Strophoids – cubic curves with remarkable properties

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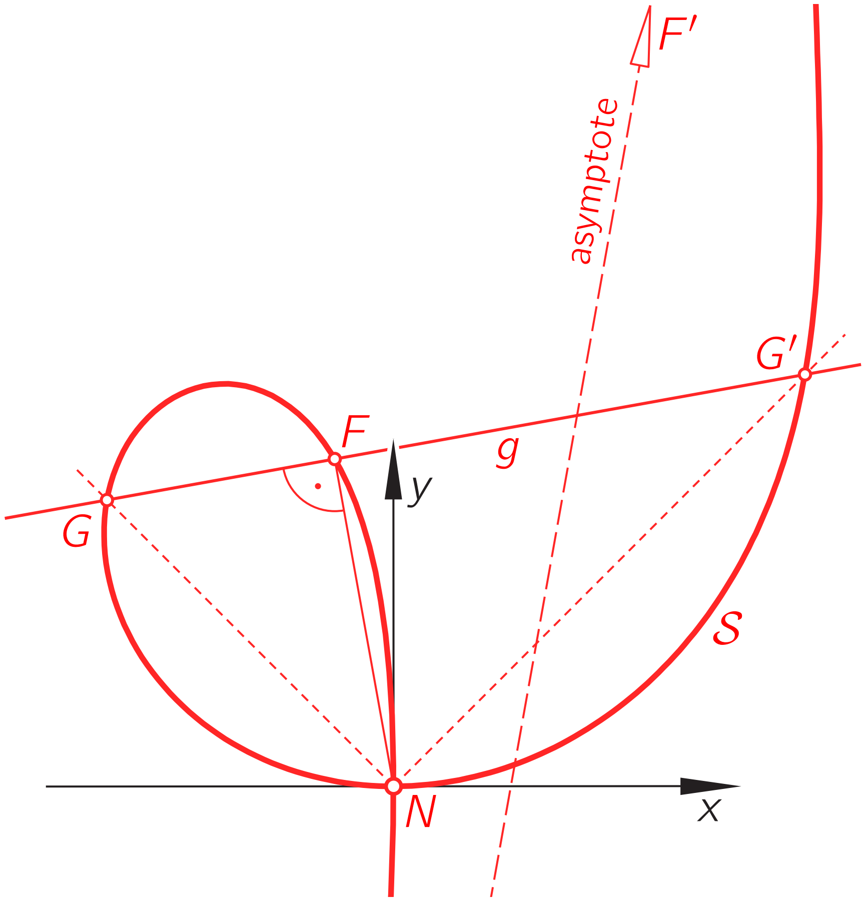
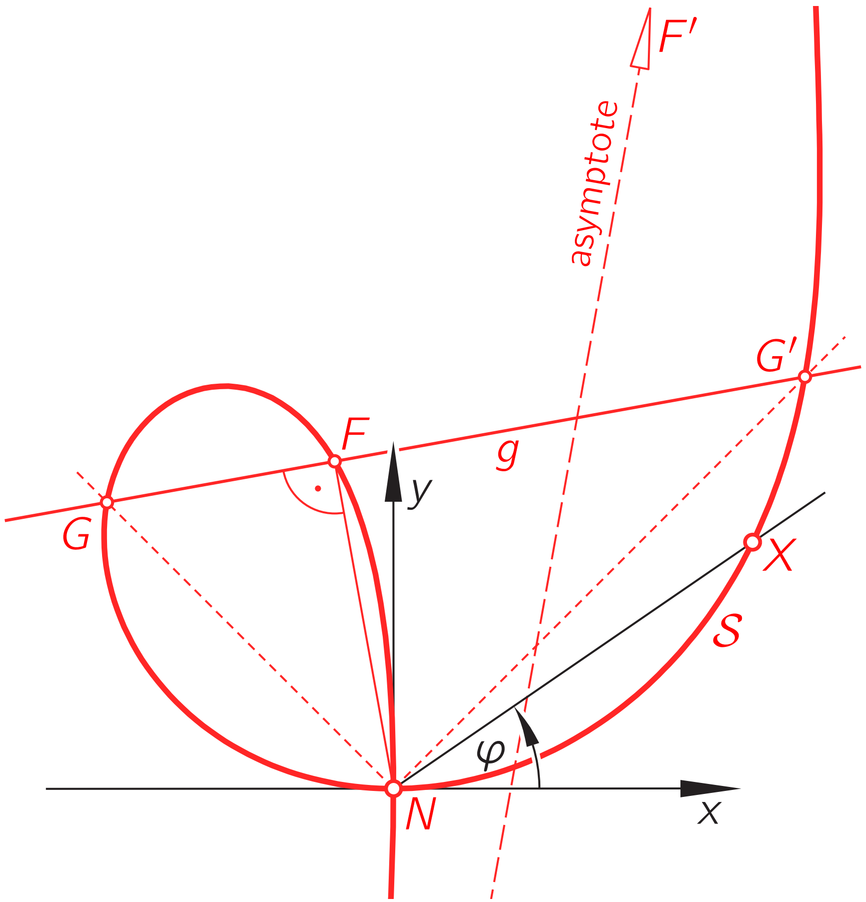


Definition: An irreducible cubic is called **circular** if it passes through the absolute circle-points.
A circular cubic is called **strophoid** if it has a double point (= node) with orthogonal tangents.
A strophoid without an axis of symmetry is called oblique, otherwise right.

$$\mathcal{S}: \ (x^2 + y^2)(ax + by) - xy = 0$$

with $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$. In fact, $\mathcal{S}$ intersects the line at infinity at $(0:1:\pm i)$ and $(0:b:-a)$.

1. Definition of Strophoids



The line through N with inclination angle φ intersects $\mathcal{S}$ in the point

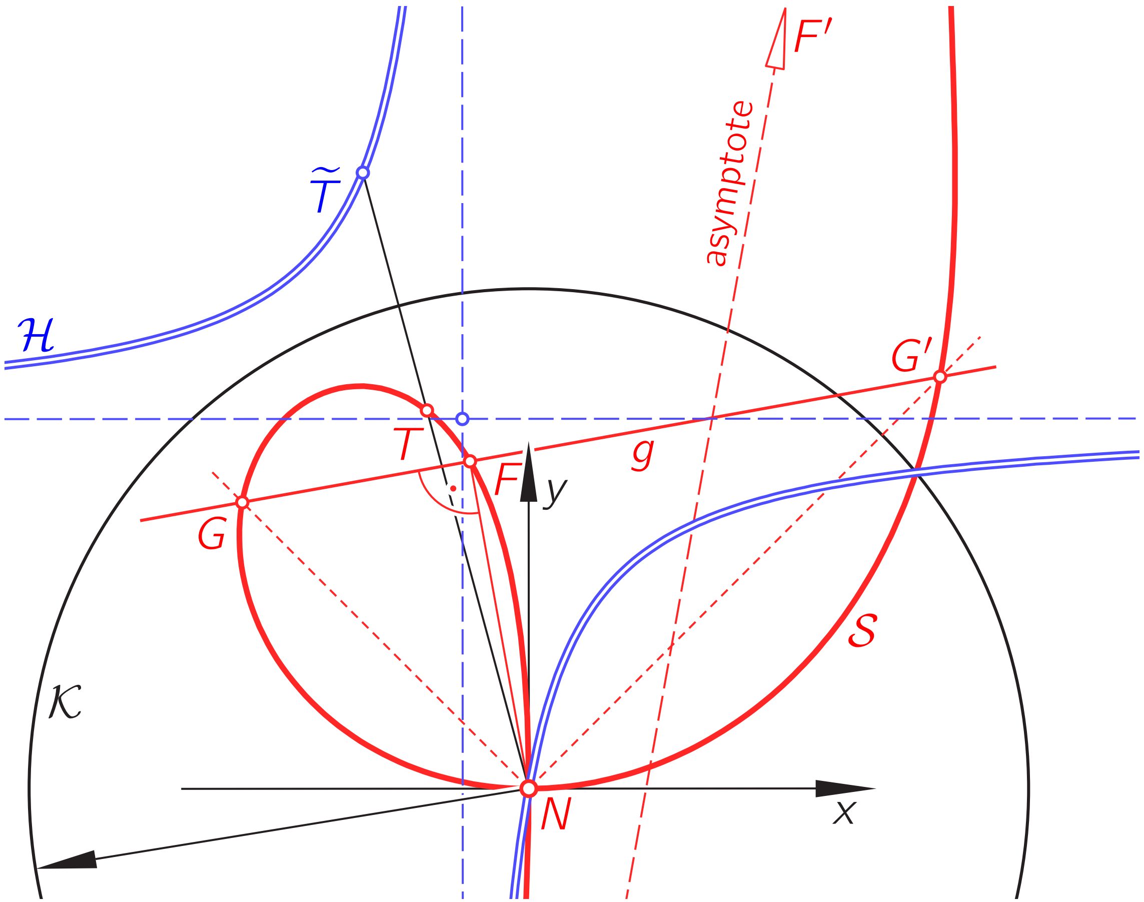
$$X = \left(\frac{\mathrm{s}\varphi\ \mathrm{c}^2\varphi}{a\,\mathrm{c}\varphi + b\,\mathrm{s}\varphi},\ \frac{\mathrm{s}^2\varphi\ \mathrm{c}\varphi}{a\,\mathrm{c}\varphi + b\,\mathrm{s}\varphi} \right).$$

This yields a parametrization of $\mathcal{S}$.

$\varphi = \pm 45°$ gives the points G, G'.

The tangents at the absolute circle-points intersect in the focus F.

1. Definition of Strophoids



The polar equation of $\mathcal{S}$ is

$$\mathcal{S}:\ r = \frac{1}{\frac{a}{\sin\varphi} + \frac{b}{\cos\varphi}}.$$

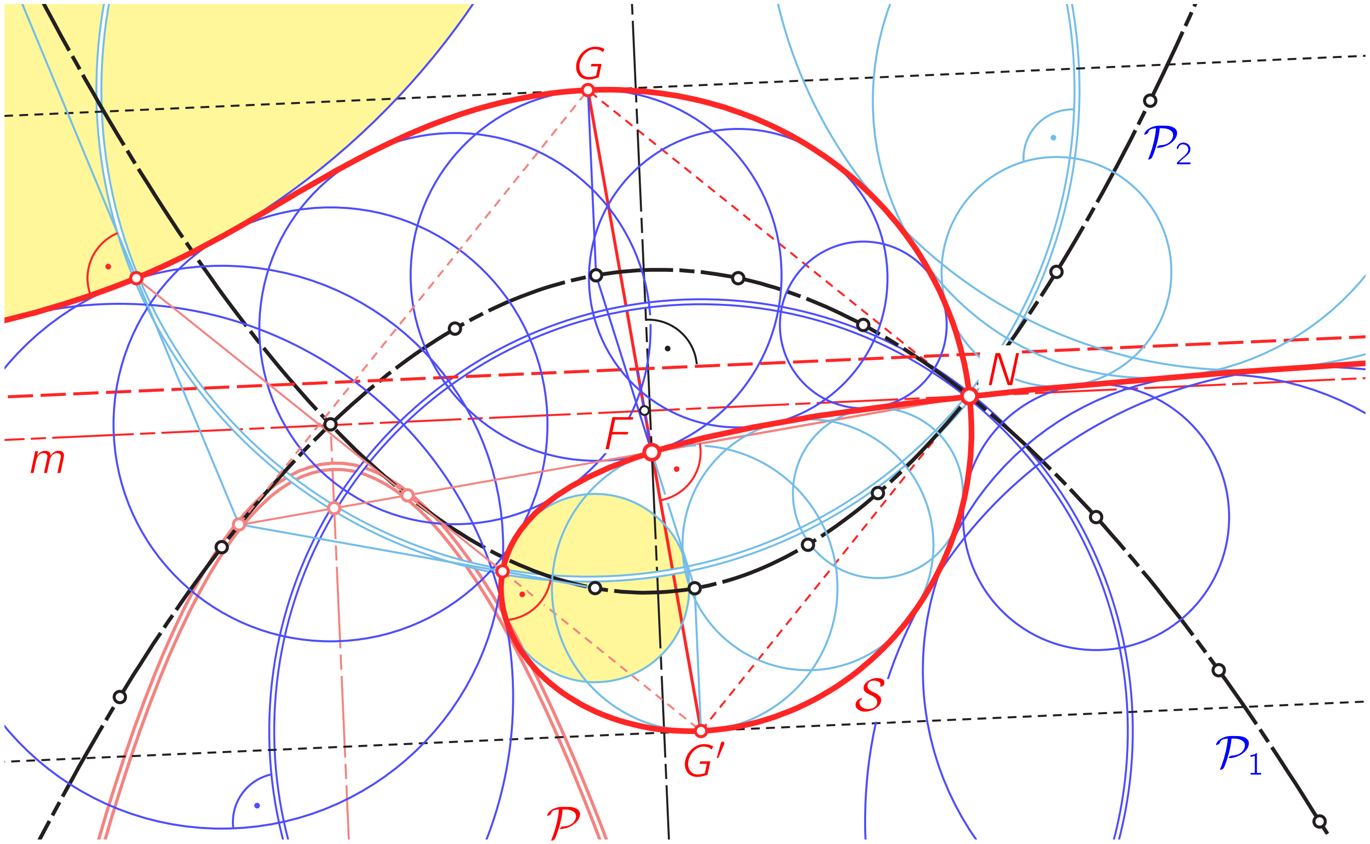
The inversion in the circle $\mathcal{K}$ transforms $\mathcal{S}$ into the curve $\mathcal{H}$ with the polar equation

$$\mathcal{H}:\ r = \frac{a}{\sin\varphi} + \frac{b}{\cos\varphi}.$$

This is an equilateral hyperbola which satisfies

$$\mathcal{H}:\ (x-b)(y-a) = ab.$$

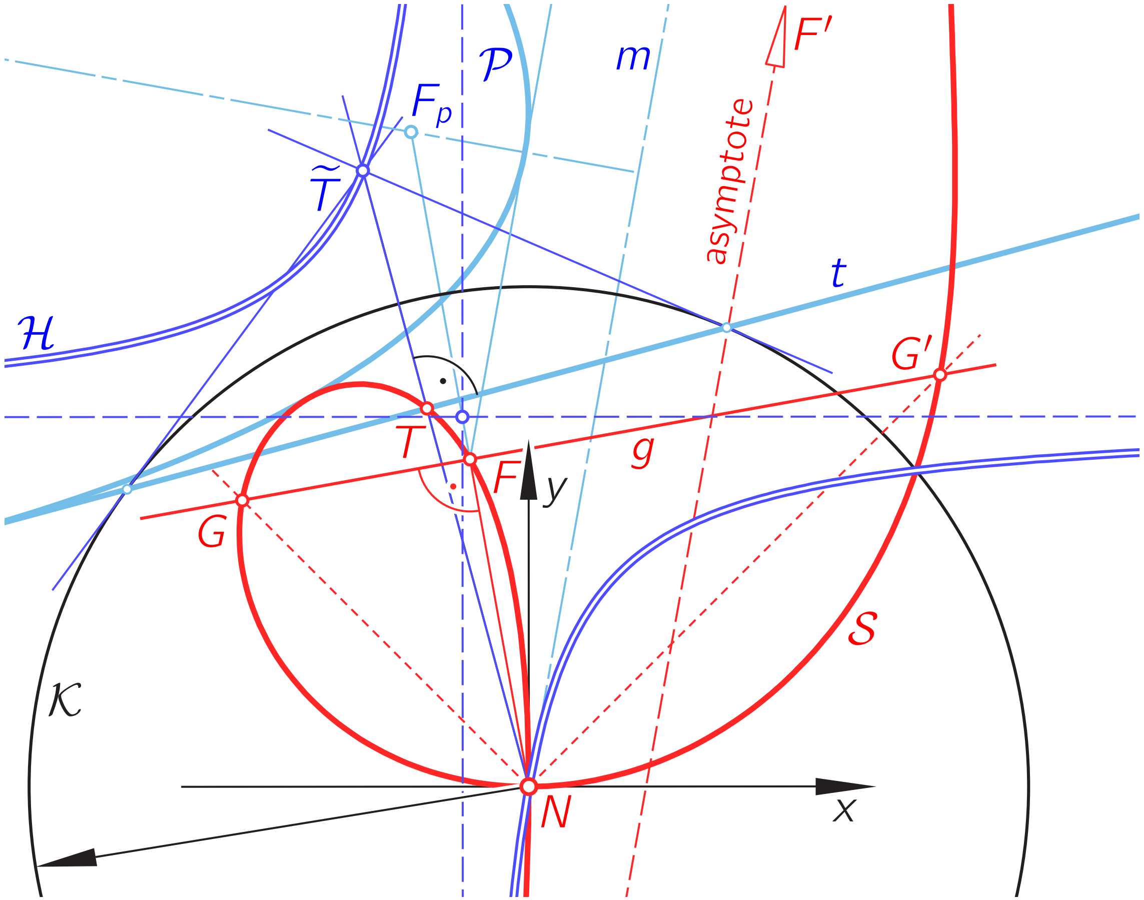
1. Definition of Strophoids



$\mathcal{H}$ has two axes of symmetry $\Longrightarrow$ the inverse curve $\mathcal{S}$ is self-invers w.r.t. two circles through N with centers G, G'.

$\mathcal{S}$ is the envelope of circles centered on confocal parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

1. Definition of Strophoids



The product of the polarity and the inversion in $\mathcal{K}$ is the pedal transformation $t \mapsto T$ w.r.t. N.

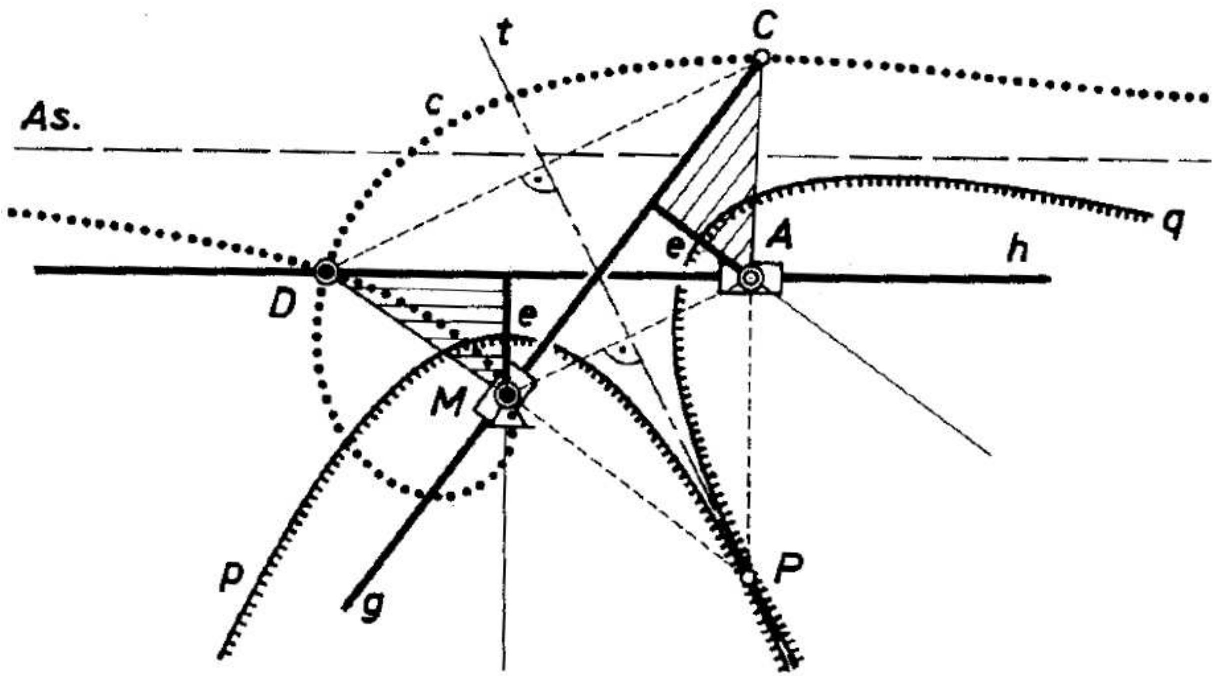
Polar to $\mathcal{H}$ is the parabola $\mathcal{P}$.

Theorem: *The strophoid $\mathcal{S}$ is the pedal curve of the parabola $\mathcal{P}$ with respect to N.*

The parabola's directrix m is parallel to the asymptote of $\mathcal{S}$.

F (on the tangent at the vertex of $\mathcal{P}$) is the midpoint between N and the parabola's focus F_p.

1. Definition of Strophoids

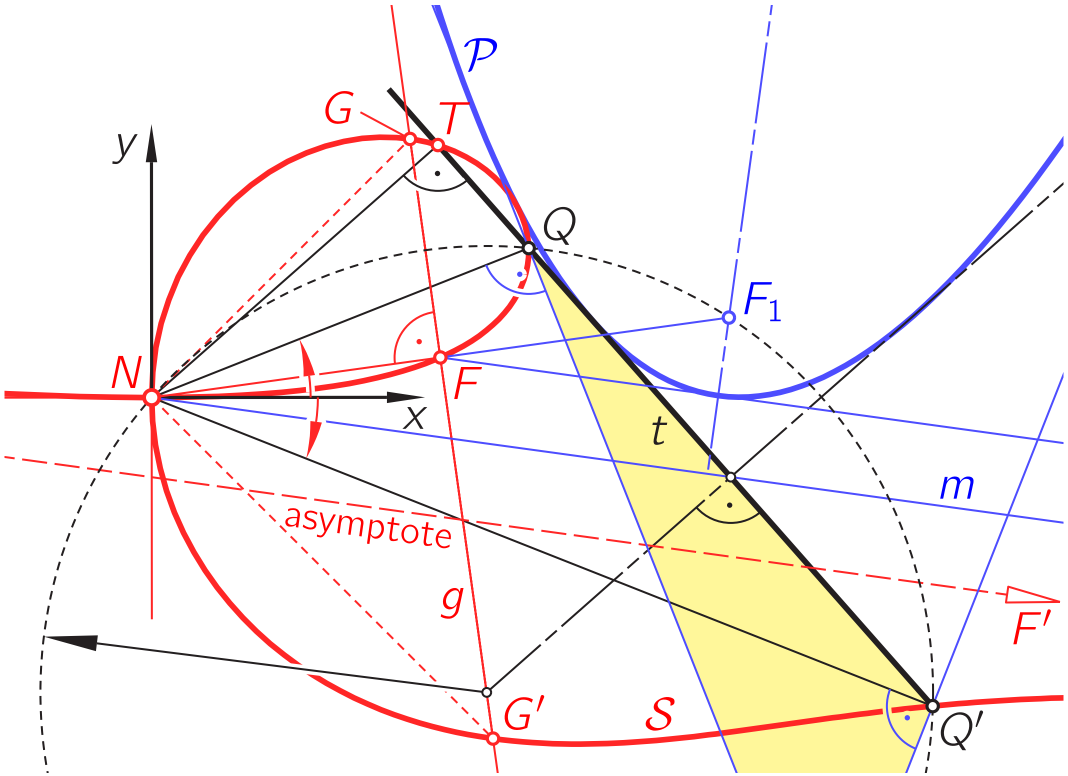


W. Wunderlich:
Ebene Kinematik, p. 115

as a particular pedal curve of a parabola,

the **strophoid** is a particular trajectory during a blau symmetric rolling of parabolas

1. Definition of Strophoids



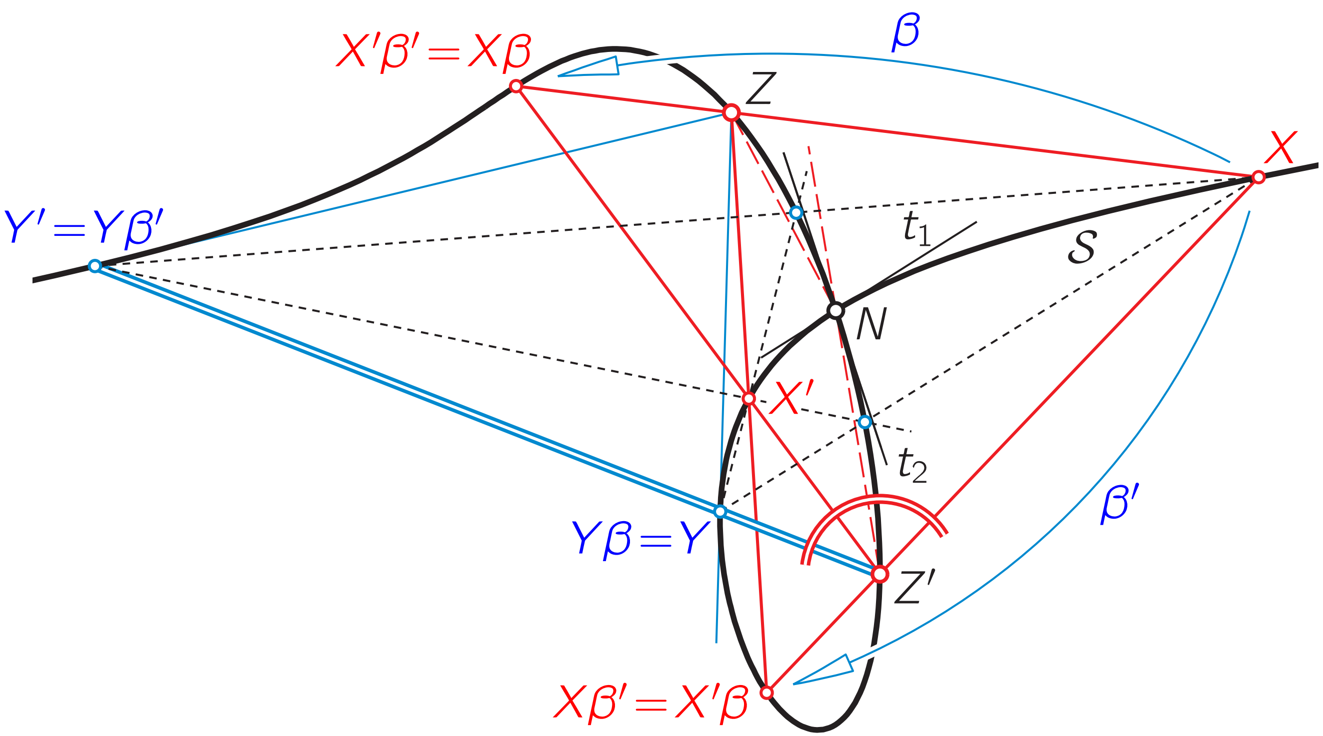
Tangents t of the parabola $\mathcal{P}$ intersect $\mathcal{S}$ beside the pedal point T in two real or conjugate complex points Q and Q'.

Definition: Q and Q' are called **associated points** of $\mathcal{S}$.

Q and Q' are associated iff the lines QN and $Q'N$ are harmonic w.r.t. the tangents at $\overline{A}$.

For given t, the points Q and Q' lie on a circle centered on g.

2. Associated Points



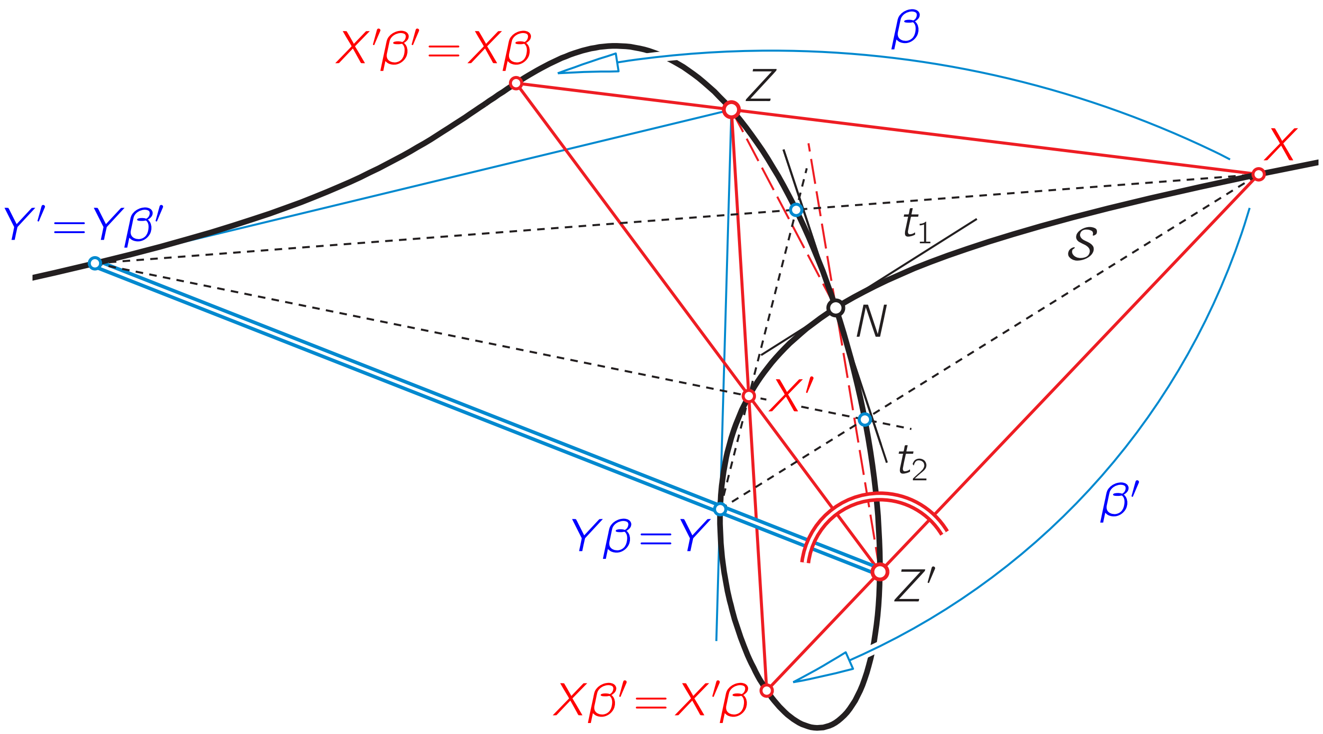
Projective properties of cubics with a node:

There is a 1-1 correspondance between $\mathcal{S}$ and lines through N, except N corresponds to t_1 and t_2.

The involution α which fixes t_1, t_2 determines pairs X, X' of associated points.

Involutions which exchange t_1 and t_2 determine involutions β on $\mathcal{S}$ with $N \mapsto N$ and several properties, e.g., there exists an 'associated' involution $\beta' = \alpha \circ \beta = \beta \circ \alpha$.

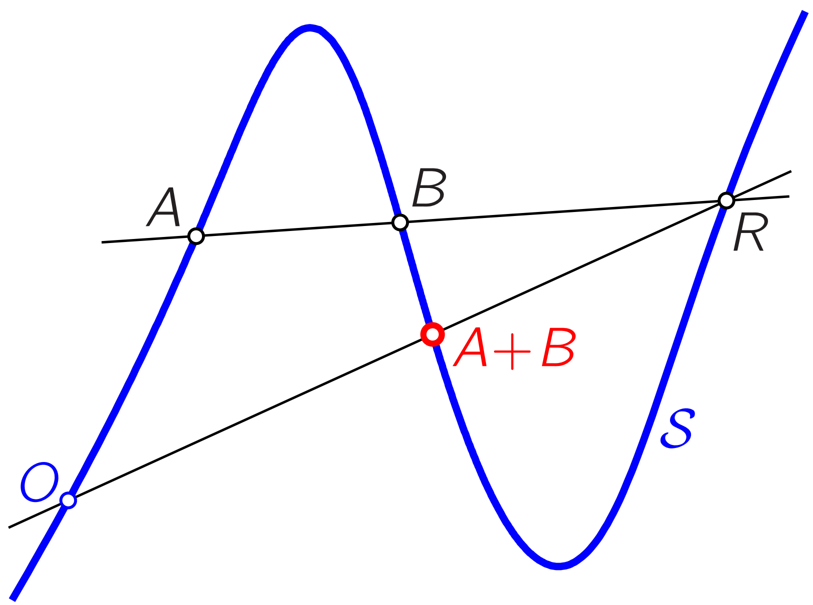
2. Associated Points



- β has a center Z such that $X, X\beta, Z$ are collinear.
- The centers Z of β and Z' of β' are associated.
- The lines $Z'X$, $Z'X\beta$ correspond in an involution which fixes $Z'N$ and the line through the fixed points Y, Y' of β.

- For associated points, the diagonal points $XY \cap X'Y'$ and $XY' \cap X'Y$ are again on $\mathcal{S}$. The tangents at corresponding points intersect on $\mathcal{S}$.

2. Associated Points



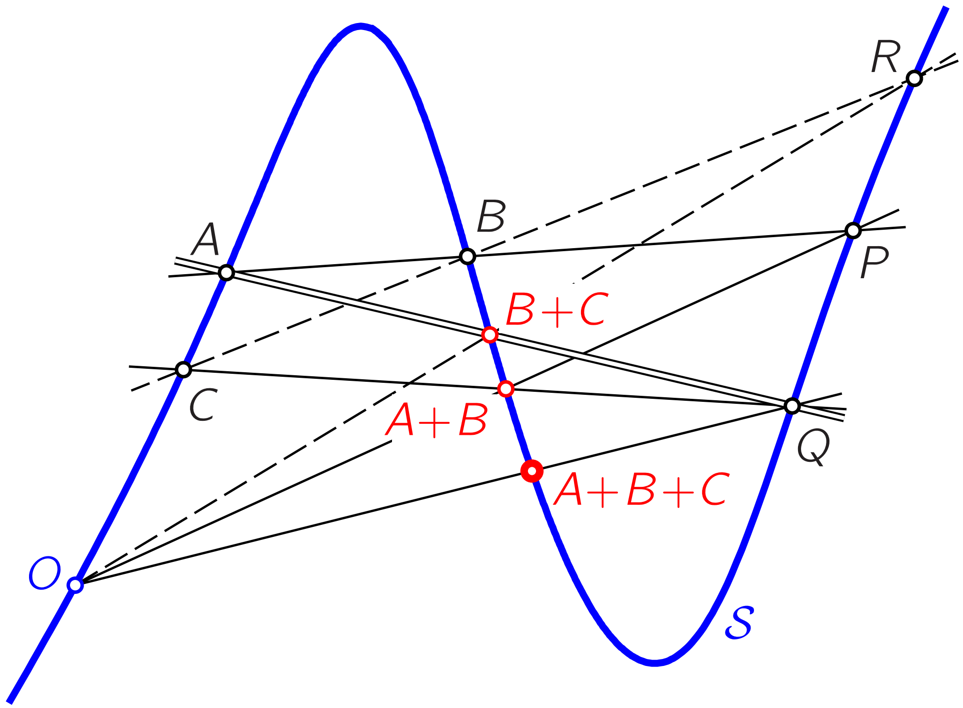
Addition of points

Theorem:

On each irreducible cubic $\mathcal{S}$ a commutative group can be defined with an arbitrary chosen point O as neutral element.

Conversely, point B is uniquely defined by A and $A + B$.

2. Associated Points



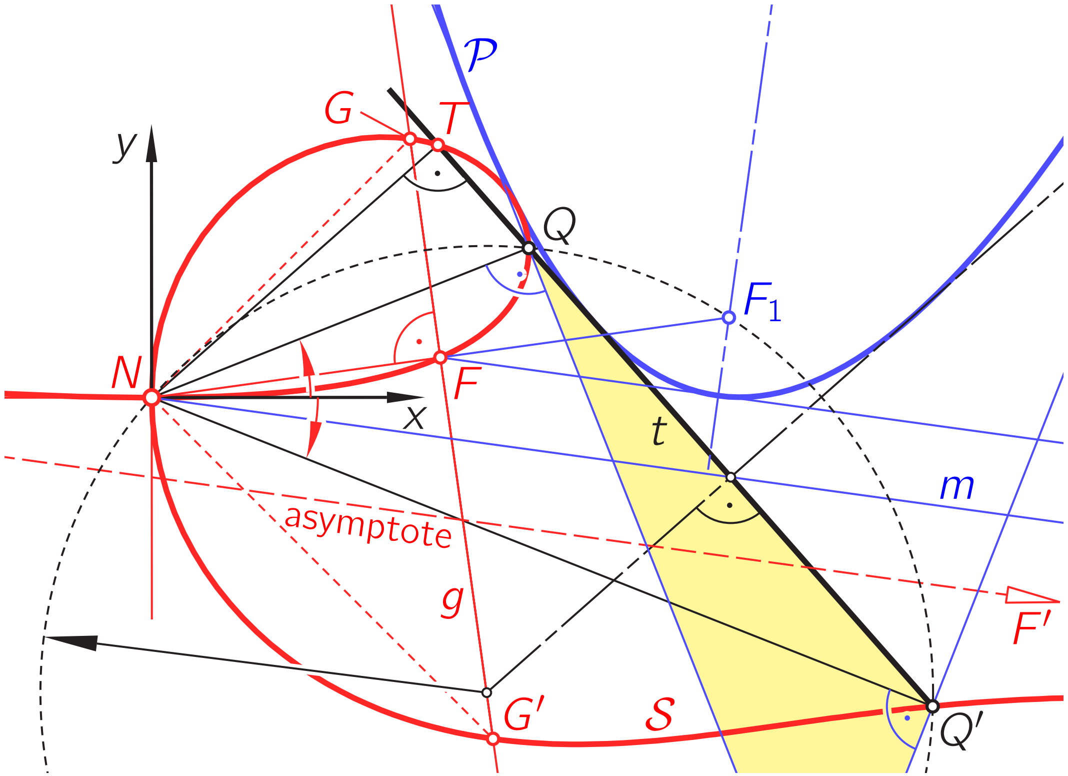
associative law

Theorem:

On each irreducible cubic $\mathcal{S}$ a commutative group can be defined with an arbitrary chosen point O as neutral element.

At cubics with a node the group is isomorphic to $(\mathbb{R} \setminus \{0\}, \cdot\,)$. Pairs of associated points differ only by their sign. N corresponds to zero.

2. Associated Points



On the equicevian cubic $\mathcal{S}$, the following pairs of points are associated:

- Q, Q',
- the absolute circle-points,
- The focal point F and the point F' at infinity,
- G, G' on the line $g \perp NF$.

2. Associated Points

Theorem:

• For each pair (Q, Q') of associated points, the lines NQ, NQ' are symmetric w.r.t. the bisectors t_1, t_2 of $\sphericalangle BNC$.

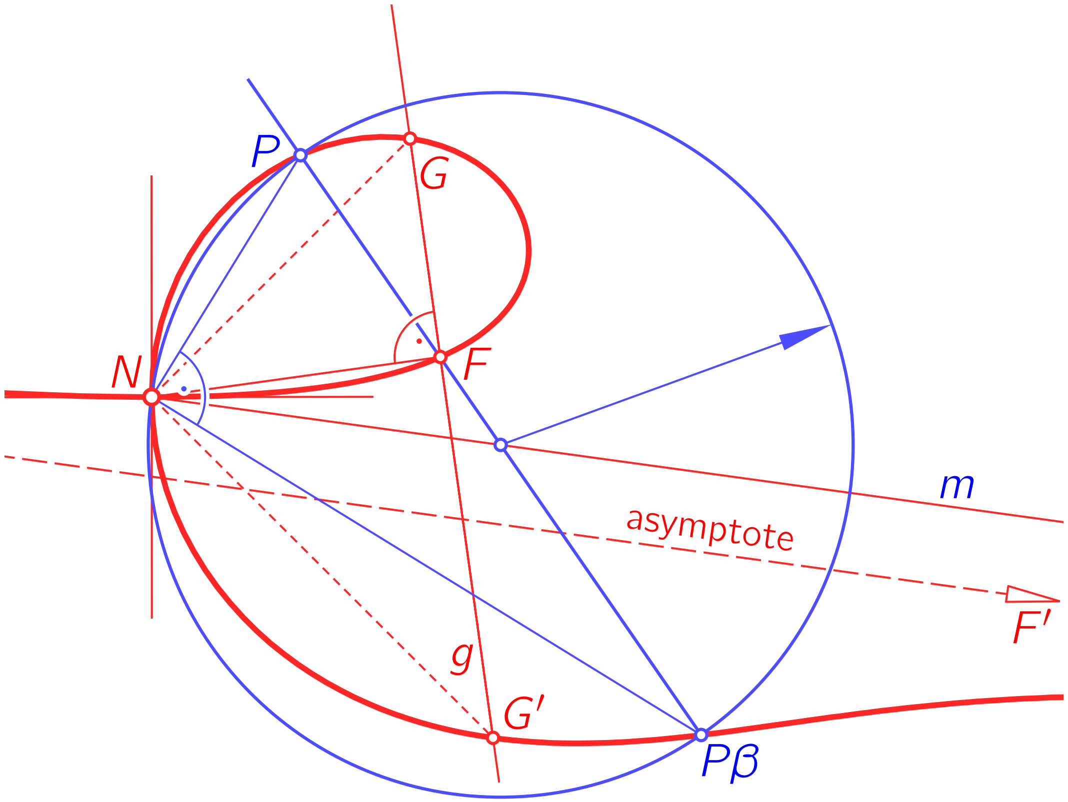
• The midpoint of associated points Q, Q' lies on the median $m = NF'$.

• The tangents of $\mathcal{S}$ at associated points meet each other at the point $T' \in \mathcal{S}$ associated to the pedal point T on $t = QQ'$.

• For each point $P \in \mathcal{S}$, the lines PQ and PQ' are symmetric w.r.t. PN.

Other consequences: For each pair (Q, Q') of associated points (as Laguerre-points) the represented two complex conjugate points are again associated points of $\mathcal{S}$.

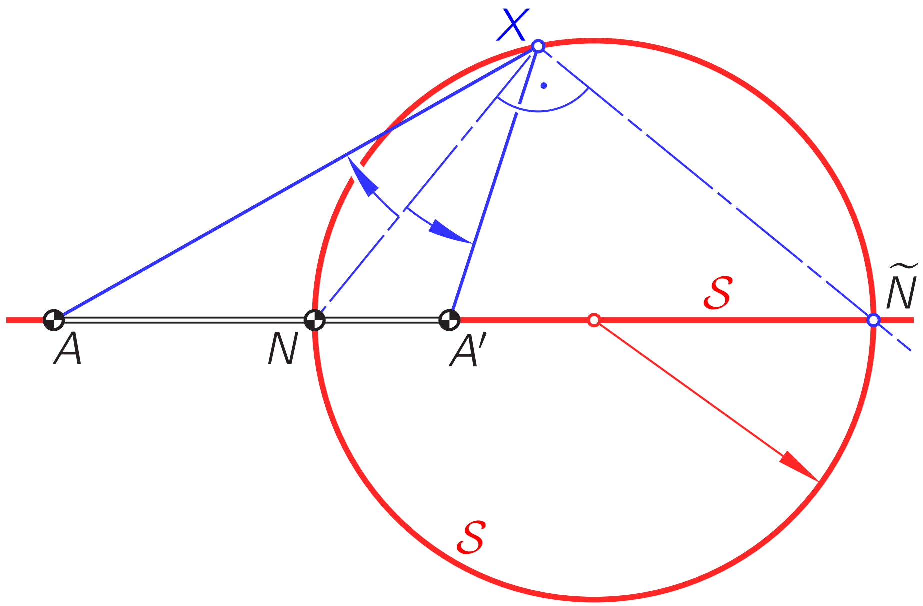
2. Associated Points



The right-angle involution at N induces an involution β on $\mathcal{S}$. Corresponding points P, $P\beta$ are collinear mit F; their midpoint is on m.

Given m, $N \in m$ and $F \notin m$, the strophoid $\mathcal{S}$ is the locus of intersection points between circles through N and centered on m with diameter lines through F.

2. Associated Points

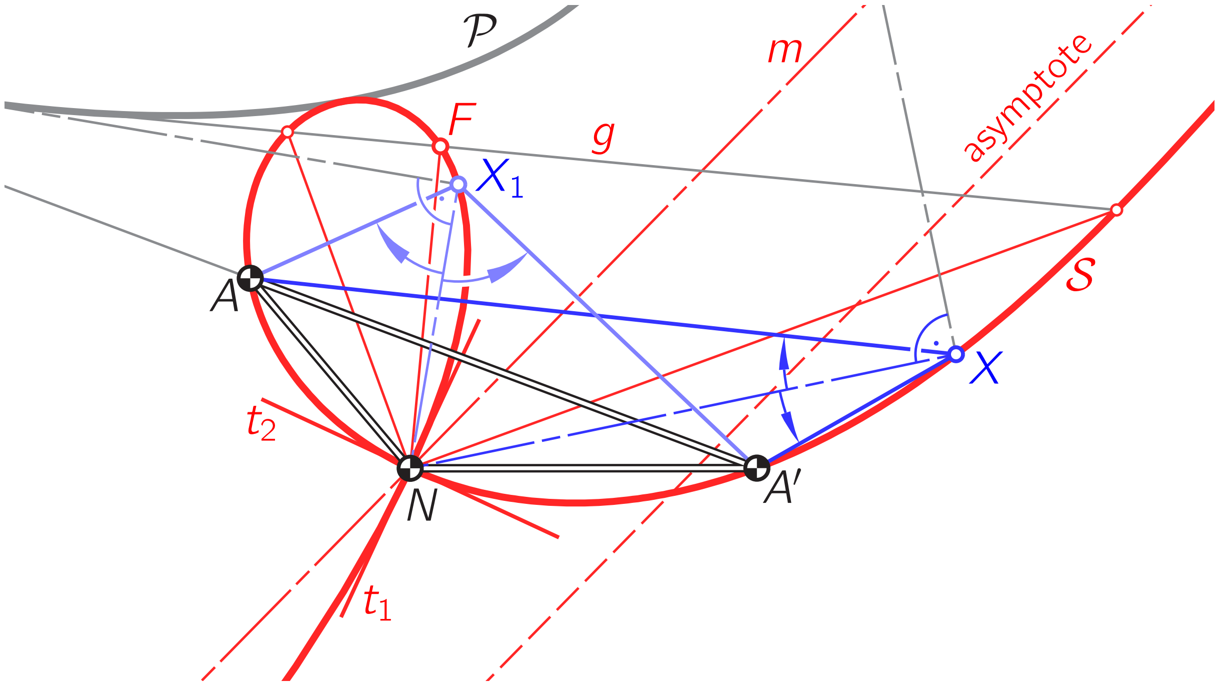


We recall:

Theorem: Given three aligned points A, A' and N, the locus of points X such that the line XN bisects the angle between XA and XA', is the Apollonian circle.

The second angle bisector passes through the point $\widetilde{N}$ harmonic to N w.r.t. A, A'.

2. Associated Points



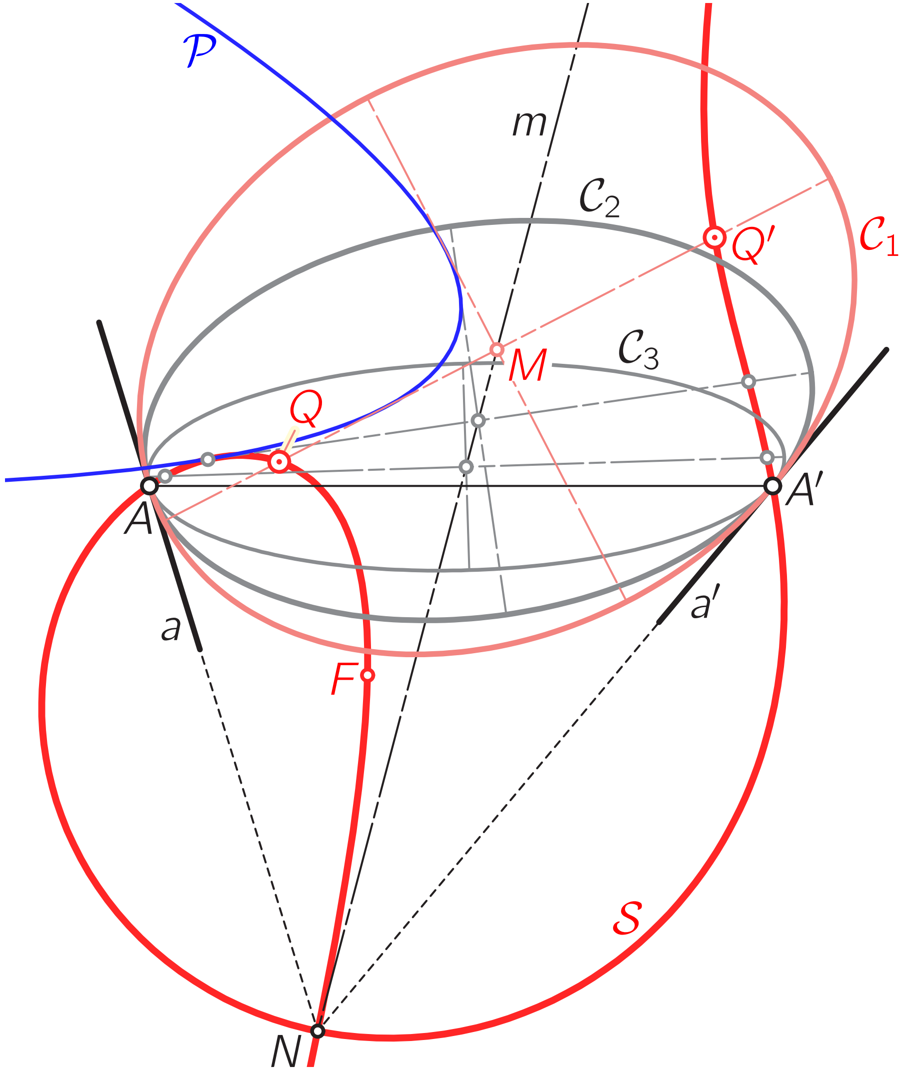
Theorem: Given the non-collinear points A, A' and N, the locus of points X such that the line XN bisects the angle between XA and XA', is a strophoid with node N and associated points A, A'.

This holds also when A is at infinity.

The respectively second angle bisectors are tangent to the parabola $\mathcal{P}$.

When N is at infinity, then the locus is an equilateral hyperbola.

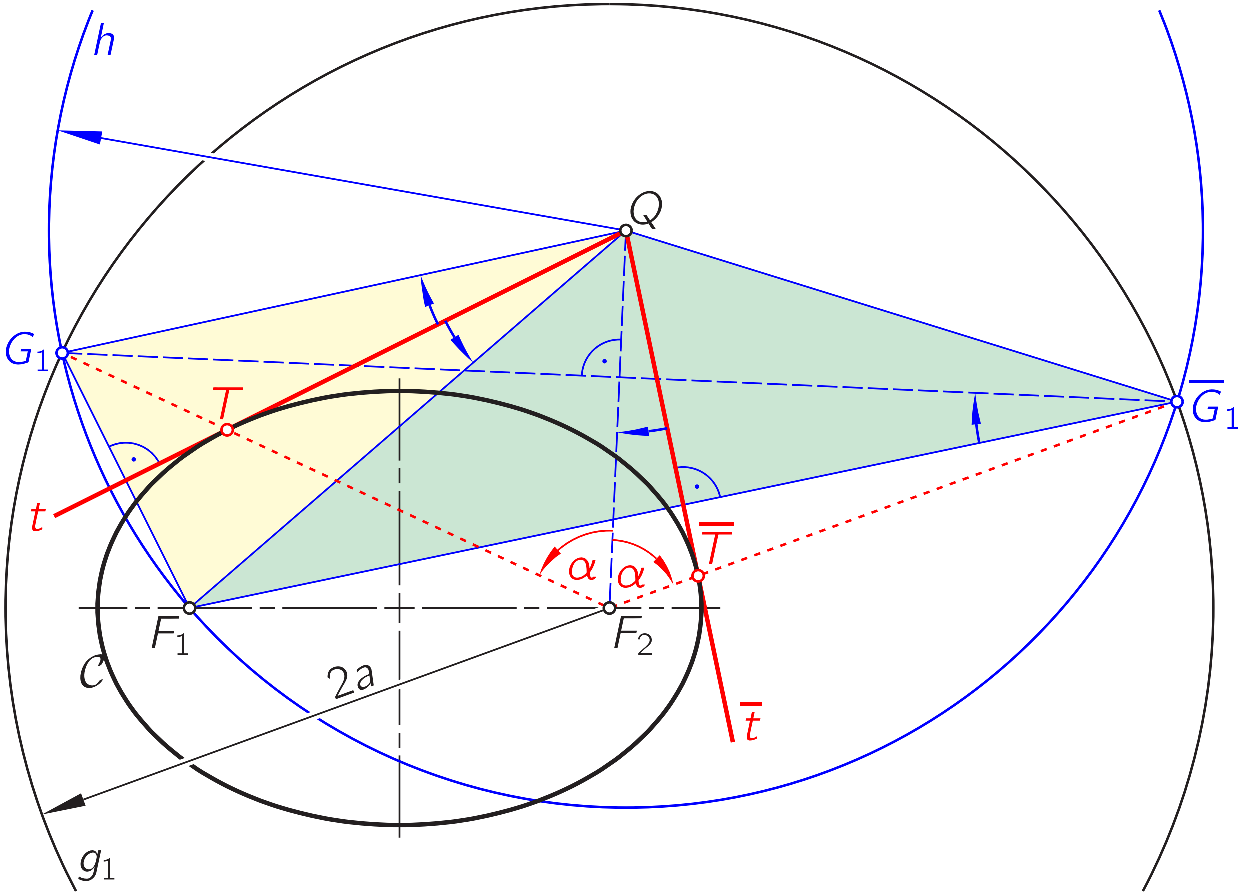
3. Strophoids as a Geometric Locus



Theorem: The strophoid $\mathcal{S}$ is the locus of focal points (Q, Q') of conics $\mathcal{N}$ which contact line AN at A and line $A'N$ at A'.

The axes of these conics are tangent to the negative pedal curve, the parabola $\mathcal{P}$. Therefore, the real focal points are associated — as well as the complex conjugate point.

3. Strophoids as a Geometric Locus

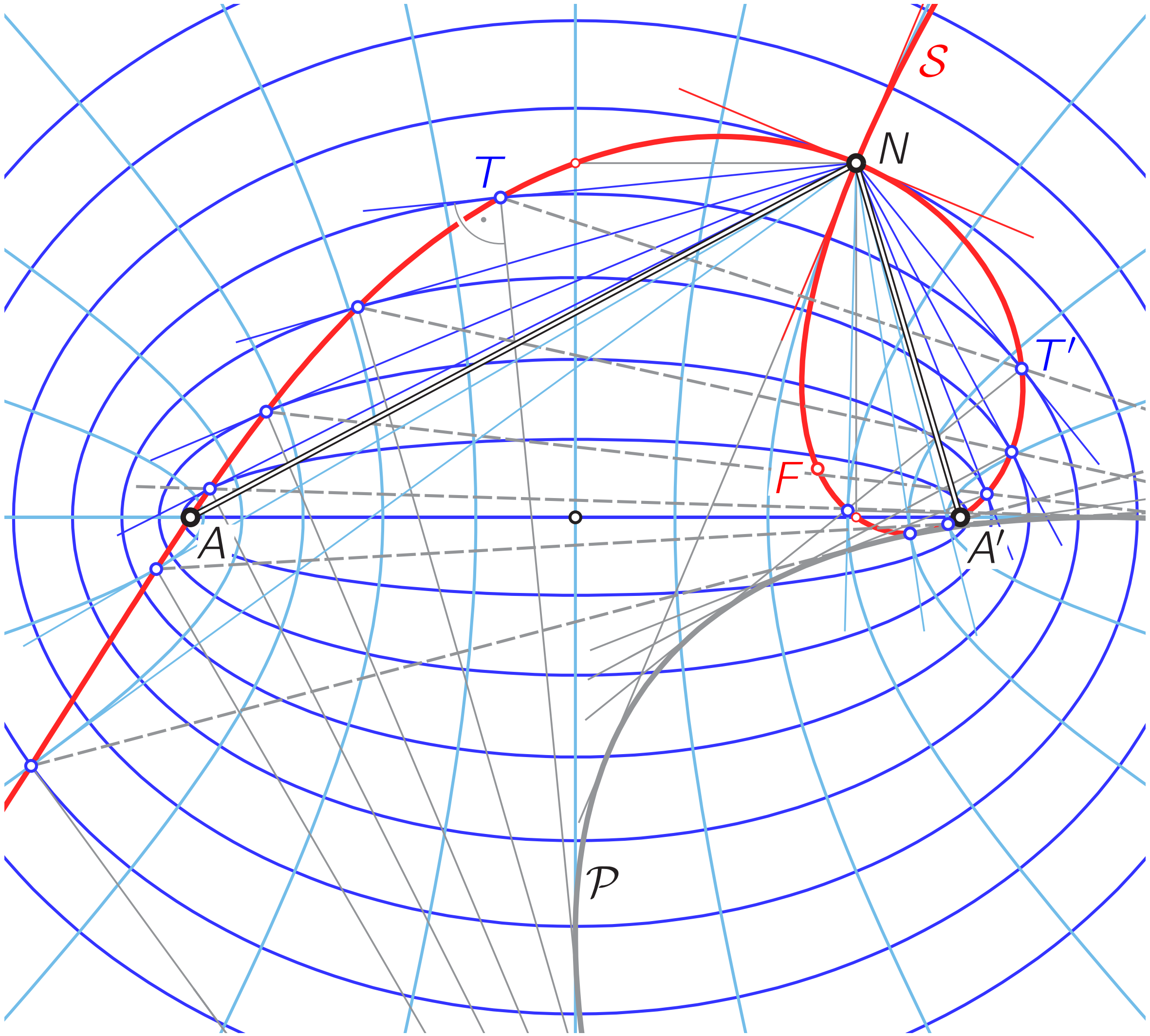


Let the tangents to $\mathcal{C}$ at T and T' intersect at Q. Then

$$\alpha = \sphericalangle TF_2Q = \sphericalangle QF_2\overline{T}.$$

On the other hand, the tangent t at T bisects the angle between TF_1 and TF_2.

3. Strophoids as a Geometric Locus

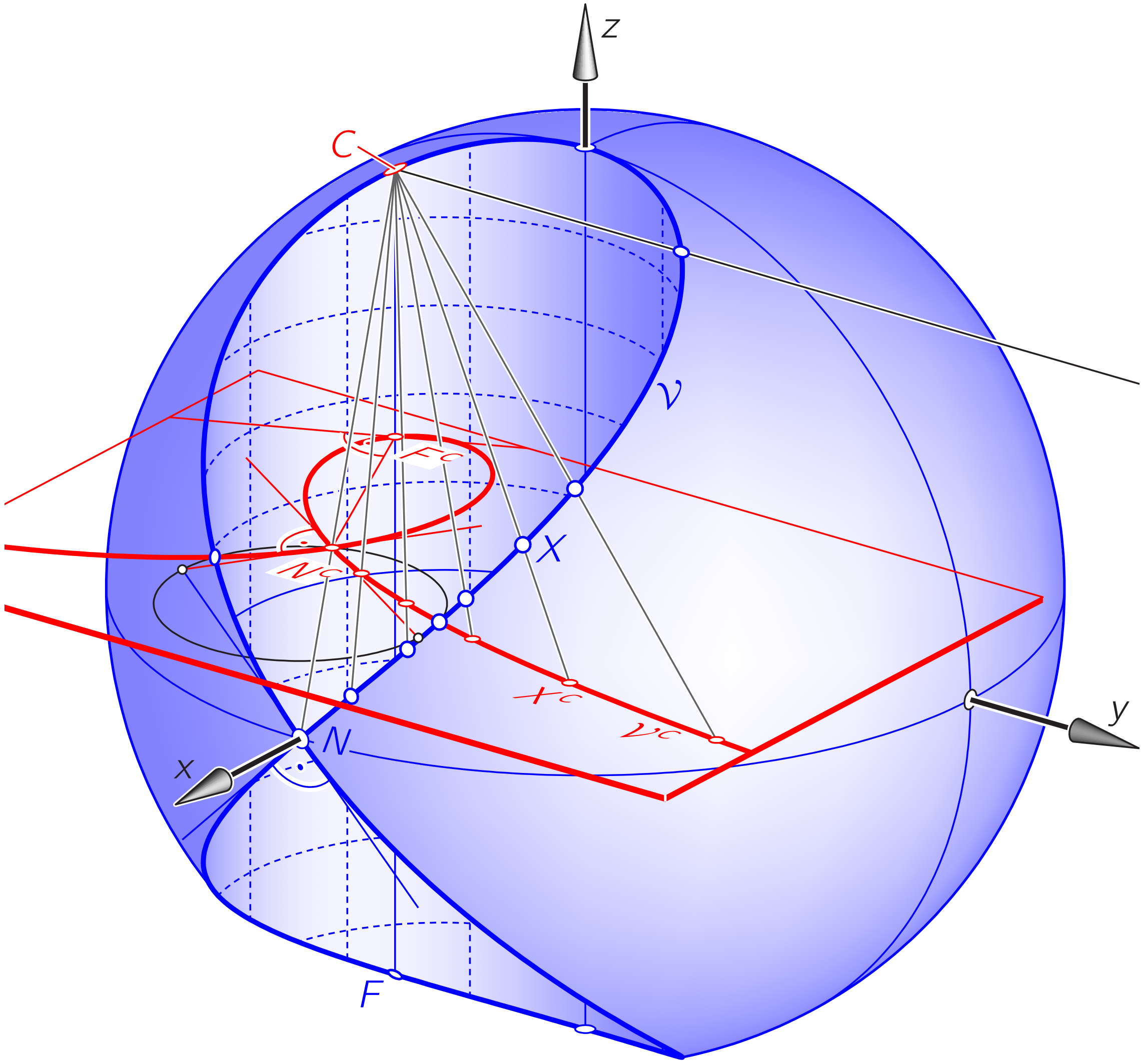


The points of contact of tangents drawn from a fixed point N to confocal conics as well as the

pedal points of normals drawn through N lie on a strophoid.

The strophoid intersects any conic in 2 points of contact and 4 pedal points.

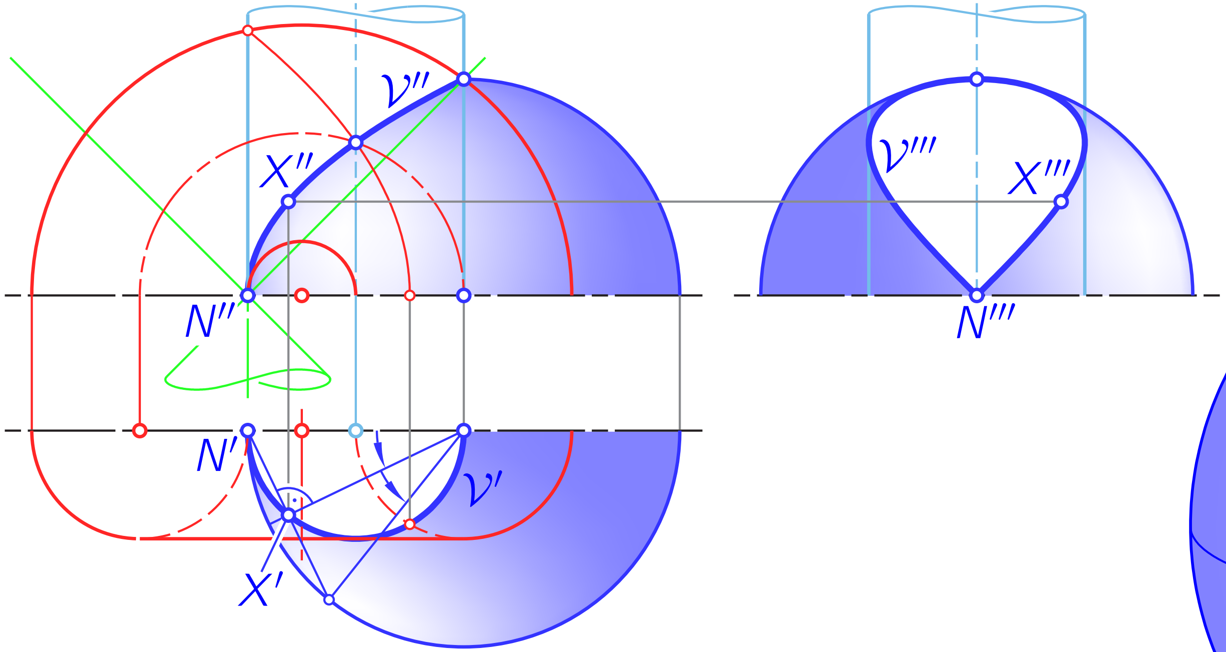
3. Strophoids as a Geometric Locus

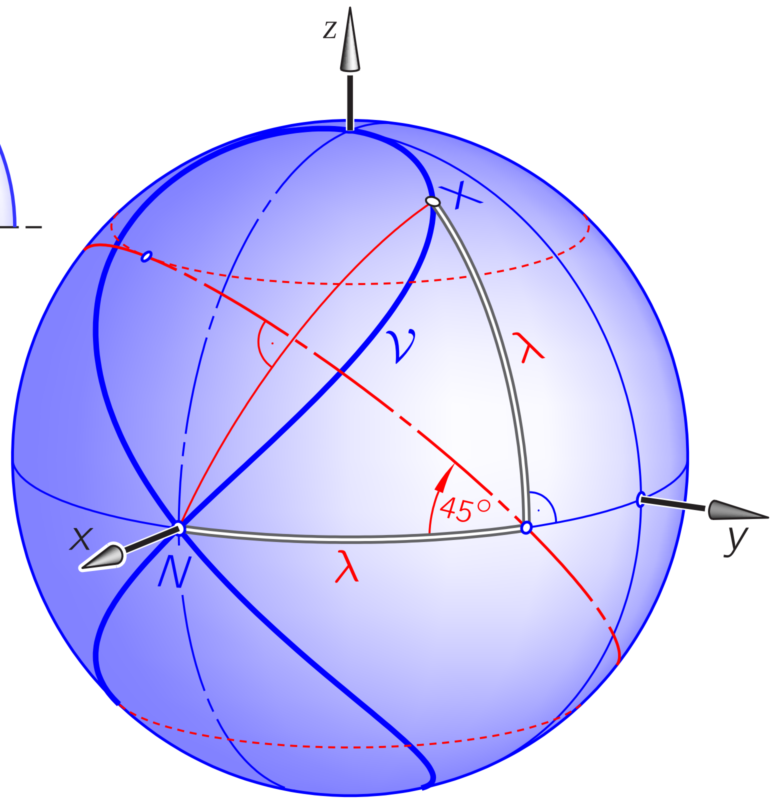


The curve $\mathcal{V}$ of intersection between the sphere (radius $2r$) and the vertical right cylinder (radius r) is called Viviani's window.

Central projections with center $C \in \mathcal{V}$ and a horizontal image planes map $\mathcal{V}$ onto a strophoid.

3. Strophoids as a Geometric Locus

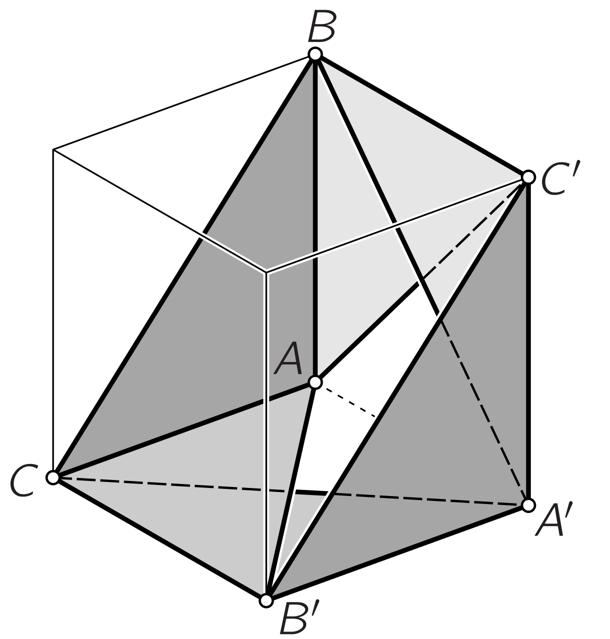


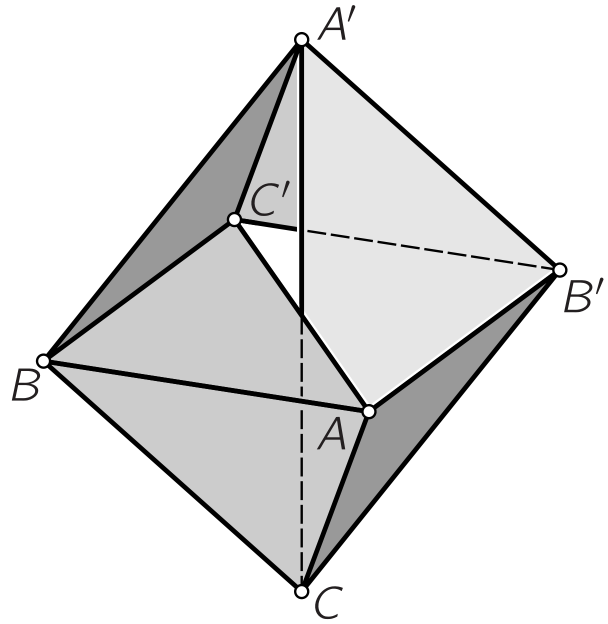


$\mathcal{V}$ lies also on a cone of revolution and on a torus.

Points of $\mathcal{V}$ have equal geographic longitude and latitude.

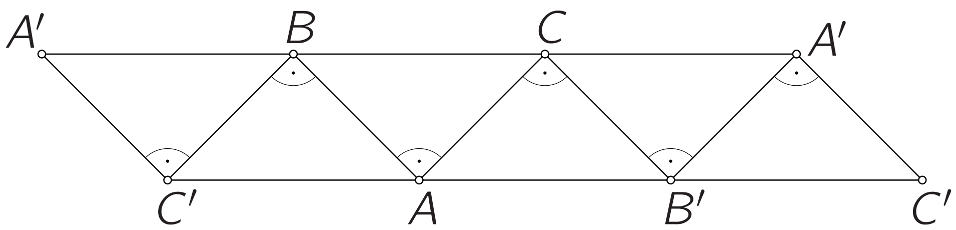
3. Strophoids as a Geometric Locus

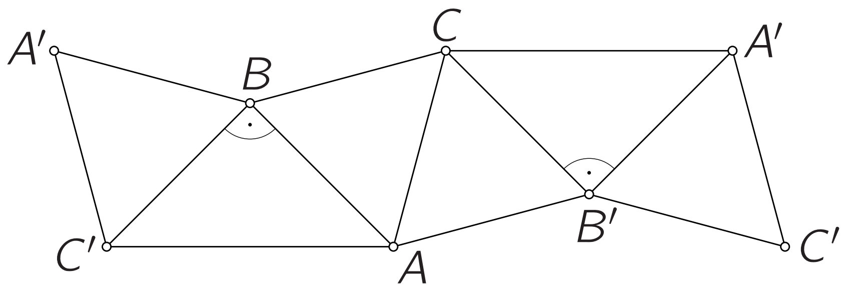




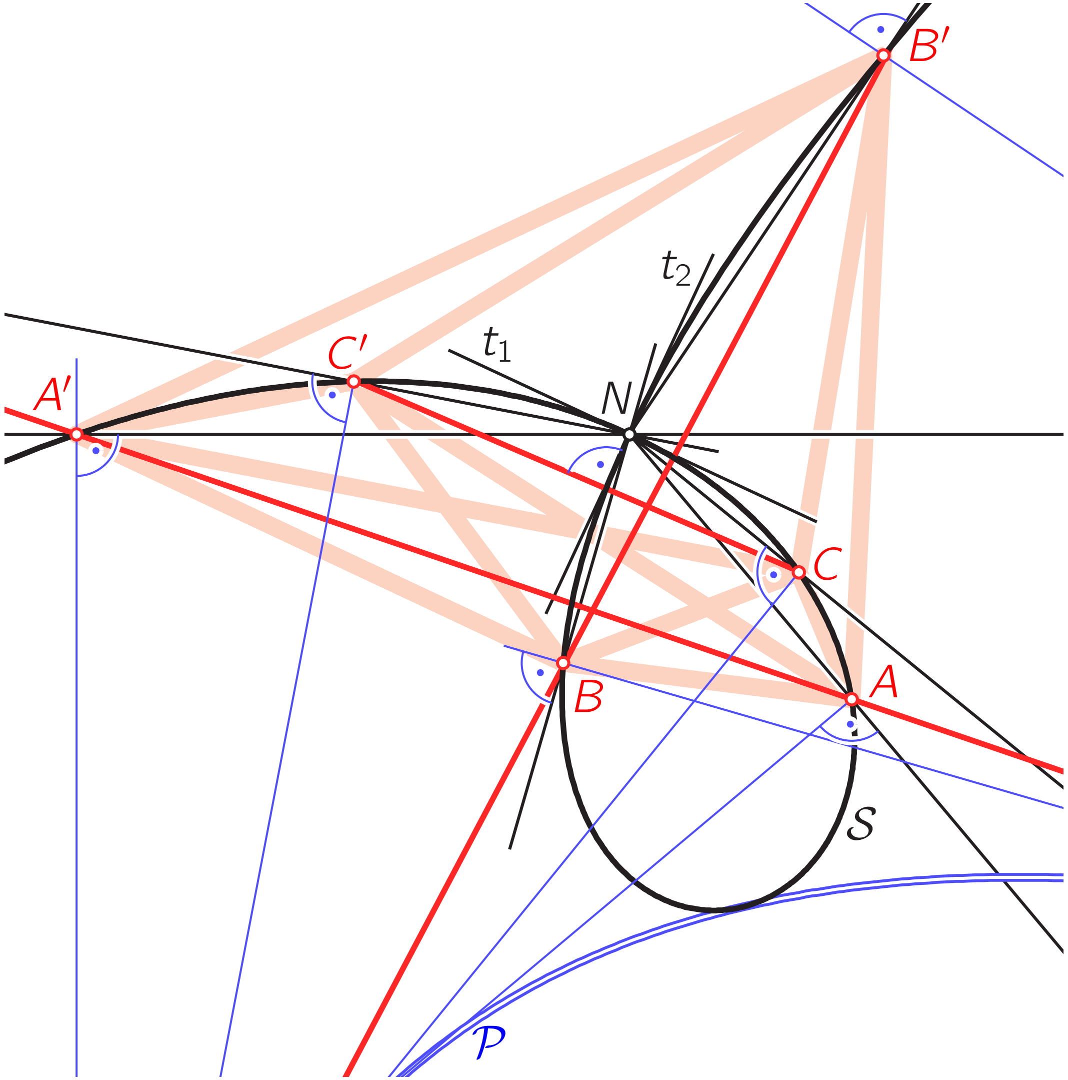
Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry (types 1 and 2)

Below: Nets of the two octahedra.



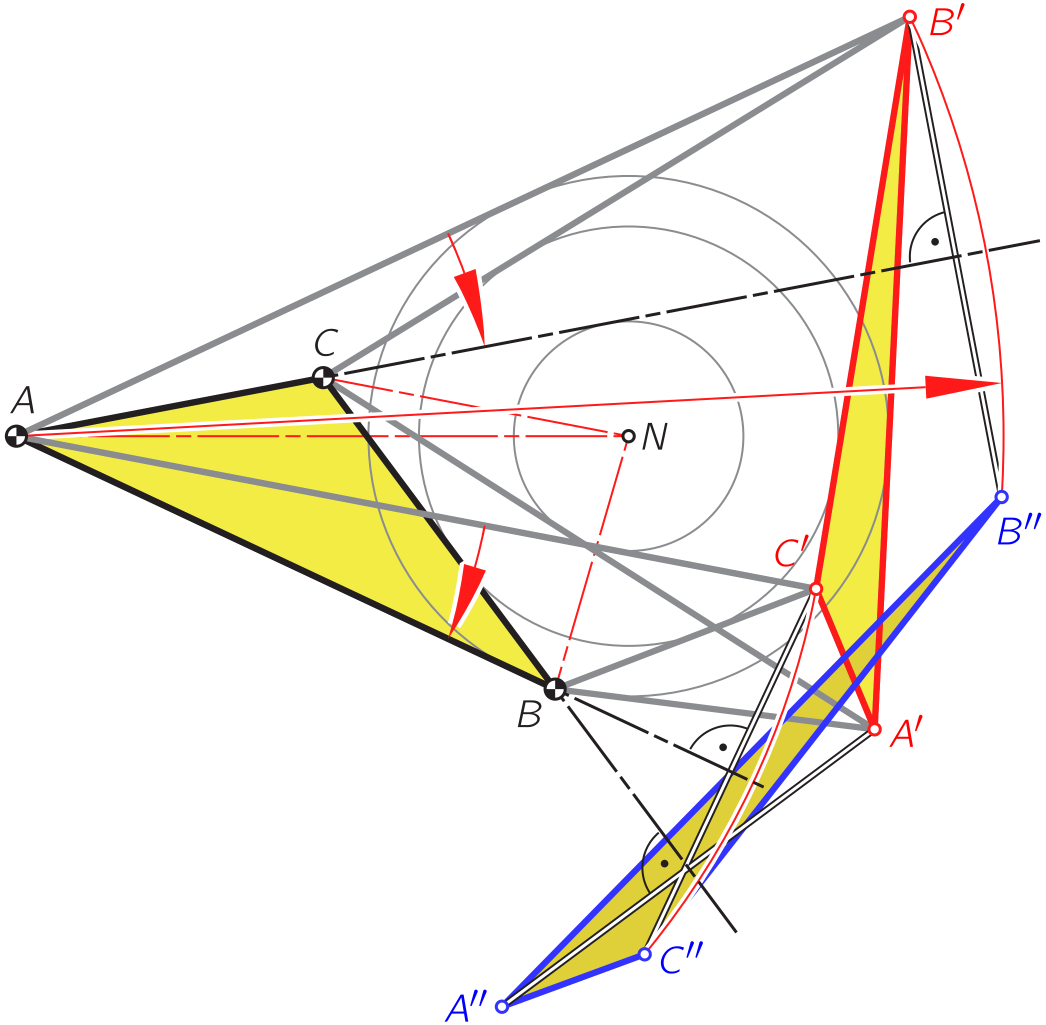


3. Strophoids as a Geometric Locus



According to R. Bricard there are 3 types of flexible octahedra (four-sided double-pyramids). Those of type 3 admit two flat poses. In each such pose, the pairs (A, A'), (B, B'), and (C, C') of opposite vertices are associated points of a strophoid $\mathcal{S}$.

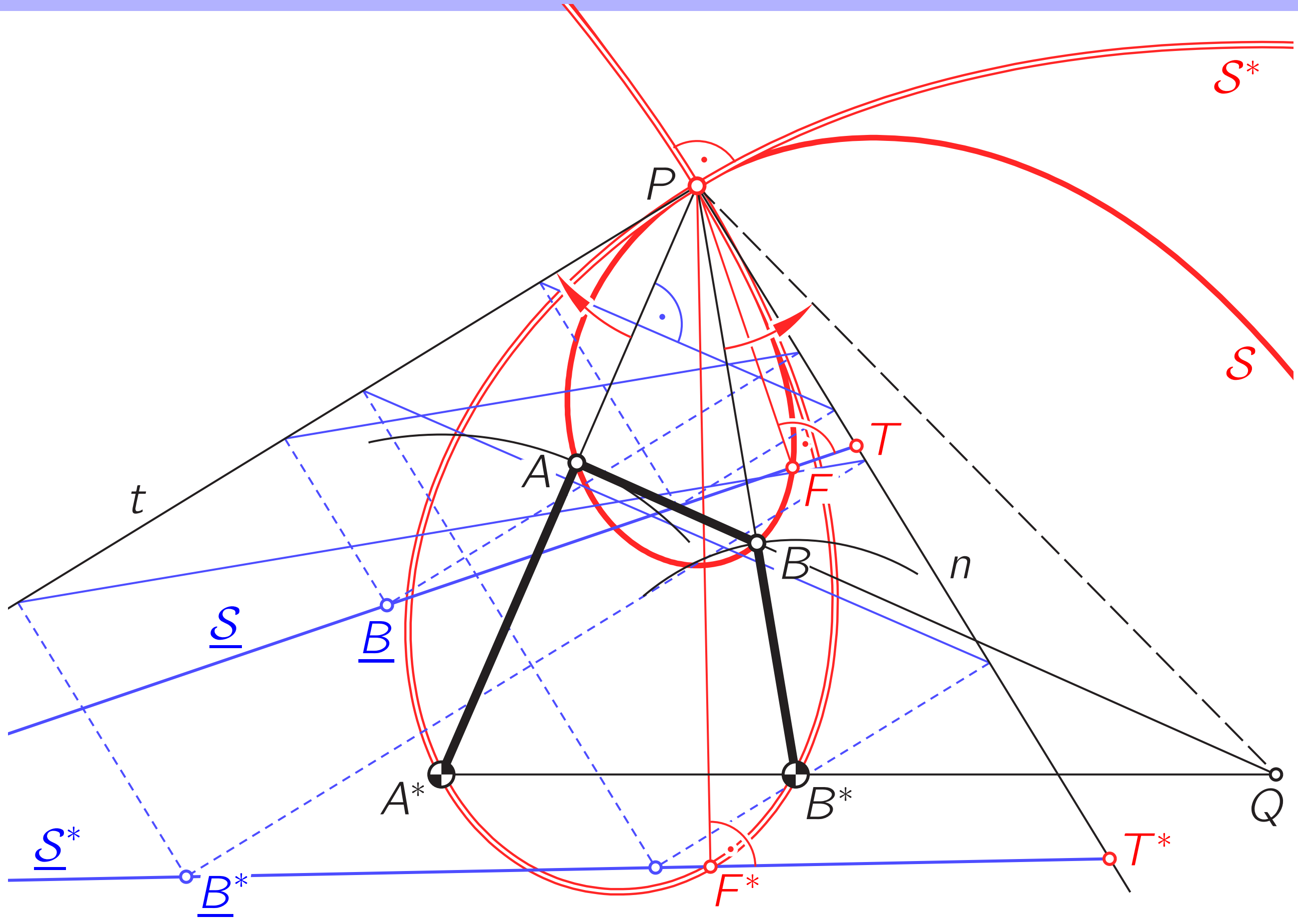
3. Strophoids as a Geometric Locus



According to Bricard's construction, all bisectors must pass through the midpoint N of the concentric circles.

The two flat poses of a type-3 flexible octahedron, when ABC remains fixed.

3. Strophoids as a Geometric Locus



In plane kinematics, points with trajectories of stationary curvature is a **strophoid** $\mathcal{S}$ as well as the locus $\mathcal{C}$ of corresponding centers of curvature. They are images of lines in a cubic transformation.

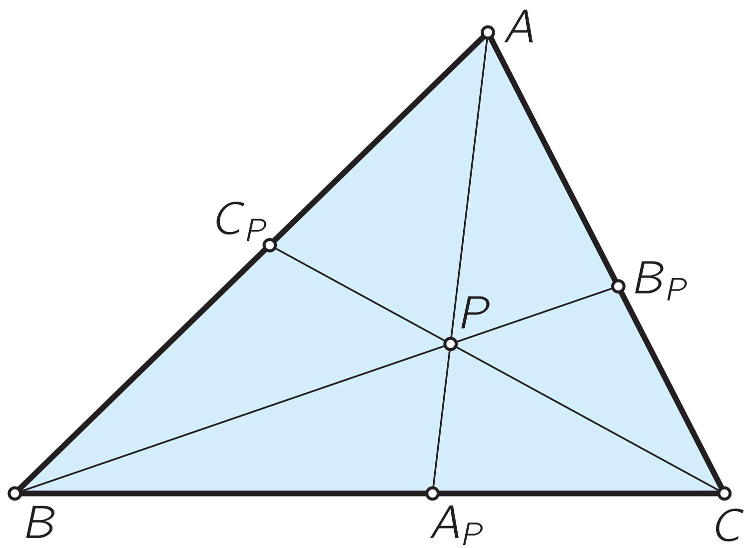
3. Strophoids as a Geometric Locus

The *elementary geometry of triangles* seems to be an endless story.

• Clark Kimberling's *Encyclopedia of Triangle Centers* shows a list of **8.116** remarkable points (available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`)

• Bernard Gibert's *Cubics in the Triangle Plane* shows a list of **724** related cubics (available at `http://bernard.gibert.pagesperso-orange.fr/index.html`)

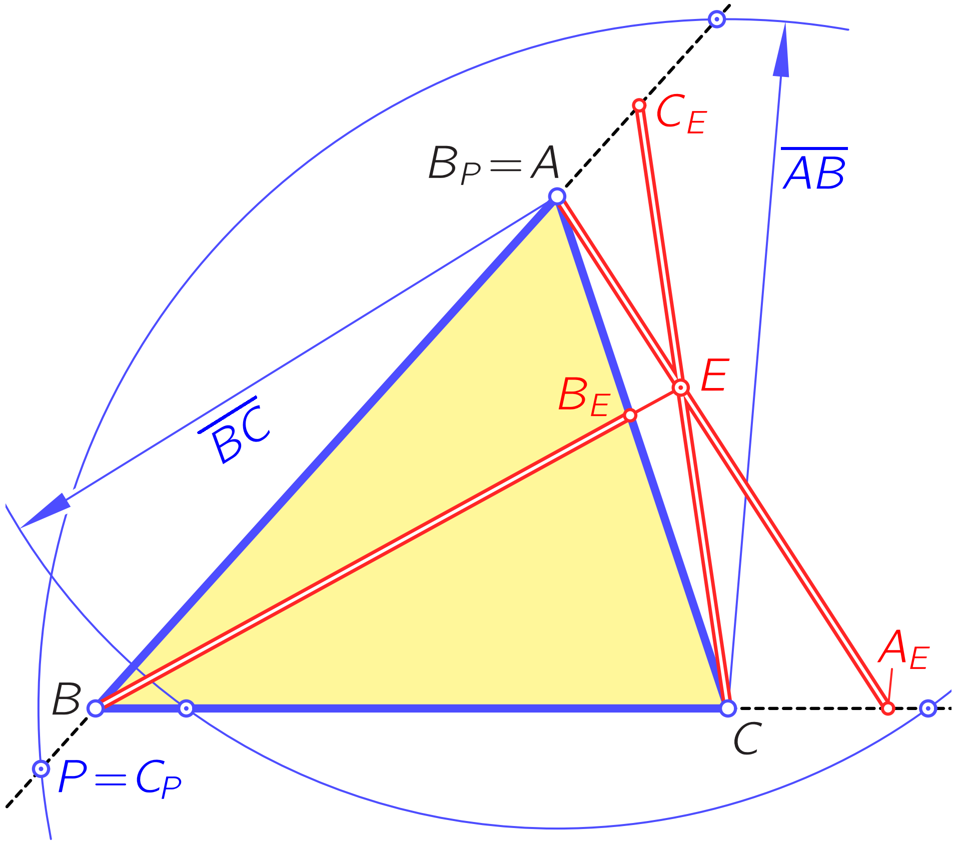
3. Strophoids as a Geometric Locus



For any point $P \neq A, B, C$ the segments AA_P, BB_P, and CC_P, are called **cevians** of the point P.

Giovanni Ceva, 1647-1734, Milan/Italy.

3. Strophoids as a Geometric Locus

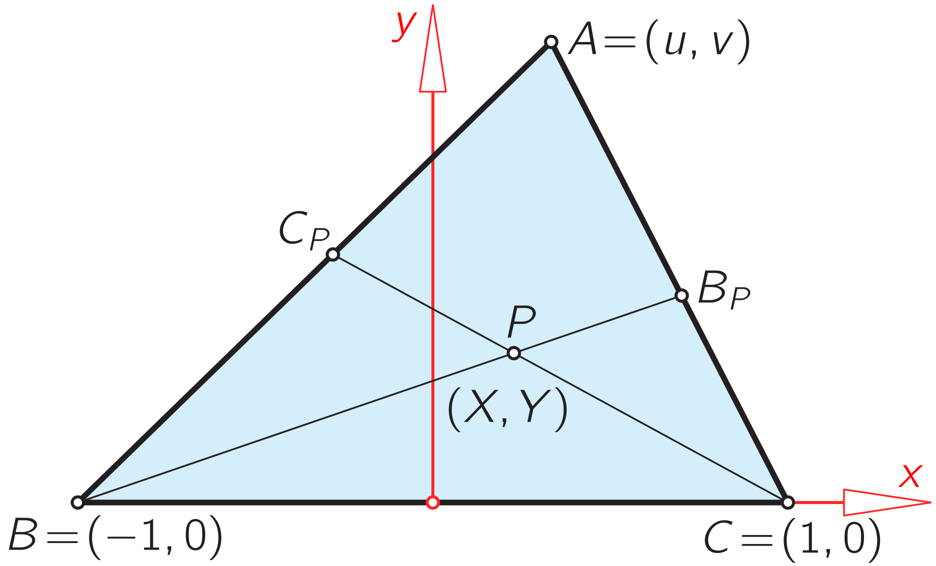


The point E is called **equicevian**, if its three cevians have the same lengths, i.e., $\overline{AA_E} = \overline{BB_E} = \overline{CC_E}$.

An equicevian point is called **improper** if it lies on one side line of the triangle (like P), otherwise **proper** (like E).

There exist ≤ 6 improper equicevian points. We focus in the sequel on proper ones.

3. Strophoids as a Geometric Locus



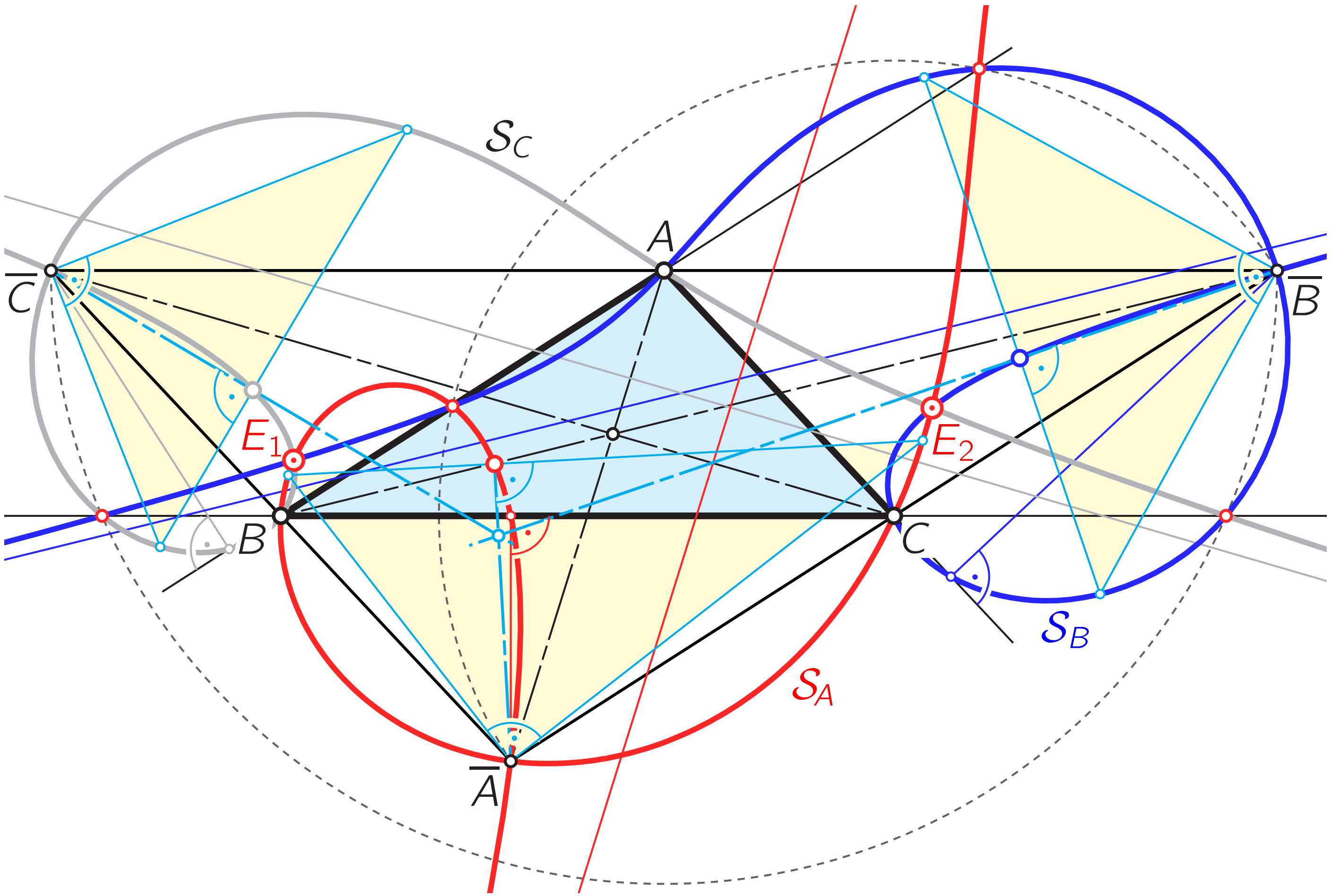
A point P is called *A-equicevian* iff $\overline{BB_P} = \overline{CC_P}$.

Theorem: *All A-equicevian points lie on the line BC or on the* **A-equicevian cubic** $\mathcal{S}_A$: $H_A(X, Y) = 0$, *where*

$$H_A(X,Y) = (vX - uY)(X^2 + Y^2) + uv(X^2 - Y^2) - (u^2 - v^2 + 1)XY - (vX + uY) - uv.$$

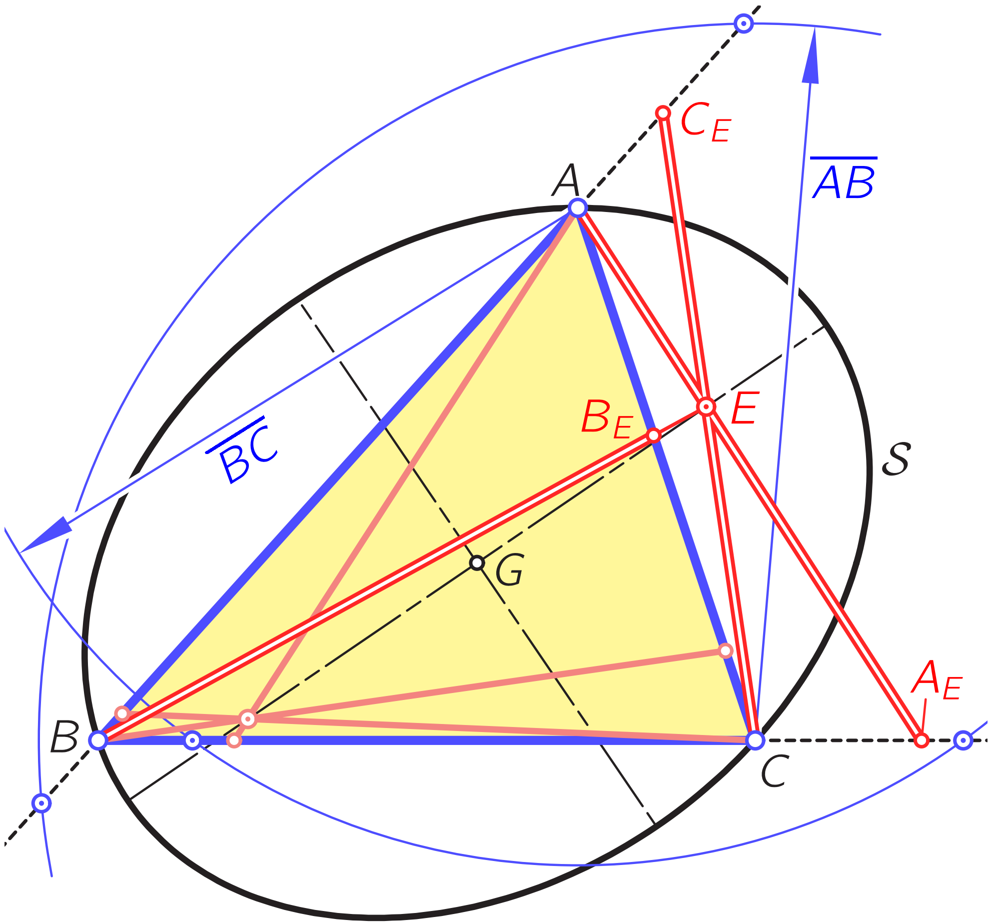
Analogue B-equicevian points ($\overline{AB_P} = \overline{CC_P}$) and C-equicevian points ($\overline{AB_P} = \overline{BC_P}$).

3. Strophoids as a Geometric Locus



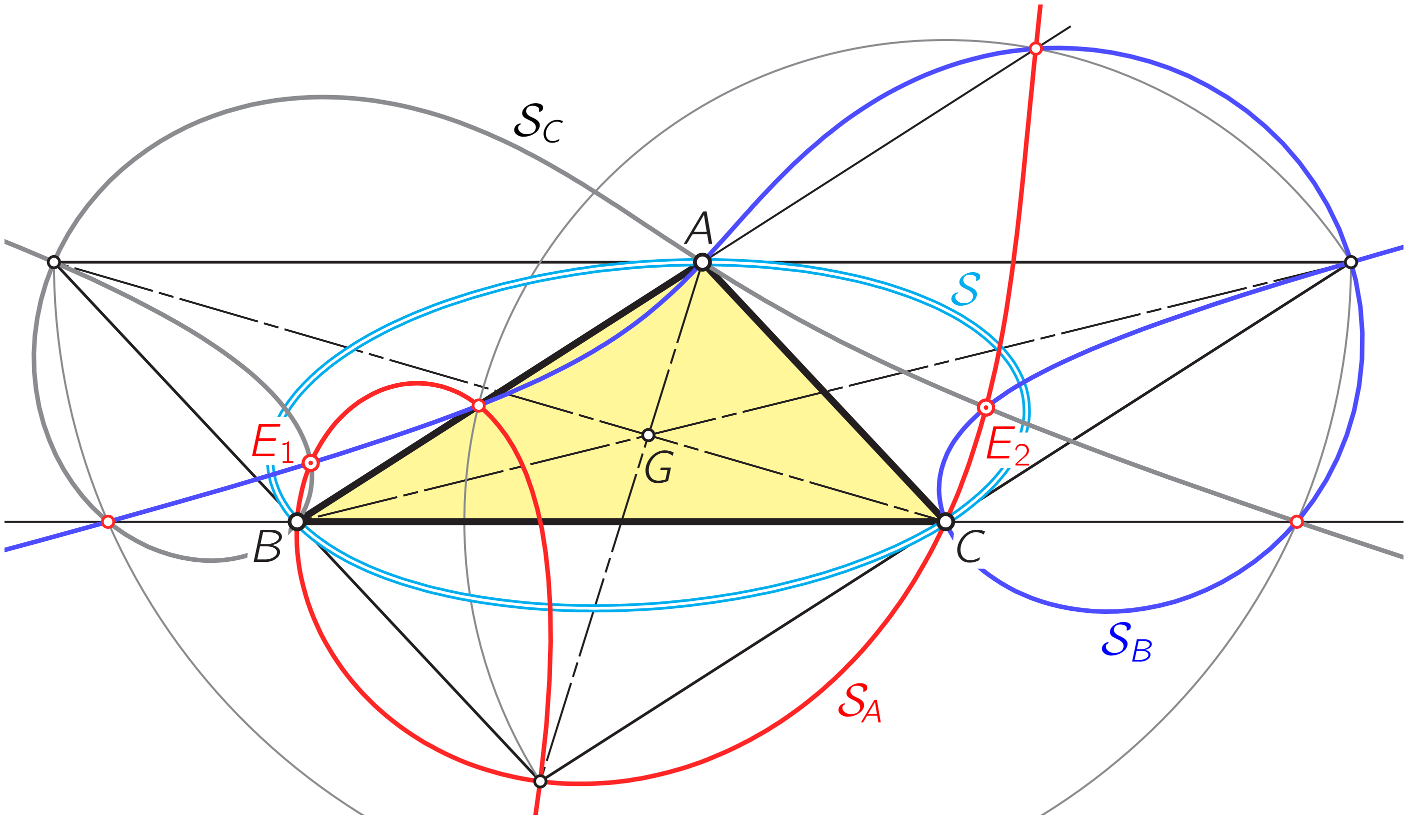
All equicevian cubics are oblique strophoids. E_1, $E_2 \in \mathcal{S}_A \cap \mathcal{S}_B \cap \mathcal{S}_C$ are proper equicevian points.

3. Strophoids as a Geometric Locus



Theorem: For each triangle ABC, the remaining equicevian points are identical with the two real and two complex conjugate focal points of the Steiner circumellipse $\mathcal{S}$.

The Steiner circumellipse $\mathcal{S}$ of ABC is the (unique) ellipse centered at the centroid G and passing through its vertices.



Theorem: When a and b, with $a \geq b$, denote the semiaxes of the Steiner circumellipse $\mathcal{S}$ of ABC, the cevians of the real foci have the length $3a/2$. The length of the cevians through the imaginary foci is $3b/2$.

S. Abu-Saymeh, M. Hajja, H.S.: *Equicevian Points of a Triangle.* Amer. Math. Monthly (to appear)

G. Brocard: *Centre de transversales angulaires éqales.* Mathésis, Ser. 2, **6** (1896) 217–221

Thank you

for your attention !

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