Henrici's flexible hyperboloid and snapping spatial four-bars

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Abstract. Henrici's flexible hyperboloid consists of rods chosen as generators of both reguli of a one-sheeted hyperboloid. If each crossing point between two rods is materialized as a spherical joint, then the hyperboloid can vary within a confocal family terminated by two flat poses. After presenting the main properties of Henrici's hyperboloid, we restrict the variation to a quadrangle with sides along generators. This induces a one-parameter transformation of the quadrangle where the side lengths are preserved. When we connect points of opposite sides by taut strings along additional generators, then the strings remain taut during the transformation. However, a continuous transformation is possible only with spherical joints at the four vertices.
As an alternative, we can pick out two sufficiently close poses. Then, it is possible to find appropriate revolute joints at the vertices that enable a physical model of this spatial four-bar to snap from one pose into the other, provided that the material of the bars and clearances of the joints admit tiny deformations. Also a converse is true: For each snapping four-bar we find a hyperboloid such that the two poses originate from a Henrici flex. Consequently, a net of taut strings spanned along additional generators of the hyperboloid is compatible with the snapping of the quadrangular frame.

Keywords: Henrici's hyperboloid, confocal quadrics, spatial four-bar, snapping structure.

1 Introduction

At the turn from the 19$^{\text{th}}$ to the 20$^{\text{th}}$ century it became fashion to produce physical models for visualizing mathematical objects and phenoma. These models were spread all over the world due to catalogues in form of books [5]. Today only at a few places some of these historical models are available, e.g., at the Vienna Institute of Technology under `https://www.geometrie.tuwien.ac.at/modelle/`.

We begin with the explanation of Henrici's hyperboloid (Figure 1). It was found in 1874 by Olaus Henrici, a German mathematician, who became director of the Laboratory of Mechanics at the University College London. The flexibility of this structure follows from properties of confocal central quadrics (see, e.g., [2]). A similar structure exists for hyperbolic paraboloids.

The restriction of the hyperboloid to a quadrilateral of four generators leads to a flexible quadrangle where the flex is compatible with a net of taut strings spanned along additional generators between opposite sides.

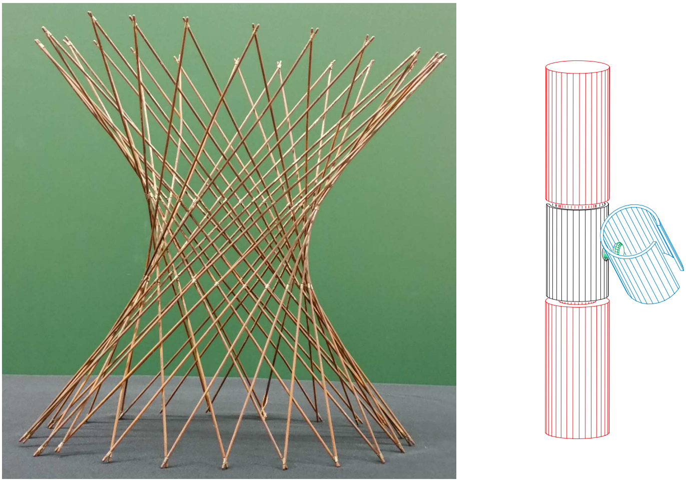

Fig. 1: Left: Henrici's flexible hyperboloid. Right: H. Wiener's design of approximate spherical joints.

We discuss the question whether the spherical joints can be replaced by revolute joints which extends the quadrangle to a *spatial four-bar*, i.e., a closed kinematic 4R-chain. It is well-known that a spatial four-bar is continuously flexible only in three cases: Either the axes are parallel or concurrent, or the common perpendiculars between neighboring axes form an isogram, i.e., a quadrangle with opposite sides of equal lengths. In the latter case we obtain a Bennet mechanism (see, e.g., [2, p. 555–559]).

2 Henrici's hyperboloid

The one-parameter family of quadrics being confocal with the triaxial ellipsoid $\mathcal{E}$ with semiaxes a, b, c is given as

$$F_k(x, y, z) := \frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1 \tag{1}$$

with the parameter $k \in \mathbb{R} \setminus \{-a^2, -b^2, -c^2\}$. If $a > b > c > 0$, then this family contains (see Figure 2)

$$\text{for} \begin{cases} -c^2 < k < \infty & \text{triaxial ellipsoids,} \\ -b^2 < k < -c^2 & \text{one-sheeted hyperboloids,} \\ -a^2 < k < -b^2 & \text{two-sheeted hyperboloids.} \end{cases} \tag{2}$$

Their curves of intersections with the coordinate planes share the respective focal points. As the limits for $k \to -c^2$ and $k \to -b^2$ occur 'flat' quadrics bounded by one of the focal conics (see [1, p. 137]).

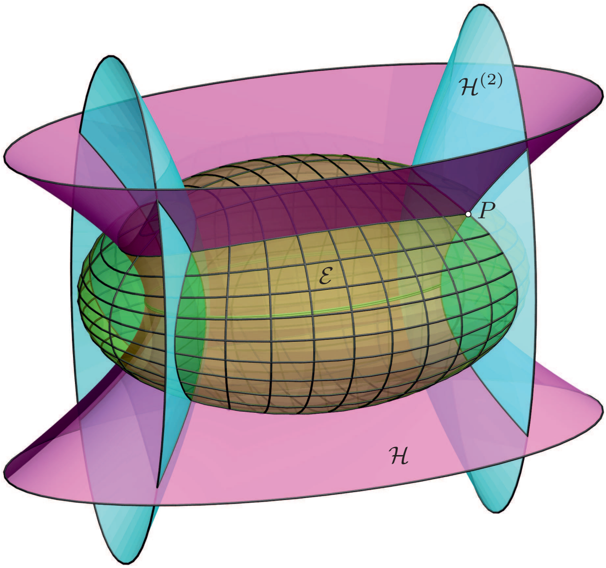

Fig. 2: Confocal central quadrics. The curves of intersection between the triaxial ellipsoid $\mathcal{E}$ and the confocal hyperboloids $\mathcal{H}$ and $\mathcal{H}^{(2)}$ are lines of curvature on $\mathcal{E}$.

Through each given point P in space away from the common planes of symmetry, the confocal family sends three mutually orthogonal surfaces, one of each type (Figure 2). Due to Dupin's theorem, the confocal surfaces intersect each other along curvature lines. The respective parameters (k_1, k_2, k_3) of the ellipsoid, the one-sheeted and the two sheeted hyperboloid passing through P are called *elliptic coordinates* of P, where $k_1 > -c^2 > k_2 > -b^2 > k_3 > -a^2$. They are related to the Cartesian coordinates (x, y, z) via

$$\begin{gathered} x^2 = \frac{(a^2+k_1)(a^2+k_2)(a^2+k_3)}{(a^2-b^2)(a^2-c^2)}, \quad y^2 = \frac{(b^2+k_1)(b^2+k_2)(b^2+k_3)}{(b^2-c^2)(b^2-a^2)}, \\ z^2 = \frac{(c^2+k_1)(c^2+k_2)(c^2+k_3)}{(c^2-a^2)(c^2-b^2)}. \end{gathered} \tag{3}$$

Apparently, eight points in space, symmetrically placed w.r.t. the coordinate frame, share the elliptic coordinates.

Suppose that the coordinate k_2 varies, while k_1 and k_3 and the signs of Cartesian coordinates remain constant. Then, this induces a smooth transformation of the one-sheeted hyperboloid $\mathcal{H}$ within the confocal family, while its points run along orthogonal trajectories of the hyperboloids.

From (3) follows that aligned points of $\mathcal{H}$, i.e., points which are mutually conjugate with respect to $\mathcal{H}$, remain aligned during the flex. Thus, $\mathcal{H}$ undergoes an affine motion. During the variation of k_2 the angle between any two intersecting generators of $\mathcal{H}$ varies as well as the distribution parameter of the generators. Therefore spherical joints are necessary at all crossing points of Henrici's hyperboloid (Figure 1, right).

Theorem 1. *Referring to the elliptic coordinates as explained before, when k_2 varies, while k_1, k_3 remain unchanged, then during the induced smooth transformation of the hyperboloid $\mathcal{H}$ the points placed on any generator remain aligned and their mutual distances are preserved.*

A similar result holds for confocal hyperbolic paraboloids.

3 Snapping spatial four-bars

Let $P_1P_2P_3P_4$ be a quadrangle with sides located on generators of the hyperboloid $\mathcal{H}$. Then during the flex of $\mathcal{H}$ we obtain spatial quadrangles $P_1'P_2'P_3'P_4'$ with equal side lengths, i.e., $\overline{P_iP_{i+1}} = \overline{P_i'P_{i+1}'}$ for all $i \in \{1, \dots, 4\}$ (subscripts modulo 4); we call two quadrangles with equal side lengths *isometric*. Moreover, as shown in Figure 3, in the interior of the quadrangles additional generators of $\mathcal{H}$ can be materialized as strings, and they remain taut during the flex.

For any given quadrangle, there exists a two-parametric set of isometric quadrangles, up to rigid motions, since there is a free choice of the interior angle $\sphericalangle P_4P_1P_2$ and of the bending angle along the diagonal P_4P_2 between the planes $[P_4, P_1, P_2]$ and $[P_2, P_3, P_4]$. The pairs of isometric quadrangles $P_1 \dots P_4$ and $P_1' \dots P_4'$ obtained by Henrici's movement are in a particular relative position: The pedal points of the common perpendicular between the lines $[P_i, P_{i+1}]$ and $[P_i, P_{i+1}]$ are corresponding under the induced isometry between the generators. Also a converse is true.

Theorem 2. *For any two isometric quadrangles $P_1 \dots P_4$ and $P_1' \dots P_4'$ there exists either a hyperboloid $\mathcal{H}$ or a hyperbolic paraboloid $\mathcal{P}$ passing through the sides of $P_1 \dots P_4$ such that we obtain a congruent copy of $P_1' \dots P_4'$ according to Henrici's transition from $\mathcal{H}$ or $\mathcal{P}$ to a confocal quadric.*

The proof is based on the singular-value decomposition of the unique affine transformation the sends $P_1 \dots P_4$ and $P_1' \dots P_4'$. This result reveals that Henrici's flexing hyperboloid is not such a particular event as one could presume.

From now on we concentrate on two poses of a flexing isometric quadrangle and ask if the spherical joints at the vertices can be replaced by revolute joints. This results in a spatial four-bar which possibly can snap

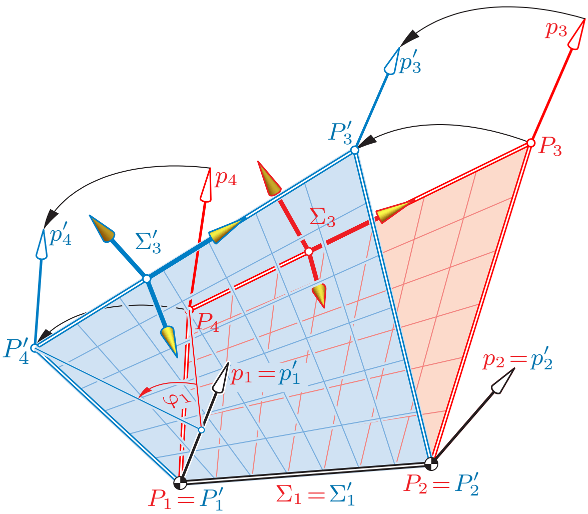

Fig. 3: Snapping spatial four-bar with the poses $P_1 \dots P_4$ and $P'_1 \dots P'_4$, while the base Σ_1 with P_1P_2 and the axes p_1 and p_2 is kept fixed.

between two poses, due to slight deformations of the sides and clearances of the joints. We speak of a *snapping four-bar* (Figure 4).

Contrary to the situation in the plane or on the sphere, in space there is a difference between spatial quadrangles and spatial four-bars: The quadrangle $P_1 \dots P_4$ consists of four sides only, while the four-bar is a loop of four rigid bodies $\Sigma_1, \dots, \Sigma_4$ with four revolute joints. We assume that the body Σ_i for $i \in \{1, \dots, 4\}$ contains the vertices P_i and P_{i+1}, and the corresponding revolute axes p_i and p_{i+1}. Then, the two poses of any snapping four-bar define four displacements

$$\delta_i \colon\ \Sigma_i \to \Sigma'_i,\ \ P_i \mapsto P'_i,\ \ P_{i+1} \mapsto P'_{i+1},\ \ p_i \mapsto p'_i,\ \ p_{i+1} \mapsto p'_{i+1}.$$

The composition $\delta_{i-1}^{-1} \circ \delta_i$ must be a rotation as it keeps the point P_i fixed. Consequently, once the displacements $\delta_1, \dots, \delta_4$ are defined, the axis p_i is uniquely determined as the line which remains pointwise fixed under $\delta_{i-1}^{-1} \circ \delta_i$.

This leads to a four-parametric choice of revolute axes for any two given isometric quadrangles. We emphasize one solution which is based on the Henrici-position according to Theorem 2. We define δ_i for all i as the screw motion along the common perpendicular of $[P_i, P_{i+1}]$ and $[P'_i, P'_{i+1}]$, that sends the quadrangle's side P_iP_{i+1} to $P'_iP'_{i+1}$.

As shown in Figure 3 with the fixed body Σ_1, the displacement of Σ_3

Fig. 4: Physical model of a snapping four-bar, produced by D. Huczala

to Σ_3' is induced by the rotations ρ_4 about p_4 through φ_4 and ρ_1 about p_1 through φ_1. However, the same displacement is the composition of the rotation $\rho_3^{-1}$of Σ_3 against Σ_2 about p_3 through $-\rho_3$ and ρ_2^{-1} about p_2 through $-\varphi_2$. This means,

$$\rho_1 \circ \rho_4 = \rho_2^{-1} \circ \rho_3^{-1}. \tag{4}$$

Conversely, this condition implies that the displacement of Σ_3 to Σ_3' can be obtained in two ways which defines a 4R-loop-structure. This condition can easily be expressed in terms of dual unit quaternions. When expanded in coordinates, it results in necessary and sufficient conditions for snapping four-bars in the form of the solvability of an overdetermined system of six equations for the unkonwn angles $\varphi_1, \dots, \varphi_4$.

Note that this condition for snappability depends on the relative positions of the revolute axes $p_1, \dots, p_4$, but not on the specification of the vertices P_i on the respective axes p_i. With other words, one snapping four-bar implies a four-parametric set of snapping four-bars, since the four vertices of the quadrangle can independently be modified on the corresponding axes. This means, for example, that each Bennet-mechanism leads to numerous continuously flexible four-bars just by changing the vertices P_i on the corresponding axes p_i.

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