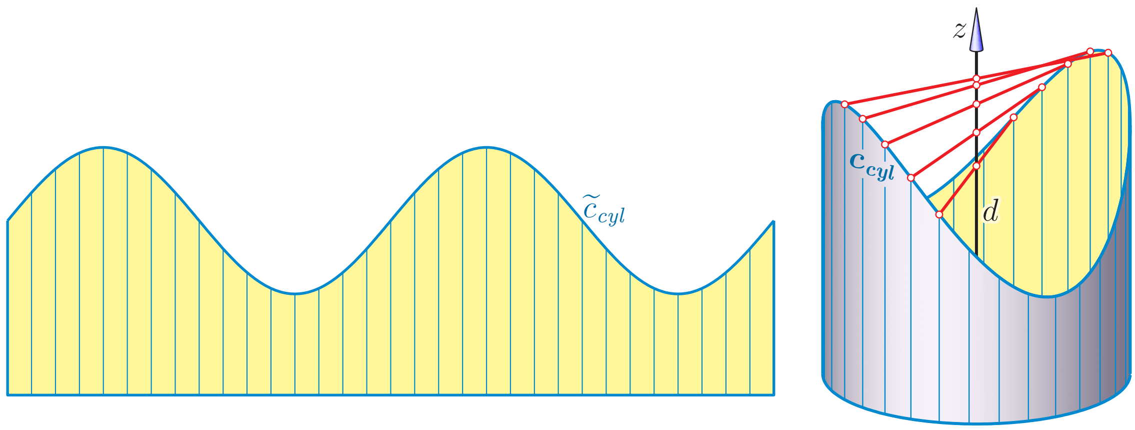

Fig. 2. The intersection c_{cyl} of Plücker's conoid $\mathcal{C}$ with a right cylinder about the double line $d = z$-axis appears in the cylinder's development as two periods of a Sine curve. The generators of $\mathcal{C}$ connect opposite points of c_{cyl} .

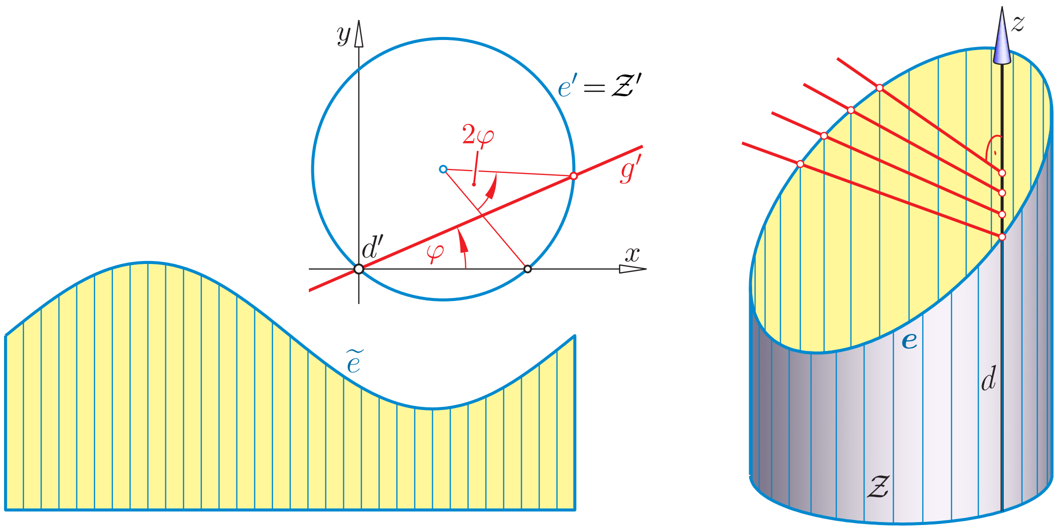

Fig. 3. The intersection e of the conoid $\mathcal{C}$ with a right cylinder $\mathcal{Z}$ through the double line d appears in the cylinder's development as one period $\widetilde{e}$ of a Sine curve. The generators of $\mathcal{C}$ meet e and intersect d orthogonally.

which the secondary vertices trace the central generators [6, p. 209].

Lemma 1.1. Let g_1, g_2, g_3 be three lines with an orthogonal transversal d such that no two of the three lines are parallel, and they are not coplanar either. Then there exists a unique Plücker conoid $\mathcal{C}$ passing through these lines.

Proof. We choose any right cylinder $\mathcal{Z}$ which passes through d and does not contact any of the given lines. Then their remaining points of intersection with $\mathcal{Z}$ span a plane that intersects $\mathcal{Z}$ along an ellipse e thus defining $\mathcal{C}$ as shown in Fig. 3. □

The intersection of $\mathcal{C}$ with the plane of any ellipse e must additionally contain a line $g \in \mathcal{C}$ passing through the common point of e and d (Fig. 1). This generator g, which is horizontal and therefore parallel to the minor axis of e, shares with e another point X. This must be the point of contact between the conoid and the plane of e. In other words: The tangent plane τ_X to $\mathcal{C}$ at X intersects $\mathcal{C}$ beside the generator g along an ellipse e which appears in the top view as a circle e' through d'.

The top view gives insight into another important property of the ellipse $e = \tau_X \cap \mathcal{C}$ (Fig. 4). For all points P in space with the top view $P' \in e'$ opposite to the top view d' of the double line, the *pedal curve* on $\mathcal{C}$, i.e., the locus of pedal points of P on the generators of $\mathcal{C}$, coincides with e. This holds since right angles enclosed with generators of $\mathcal{C}$ appear in the top view again as right angles, provided that the spanned plane is not parallel to the double line d. It means conversely that for each point of e the surface normal to $\mathcal{C}$ meets the vertical line through P'. We summarize.

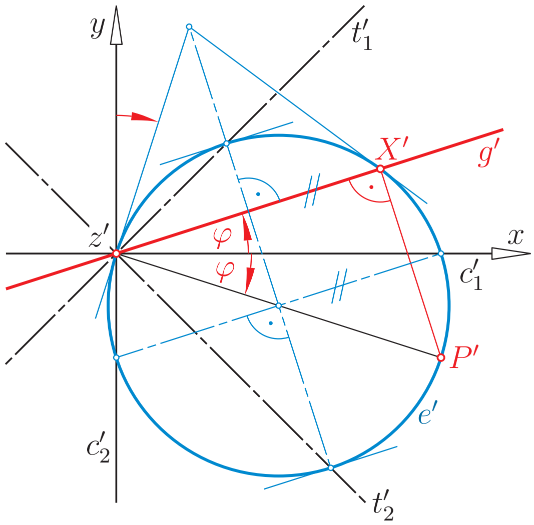

Fig. 4. X is the pedal point of g for all points P with the top view P'; the ellipse $e \in \mathcal{C}$ is the pedal curve of P.

Lemma 1.2. All pedal curves of Plücker's conoid $\mathcal{C}$ are planar. For points outside the double line the pedal curves are ellipses with the same excentricity.

Due to P. Appell [1], Plücker's conoid is the only algebraic non-torsal ruled surface with planar pedal curves (note also [6, p. 211]).

2 Bisector of two skew lines

A classical result states that the *bisector* of two skew lines ℓ_1, ℓ_2, i.e., the set of points X being equidistant to ℓ_1 and ℓ_2, is an orthogonal (or equilateral) hyperbolic paraboloid (Fig. 5). This is reported, e.g., in [8, p. 154] or [7, p. 64].

If in an appropriate coordinate system (x, y, z) the lines ℓ_1, ℓ_2 are given by $z = \pm d$ and $x \sin\alpha = \pm y \cos\alpha$, then the bisector satisfies the equation

$$\mathcal{P}:\ 2d\,z + xy\sin 2\alpha = 0. \tag{3}$$

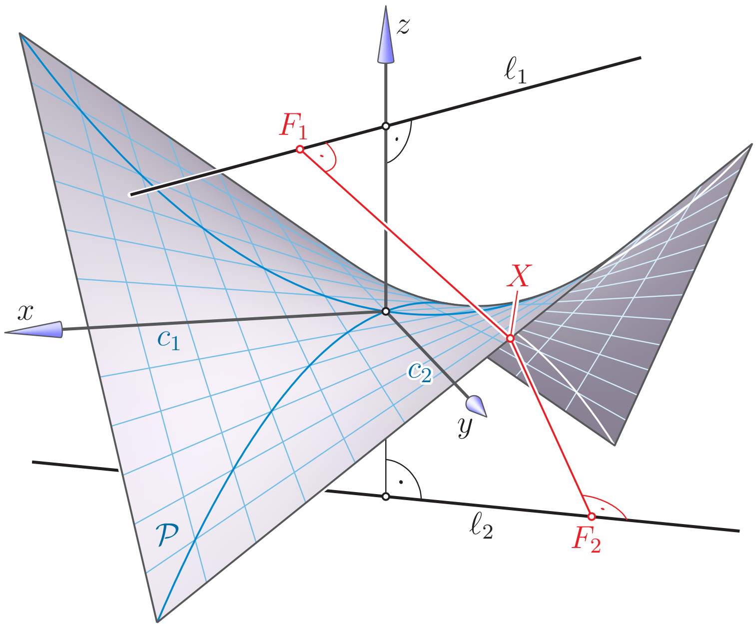

Fig. 5. The bisector of two skew lines ℓ_1 and ℓ_2 is an orthogonal hyperbolic paraboloid $\mathcal{P}$ which contains the axes of symmetry c_1, c_2 of ℓ_1 and ℓ_2 as vertex generators.

Conversely, the question for all pairs (ℓ_1, ℓ_2) of lines for which a given orthogonal hyperbolic paraboloid $\mathcal{P}$ is the bisector, was answered in [4], but already reported at the turn to the 20^{th} century in [9, p. 54]. We recall:

Lemma 2.1. All pairs of skew lines (ℓ_1, ℓ_2) which share the bisecting orthogonal hyperbolic paraboloid $\mathcal{P}$ are located on a Plücker conoid $\mathcal{C}$ in symmetric position with respect to the central generators c_1 and c_2 of $\mathcal{C}$, that coincide with the vertex generators of $\mathcal{P}$.

Proof. We refer to the coordinates of ℓ_1 and ℓ_2 as given above. Then the paraboloid $\mathcal{P}$ satisfying (3) remains the same if the quotient $d/\sin 2\alpha$ does not change. Obviously, the points $X = (r\cos\alpha,\ \pm r\sin\alpha,\ \pm d)$ $(r \in \mathbb{R})$ of ℓ_1 and ℓ_2 satisfy

$$\mathcal{C}\colon\ (x^2+y^2)\,z - 2c\,xy = 0 \ \text{ for } \ c := \frac{d}{\sin 2\alpha}\,. \tag{4}$$

This is the equation of a Plücker conoid $\mathcal{C}$ according to (2). The lines (ℓ_1, ℓ_2) are symmetric with respect to (w.r.t., for short) to the x- and y axis, i.e., to the central generators c_1 and c_2 of $\mathcal{C}$ (Fig. 7). □

As reported in [7, Theorem2.3.6], the lines ℓ_1, ℓ_2 are polar w.r.t. $\mathcal{P}$, i.e., each point $X_1 \in \ell_1$ is conjugate w.r.t. $\mathcal{P}$ to all points $X_2 \in \ell_2$, and vice versa. This follows since the coordinates $X_i = (x_i, y_i, z_i)$ for $i = 1, 2$ with

$$y_1 = x_1 \tan\alpha,\ z_1 = d, \qquad y_2 = -x_2 \tan\alpha,\ z_1 = -d$$

satisfy the polar form of $\mathcal{P}$,

$$d(z_1+z_2) + (x_1y_2 + x_2y_1)\sin\alpha\cos\alpha = 0.$$

Therefore, the polarity in the paraboloid $\mathcal{P}$ maps the Plücker conoid $\mathcal{C}$ onto itself. The ellipses e in tangent planes of $\mathcal{C}$ are polar to the quadratic cones of tangents drawn from points $X \in \mathcal{C}$ to $\mathcal{C}$. For further details see [11, Theorem 3].

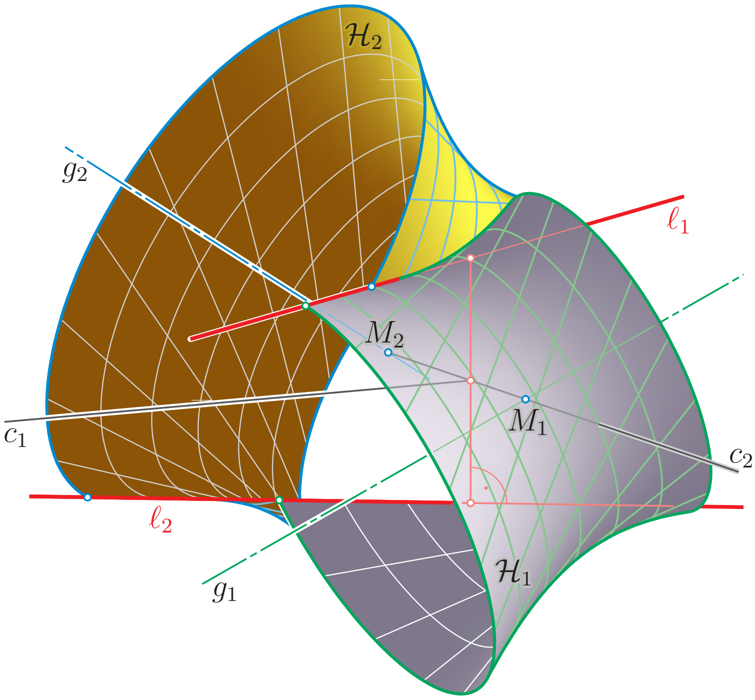

Fig. 6. Two hyperboloids of revolution $\mathcal{H}_1, \mathcal{H}_2$ through two skew lines ℓ_1 and ℓ_2. The two hyperboloids with respective centers M_1, M_2 and axes g_1, g_2 on the bisecting paraboloid (with vertex generators c_1, c_2) share the secondary semiaxis.

3 Plücker's conoid as locus of instant screw axes for skew gears

Here we report about another property of the bisecting orthogonal paraboloid $\mathcal{P}$ of two skew lines ℓ_1, ℓ_2 (see, e.g., [7, Theorem 2.3.5]: The generators of $\mathcal{P}$ are the axes of rotations which send the line ℓ_1 to the line ℓ_2 (Fig. 7). In other words: The generators of $\mathcal{P}$ are axes of one-sheeted hyperboloids of revolution passing through symmetric pairs of lines ℓ_1, ℓ_2 (Fig. 6). These hyperboloids are centered on vertex generators of $\mathcal{P}$. By the way, the two hyperboloids share the secondary semiaxis b. This follows from a result of Wunderlich [15] and Krames [5] which states that two skew generators ℓ_1, ℓ_2 of any hyperboloid of revolution define already the secondary semiaxis $b = d \cot \varphi$, where $2d = \overline{\ell_1 \ell_2}$ and $2\varphi = \sphericalangle\, \ell_1 \ell_2$ (see also [7, p. 37]).

By virtue of Lemma 2.1, the lines ℓ_1, ℓ_2 are generators of the Plücker conoid $\mathcal{C}$ and symmetric w.r.t. the central generators c_1, c_2 (Fig. 7). The limit $\ell_1, \ell_2 \to c_1$ reveals that generators of $\mathcal{P}$ being skew to c_1 are axes of hyperboloids which contact $\mathcal{C}$ along c_1. Therefore, all c_1 intersecting generators of the orthogonal hyperbolic paraboloid $\mathcal{P}$ are surface normals of $\mathcal{C}$.[3]

[3]Model XXIII, no. 10, of Schilling's famous collection of mathematical models [9] shows the pair of surfaces $\mathcal{C}$ and $\mathcal{P}$ (see, e.g., `https://www.geometrie.tuwien.ac.at/modelle/models_show.php?mode=`

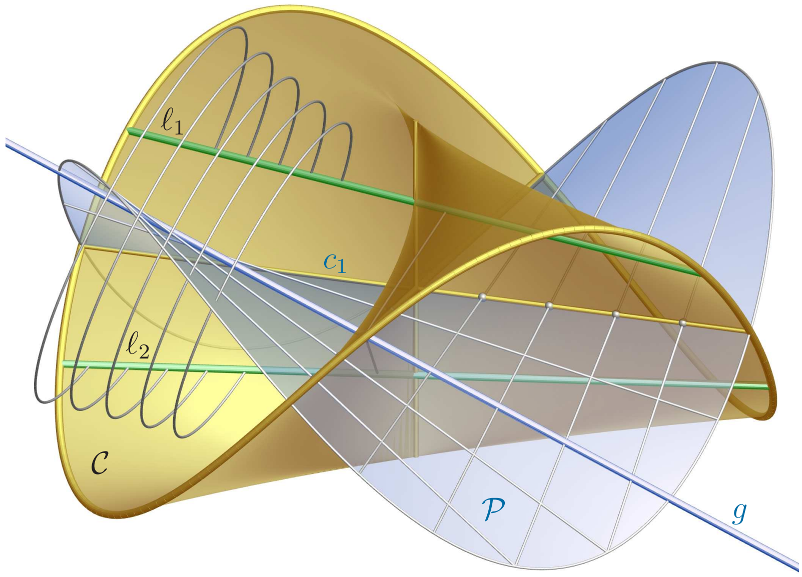

Fig. 7. All pairs of skew lines (ℓ_1, ℓ_2) which share the bisecting orthogonal hyperbolic paraboloid $\mathcal{P}$ are located on a Plücker conoid $\mathcal{C}$. Generators g of $\mathcal{P}$ are axes of rotations with $\ell_1 \mapsto \ell_2$ (courtesy: G. GLAESER).

Two generators of the $\mathcal{P}$-regulus through c_2 are axes of hyperboloids of revolution with mutual contact along the other vertex generator c_1, since both hyperboloids contact $\mathcal{C}$. However, also a converse statement holds true: Lines $\ell_1, \ell_2 \in \mathcal{C}$ which are symmetric w.r.t. the central generators c_1, c_2, are axes of hyperboloids of revolution with mutual contact along c_1. This follows immediately by a $180°$-rotation about c_1, which exchanges c_1 and c_2 and transforms one hyperboloid in the other.

It is surprising that this remains true if c_1 is replaced by any other generator of the Plücker conoid $\mathcal{C}$. This result is well-known in spatial kinematics: the Plücker conoid $\mathcal{C}$ is the locus of instant axes ℓ_{12} of the relative screw motion of two wheels that rotate with respectively constant velocities ω_1 and ω_2 about fixed axes $\ell_1, \ell_2 \subset \mathcal{C}$ being symmetric w.r.t. the central generators c_1, c_2 of $\mathcal{C}$. The axodes of the relative screw motion are the hyperboloids of revolution $\mathcal{H}_1$, $\mathcal{H}_2$ with mutual contact along ℓ_{12} as mentioned before (see, e.g., [7, Figs. 2.19, 2.20]).[4]

Thus, the conoid $\mathcal{C}$ provides the solution of a purely geometric problem: For given skew axes ℓ_1, ℓ_2, find pairs of hyperboloids of revolution which contact each other along a line.

A standard proof of this result uses dual vectors for the representation of oriented lines and screws (see, e.g., [2]). Here we present a synthetic proof of an equivalent statement.

Theorem 3.1. Given two skew lines ℓ_1, ℓ_2, the vertex generators of all orthogonal hyperbolic paraboloids through ℓ_1 and ℓ_2 belong to a Plücker conoid $\mathcal{C}$. The axes of symmetry of ℓ_1, ℓ_2 are the central generators of $\mathcal{C}$.

Proof. Why this theorem is equivalent to the statement on pairs of hyperboloids $(\mathcal{H}_1, \mathcal{H}_2)$ of

`2&n=100&id=0`, retrieved Sept. 2022). At this model the boundary curves of the surfaces are even congruent (note [11, Theorem 3]).

[4]There are various relations between the two fixed axes of rotations ℓ_1, ℓ_2, the relative axis ℓ_{12}, the angular velocities ω_1, ω_2, and the pitch of the relative screw motion (see [2, Fig. 7]). Due to a more general result by Plücker, the conoid is the locus of axes of linear line complexes which are contained in a pencil [6, p. 214].

revolution with a line contact? The common surface normals of two hyperboloids along their line of contact ℓ_{12} form one regulus of an orthogonal hyperbolic paraboloid $\mathcal{P}$ which passes through the axes ℓ_1 and ℓ_2. The line ℓ_{12} is the vertex generator of the complementary regulus on $\mathcal{P}$. The other vertex generator of $\mathcal{P}$ intersects all three lines ℓ_{12}, ℓ_1, and ℓ_2 orthogonally. Therefore, it is the common perpendicular d of ℓ_1 and ℓ_2.

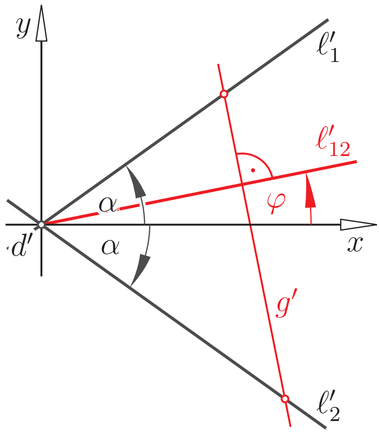

Fig. 8. Top view of an an orthogonal hyperboloid passing through ℓ_1 and ℓ_2 with d and ℓ_{12} as vertex generators and g as a generator which intersects ℓ_1, ℓ_2 and ℓ_{12} (note Theorem 3.1).

We still assume that ℓ_1 and ℓ_2 are horizontal, and we us the orthogonal projection in direction of the common orthogonal transversal d. As shown in Fig. 8, for any choice of a horizontal vertex generator ℓ_{12}, the top view shows on an ℓ_{12} intersecting generator g the affine ratio of g's meeting points with ℓ_1, ℓ_2 and ℓ_{12}. Let $\pm d$ be the given z-coordinates ℓ_1 and ℓ_2. If the signed angles between the x-axis and the lines ℓ_1, ℓ_2 and ℓ_{12} are respectively α, $-\alpha$ and φ, then the altitude of ℓ_{12} is

$$\begin{aligned} z &= -d + 2d\,\frac{\tan(\alpha+\varphi)}{\tan(\alpha-\varphi)+\tan(\alpha+\varphi)} = d\,\frac{\tan(\alpha+\varphi)-\tan(\alpha-\varphi)}{\tan(\alpha-\varphi)+\tan(\alpha+\varphi)} \\ &= \frac{d}{\sin 2\alpha}\,\sin 2\varphi\,, \end{aligned}$$

which confirms by (1) the claim. □

4 Concyclic generators of Plücker's conoid

Given any Plücker conoid $\mathcal{C}$, let $e \subset \mathcal{C}$ be the pedal curve of any given point P. Suppose that some generators of $\mathcal{C}$ are tangents to a sphere $\mathcal{S}$ centered at P. Then their pedal points w.r.t. P must have equal distances to P. Since they are located in the plane of e, they belong to a circle $k \subset \mathcal{S}$ with an axis through P (compare with Fig. 9). The circle k can share with the ellipse e at most four points. Therefore, at most four generators of $\mathcal{C}$ can contact any sphere with the center P. Below we discuss the converse problem: Is the center P of a contacting sphere unique?

Definition 4.1. Four mutually different lines $g_1, \ldots, g_4$ are called *concyclic* if they belong to a Plücker conoid $\mathcal{C}$ and their points of intersection with any tangent plane τ_X to $\mathcal{C}$ are concyclic, i.e., located on a circle.

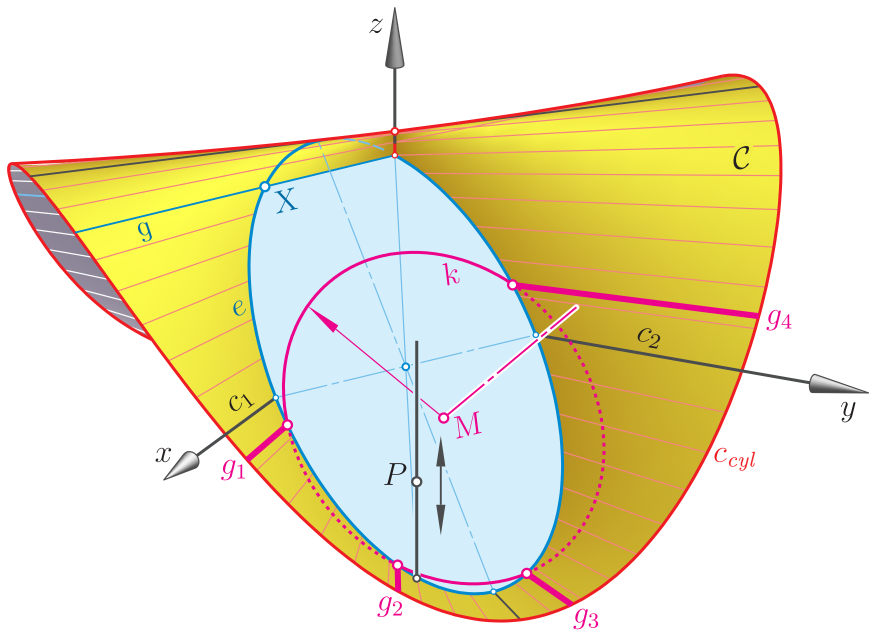

Fig. 9. The ellipse e is the pedal curve of $\mathcal{C}$ w.r.t. the point P. There are infinitely many spheres contacting the four concyclic generators $g_1, \ldots, g_4$ of $\mathcal{C}$. The center of a contacting sphere must be located on the vertical line through P and on the axis of the circle k.

Suppose that two out of four concyclic lines are intersecting. Then also the remaining two must be intersecting, since in this case the center of the circumcircle k must be located on the principal axis of the ellipse $e = \tau_X \cap \mathcal{C}$. We call this the *symmetric case*.

Lemma 4.1. If the generators $g_1, \ldots, g_4 \subset \mathcal{C}$ are concyclic, then they intersect all tangent planes τ_X to $\mathcal{C}$ at four concyclic points, provided that in the particular case $g_i \subset \tau_X$ the point of contact X with $\mathcal{C}$ serves as the point of intersection.

Proof. We compare two ellipses $e_1, e_2 \subset \mathcal{C}$. The top view (Fig. 10) shows that lines h' through d' intersect e_1' and e_2' at points H_1', H_2' which define a similarity $e_1' \to e_2'$. In particular, the top view of the generator g_1 in the plane of e_1 intersects e_1' at the top view X_1' of the point of contact between the plane of e_1 and $\mathcal{C}$.

The similarity $e_1' \to e_2'$ induces in space an affine correspondence α_{12} between the ellipses e_1 and e_2 and consequently between the corresponding planes. The torsal generators and the central generators of $\mathcal{C}$ indicate that α_{12} sends the vertices of e_1 to vertices of respectively equal types of e_2. Using appropriate coordinates (x_1, y_1) and (x_2, y_2) in the respective planes, the correspondence can be expressed in the form

$$\alpha_{12}\colon\ (x_1, y_1) \mapsto (x_2, y_2) = (\lambda x_1,\ \mu y_1)$$

with

$$e_1 :\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1\ \to\ e_2 :\ \frac{x_2^2}{\lambda^2 a^2} + \frac{y_2^2}{\mu^2 b^2} = 1.$$

If four generators of $\mathcal{C}$ intersect e_1 at the points $A_1, \ldots, D_1$ of a circle k with radius r, then the corresponding points $A_2, \ldots, D_2 \in e_2$ are located on an ellipse k_2 with axes of lengths λr, μr

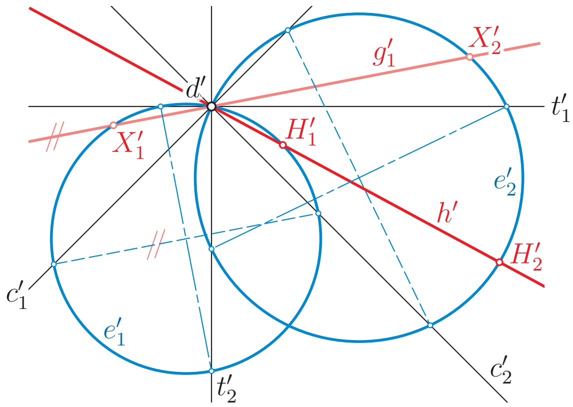

Fig. 10. The generators h of a Plücker conoid intersect different tangent planes at points H_1, H_2 that correspond each other in an affine transformation α_{12}.

parallel to the coordinate axes. There is a linear combination of the equations of e_2 and k_2 with equal coefficients of x_2^2 and y_2^2, namely

$$(\lambda^2-\mu^2)a^2b^2\left(\frac{x_2^2}{\lambda^2a^2}+\frac{y_2^2}{\mu^2b^2}+\dots\right)+(\mu^2b^2-\lambda^2a^2)r^2\left(\frac{x_2^2}{\lambda^2r^2}+\frac{y_2^2}{\mu^2r^2}+\dots\right)=0.$$

This means that the pencil spanned by e_2 and k_2 contains a circle through the base points $A_2,\dots,D_2$, as claimed.[5]

By the way, for the second coefficient in this linear combination holds

$$(\mu^2b^2-\lambda^2a^2)=b^2-a^2,$$

since e_1 and e_2 have the same eccentricity. The center of the circumcircle of $A_2,\dots,D_2\in e_2$ is independent of r and has the coordinates $(\frac{m}{\lambda},\frac{n}{\mu})$, if (m,n) are the coordinates of the center M of the preimage k in the plane of e_1 (note Fig. 9). □

Suppose that, by virtue of Lemma 1.1, the three lines g_1, g_2, g_3 with a common orthogonal transversal d define a Plücker conoid $\mathcal{C}$. If these lines contact a sphere $\mathcal{S}$ with center P, then the points of contact lie on the pedal curve e of P and have equal distances to P. The circumcircle k of the pedal points shares with e a fourth point, and consequently there exists a fourth generator $g_4\subset\mathcal{C}$ which contacts the sphere $\mathcal{S}$ as well. If $g_1,\dots,g_4$ are mutually different, then they are concyclic. However, it can happen that k contacts e at any point. Then the line g_4 coincides with one of the three given lines.

Theorem 4.1. If four lines $g_1,\dots,g_4$ are concyclic on the Plücker conoid $\mathcal{C}$, then there exist infinitely many spheres which contact these lines. The six bisecting hyperbolic paraboloids of the pairs (g_i,g_j), $i,j\in\{1,\dots,4\}$, $i\neq j$, belong to a pencil.

In the non-symmetric case, the spine curve of the enveloping canal surface $\mathcal{E}$ is a rational quartic q (Fig. 11). The top view of q is an equilateral hyperbola with the top views of the torsal generators of $\mathcal{C}$ as asymptotes (Fig. 12). In the symmetric case, the spine curve splits into two parabolas in the planes which connect the double line d of $\mathcal{C}$ with one of the torsal generators.

[5] An alternative proof based on Desargues's involution theorem [3, Sect. 7.4] is mentioned in [10, p. 60].

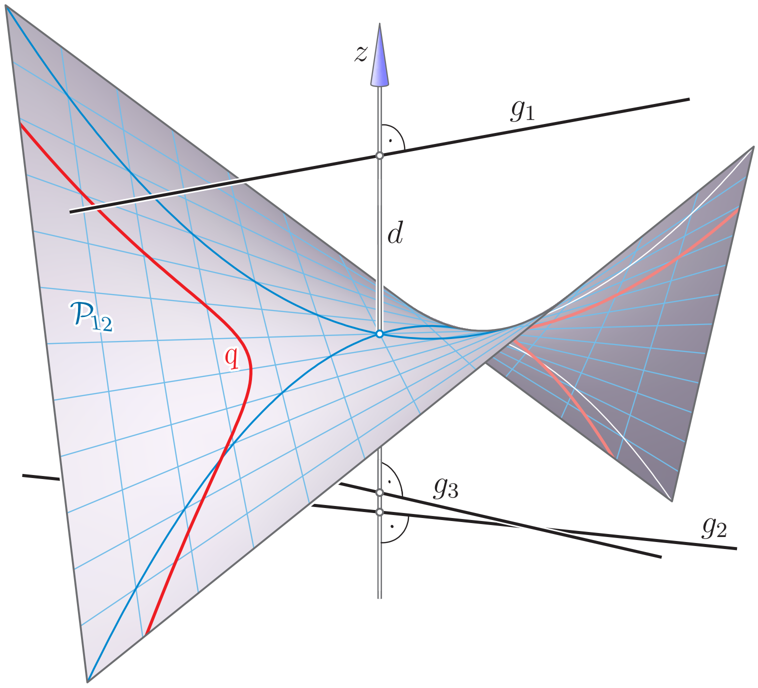

Fig. 11. Spine curve q of the enveloping canal surface of spheres that contact the lines g_1, g_2 (with bisector $\mathcal{P}_{12}$) and the line g_3.

Remark 4.1. According to [10, Satz 4], there are only two cases where four mutually skew lines have a continuum of contacting spheres: The given lines are either concyclic or belong to a hyperboloid of revolution. For similar problems see also [12].

Proof. We assume that the skew generators $g_i, g_j \subset \mathcal{C}$ for $i, j \in \{1, 2, 3\}$ have the polar angles φ_i, φ_j and the z-coordinates $z_i = c\,\sin 2\varphi_i$ and $z_j = c\,\sin 2\varphi_j$ according to (1). The distance of any space point $X = (x, y, z)$ to g_i satisfies

$$\overline{Xg_i}^{\,2} = x^2 + y^2 + (z - z_i)^2 - (x\cos\varphi_i + y\sin\varphi_i)^2.$$

The bisecting paraboloid $\mathcal{P}_{ij}$ of the generators g_i, g_j of $\mathcal{C}$ has the equation $\overline{Xg_i}^{\,2} - \overline{Xg_j}^{\,2} = 0$, i.e.,

$$\mathcal{P}_{ij}\colon\ (\sin^2\varphi_i - \sin^2\varphi_j)(x^2 - y^2) - (z_i - z_j)\left(\frac{xy}{c} + 2z\right) + (z_i^2 - z_j^2) = 0.$$

The paraboloids $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ share a quartic q, and each point $P \in q$ is the center of a sphere $\mathcal{S}$ which contacts g_1, g_2 and g_3. As explained before, $\mathcal{S}$ must also contact the line g_4 which completes the concyclic quadruple.[6]

We obtain the equation of the top view q' of q as a linear combination of the equations of $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ after the elimination of z. In the unsymmetric case, the resulting equation has the form

[6] As proved in [10], the four lines $g_1, \dots, g_4$ are concyclic if and only if the (5×4)-matrix with the rows $(1,\, z_i,\, z_i^2,\, \cos 2\varphi_i,\, \sin 2\varphi_i)$, $i = 1, \dots, 4$, has a rank ≤ 3.

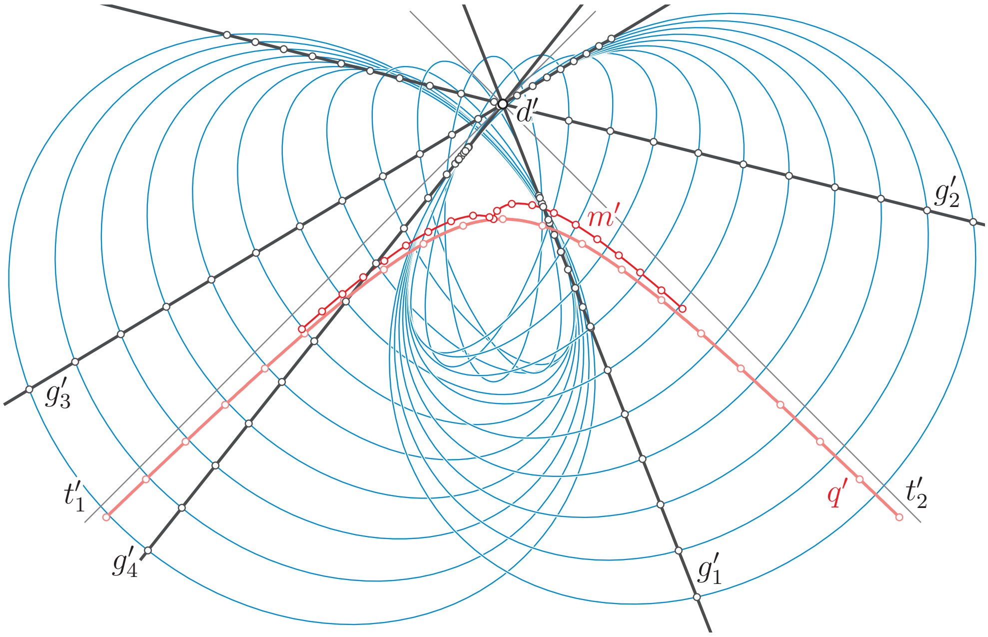

Fig. 12. Top view of a sample of circles of the canal surface through the four concyclic lines $g_1, \dots, g_4$ on the Plücker conoid with torsal generators t_1, t_2 and double line d. The hyperbola q' is the top view of the spine curve and m' that of the curve of circle centers.

$u(x^2 - y^2) = v$ with $u, v \in \mathbb{R} \setminus \{0\}$, namely

$$\begin{aligned} u &= z_1(\sin^2\varphi_2 - \sin^2\varphi_3) + z_2(\sin^2\varphi_3 - \sin^2\varphi_1) + z_3(\sin^2\varphi_1 - \sin^2\varphi_2) \\ &= \frac{c}{2}\left[\sin 2(\varphi_2 - \varphi_1) + \sin 2(\varphi_3 - \varphi_2) + \sin 2(\varphi_1 - \varphi_3)\right] \text{ and} \\ v &= z_1^2\,(z_2 - z_3) + z_2^2\,(z_3 - z_1) + z_3^2\,(z_1 - z_3). \end{aligned}$$

Consequently, q' is an equilateral hyperbola with the semiaxis $\sqrt{v/u}$ and the asymptotes $x \pm y = 0$, which are the top views of the torsal generators of $\mathcal{C}$ (Fig. 12). In order to compute the z-coordinate of the points of q, we use the equation of $\mathcal{P}_{12}$ which is linear in z. Therefore, q is rational. □

The envelope $\mathcal{E}$ of the infinitely many spheres $\mathcal{S}$, as mentioned in Theorem 4.1, must contain the four concyclic lines $g_1, \dots, g_4$. The sphere $\mathcal{S}$ with center $P \in q$ contacts the envelope $\mathcal{E}$ along the circumcircle k of the pedal points of $g_1, \dots, g_4$ w.r.t. P. The existence of this circle was confirmed in Lemma 4.1.

The shape of the envelope $\mathcal{E}$ is hard to grasp as it has singularities. This becomes apparent since on tangents g_i with a top view g_i' intersecting the equilateral hyperbola q' (like g_1' and g_4' in Fig. 12) the pedal point cannot trace the full line while P runs along one branch of the spine curve q. There needs to be a point of return. Fig. 13 shows a part of the envelope $\mathcal{E}$ which has no visible singularity. The complete canal surface contains also a second component which is obtained by a halfturn about the z-axis.

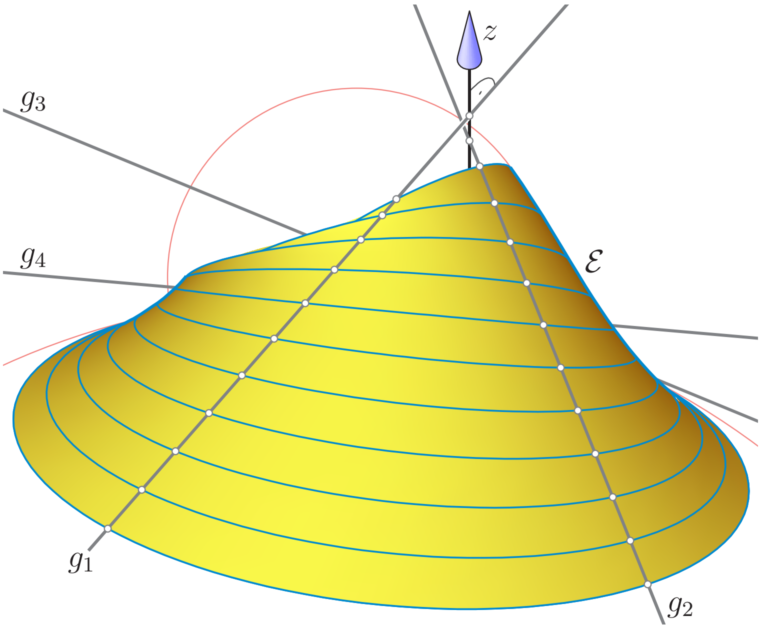

Fig. 13. A portion of the canal surface $\mathcal{E}$ through the four concyclic lines $g_1, \ldots, g_4$ along with the spheres (red) contacting $\mathcal{E}$ along the terminating circles.

References

[1] APPELL, P. Propriété caractéristique du cylindroïde. In *Bull. Soc. Math. France* 28, 1900, pp. 261–265.

[2] FIGLIOLINI, G., STACHEL, H., ANGELES, J. A new look at the Ball-Disteli diagram and its relevance to spatial gearing. In *Mech. Mach. Theory* 42/10, 2007, pp. 1362–1375.

[3] G. GLAESER, H. STACHEL, B. ODEHNAL *The Universe of Conics.* From the ancient Greeks to 21st century developments. Springer Spectrum, Berlin, Heidelberg 2016.

[4] HUSTY, M., SACHS, H. Abstandsprobleme zu windschiefen Geraden I. In *Sitzungsber.*, Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. 203, 1994, pp. 31–55.

[5] KRAMES, J. Über die in einem Strahlnetz enthaltenen Drehhyperboloide. In *Rad.* Jugosl. Akad. Znan. Umjet., Mat. Znan. 2, 1983, pp. 1–7.

[6] MÜLLER, E., KRAMES, J.L. *Vorlesungen über Darstellende Geometrie. Band III: Konstruktive Behandlung der Regelflächen.* B.G. Teubner, Leipzig, Wien 1931.

[7] ODEHNAL, B., STACHEL, H., GLAESER, G. *The Universe of Quadrics.* Springer Spectrum, Berlin, Heidelberg 2020.

[8] SALMON, G., FIEDLER, W. *Die Elemente der analytischen Geometrie des Raumes.* B.G. Teubner, Leipzig 1863.

[9] SCHILLING, M. *Catalog mathematischer Modelle.* 7. Auflage, Martin Schilling, Leipzig 1911.

[10] STACHEL, H. Unendlich viele Kugeln durch vier Tangenten. In *Math. Pannonica* 6, 1995, pp. 55–66.

[11] STACHEL, H. Plücker's conoid, hyperboloids of revolution and orthogonal hyperbolic paraboloids. In Z. JELI (ed.): *Proceedings moNGeometrija2020*, Serbian Society for Geometry and Graphics (SUGIG), ISBN 978-86-6060-046-4, pp. 51–62.

[12] THEOBALD, T. *New Algebraic Methods in Computational Geometry.* Habilitation thesis, TU München, 2003.

[13] WUNDERLICH, W. *Darstellende Geometrie II.* BI Mannheim, 1967.

[14] WUNDERLICH, W. *Ebene Kinematik.* BI Mannheim, 1970.

[15] WUNDERLICH, W. Die Netzflächen konstanten Dralls. In *Sitzungsber.*, Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. 191, 1982, pp. 59–84.

Hellmuth Stachel

Vienna University of Technology
Wiedner Hauptstr. 8-10/104, 1040 Wien, Austria
e-mail: stachel@dmg.tuwien.ac.at