Two Remarkable Spherical Arrangements of Circles

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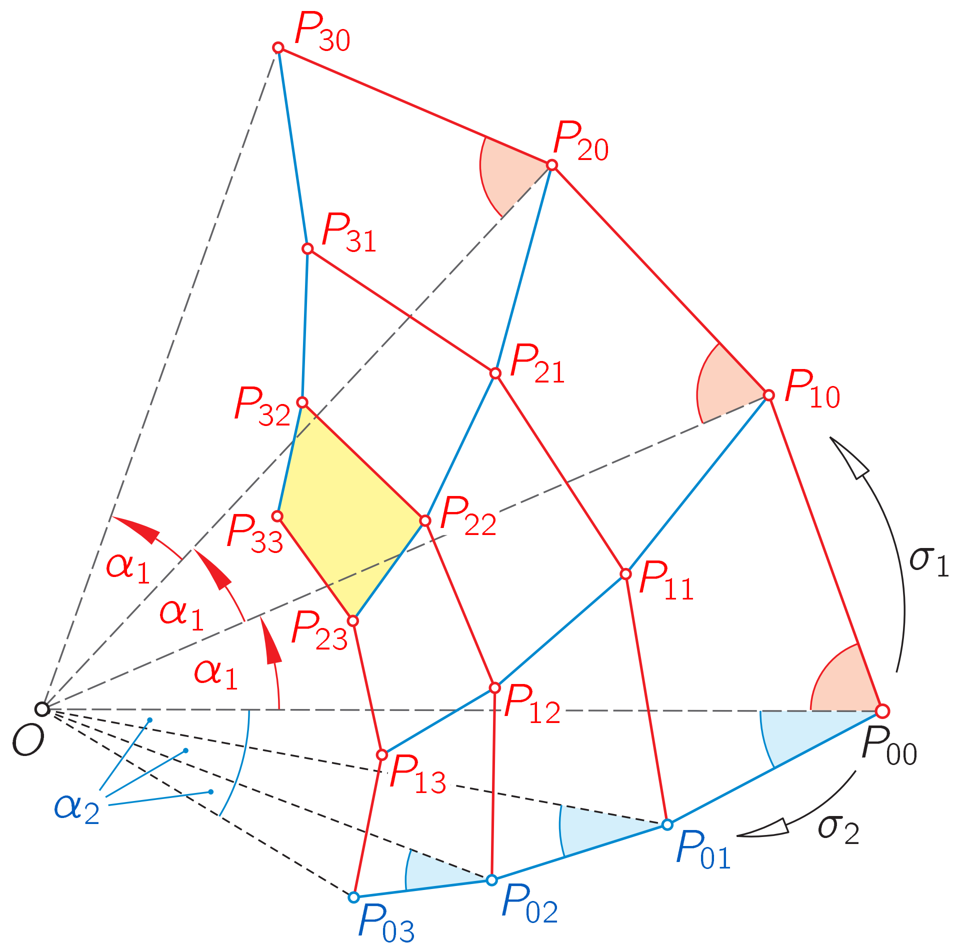
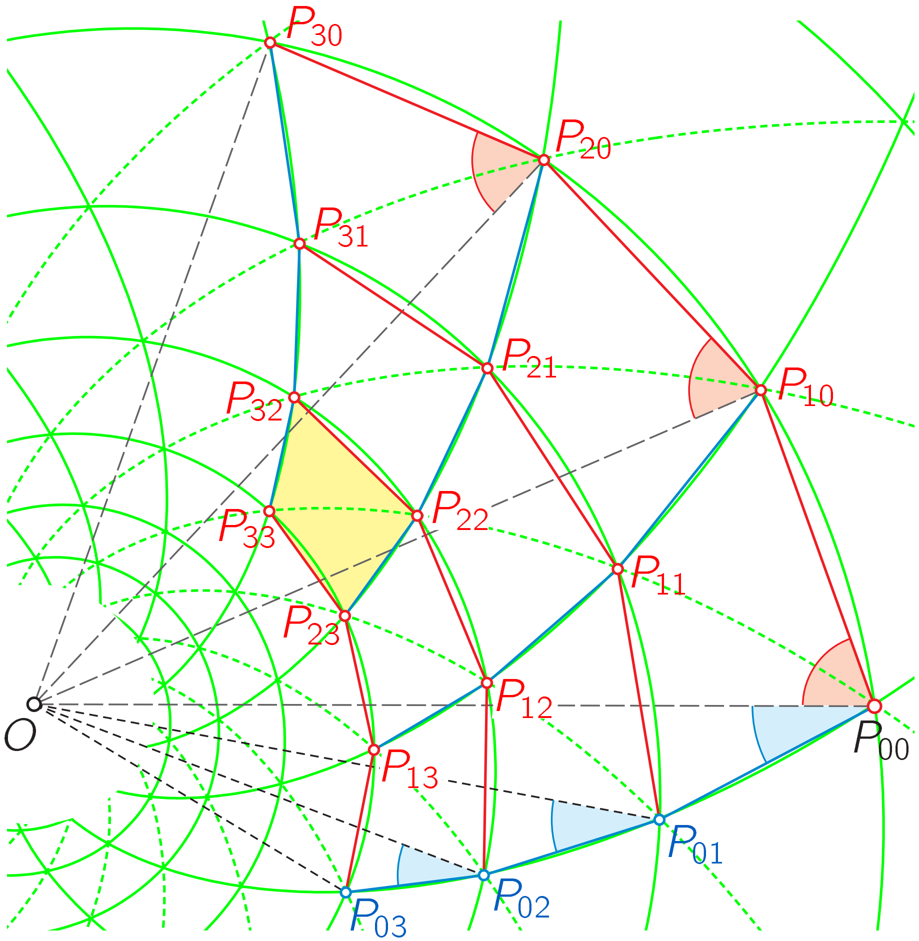


Given: two stretch-rotations σ_1, σ_2 with common center O, with respective signed angles α_1, α_2 of rotations and dilation factors $\delta_1, \delta_2 \neq 1$.

For any point P, the set of points $P_{ij} = \sigma_1^i \sigma_2^j(P)$, $i, j \in \mathbb{Z}$, is called a **spiral grid**.

The spiral grid is called closed $\iff$ $\exists\ n_1, n_2 \in \mathbb{N}$ such that $\sigma_1^{n_1}$ and $\sigma_2^{n_2}$ differ by a full rotation about O.

1. Spiral grids and spiral polyhedra



The same spiral grid can also be generated by other pairs σ_1', σ_2' of stretch rotations, when

$$\sigma_1' = \sigma_1^{c_1} \circ \sigma_2^{c_2}, \quad \sigma_2' = \sigma_1^{d_1} \circ \sigma_2^{d_2},$$

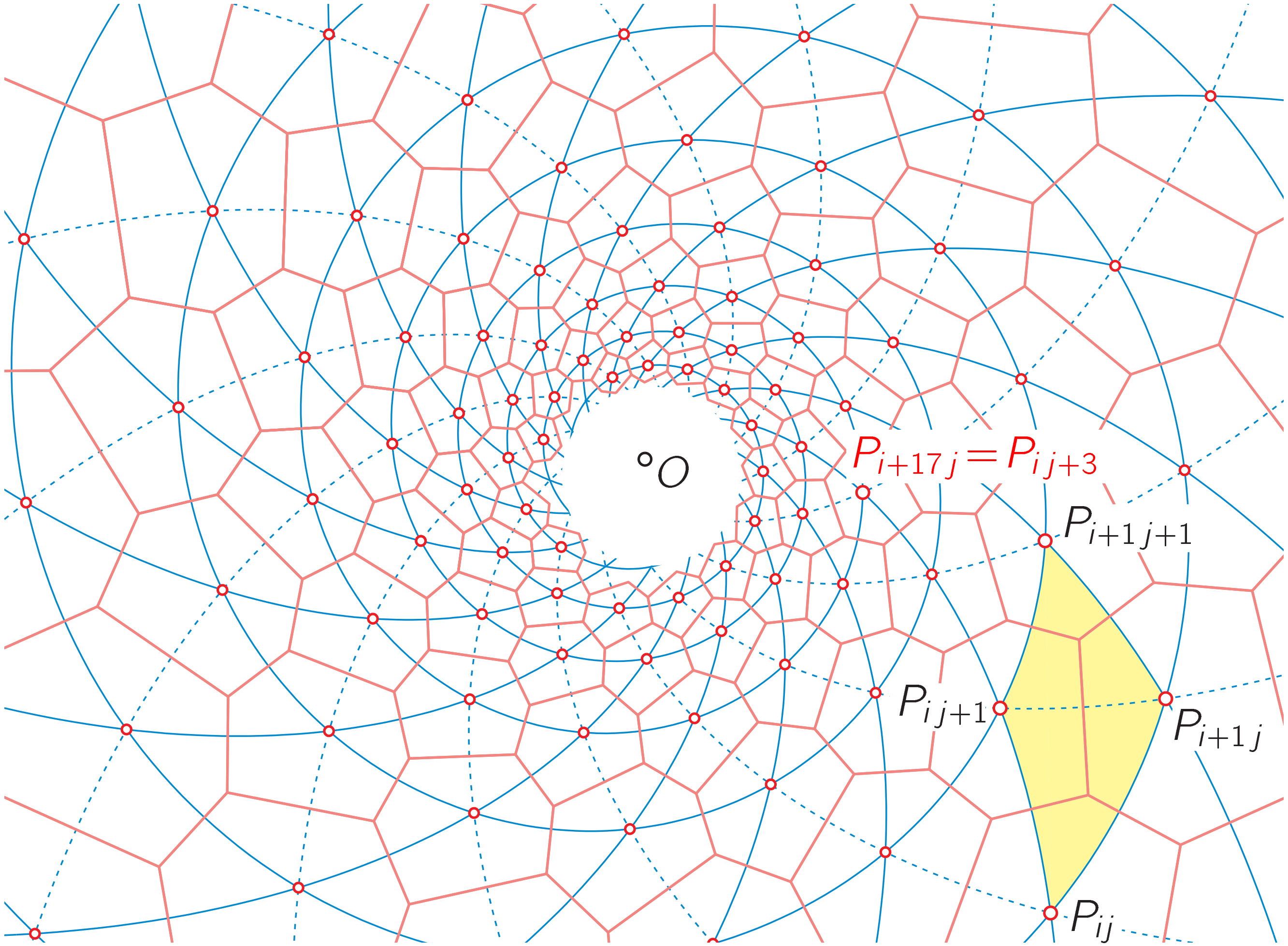
$$c_1, c_2, d_1, d_2 \in \mathbb{Z}, \quad c_1 d_2 - c_2 d_1 = 1.$$

1. Spiral grids and spiral polyhedra

Spiral grids play a role in *Phyllotaxis*, a topic of plantmorphogenesis.

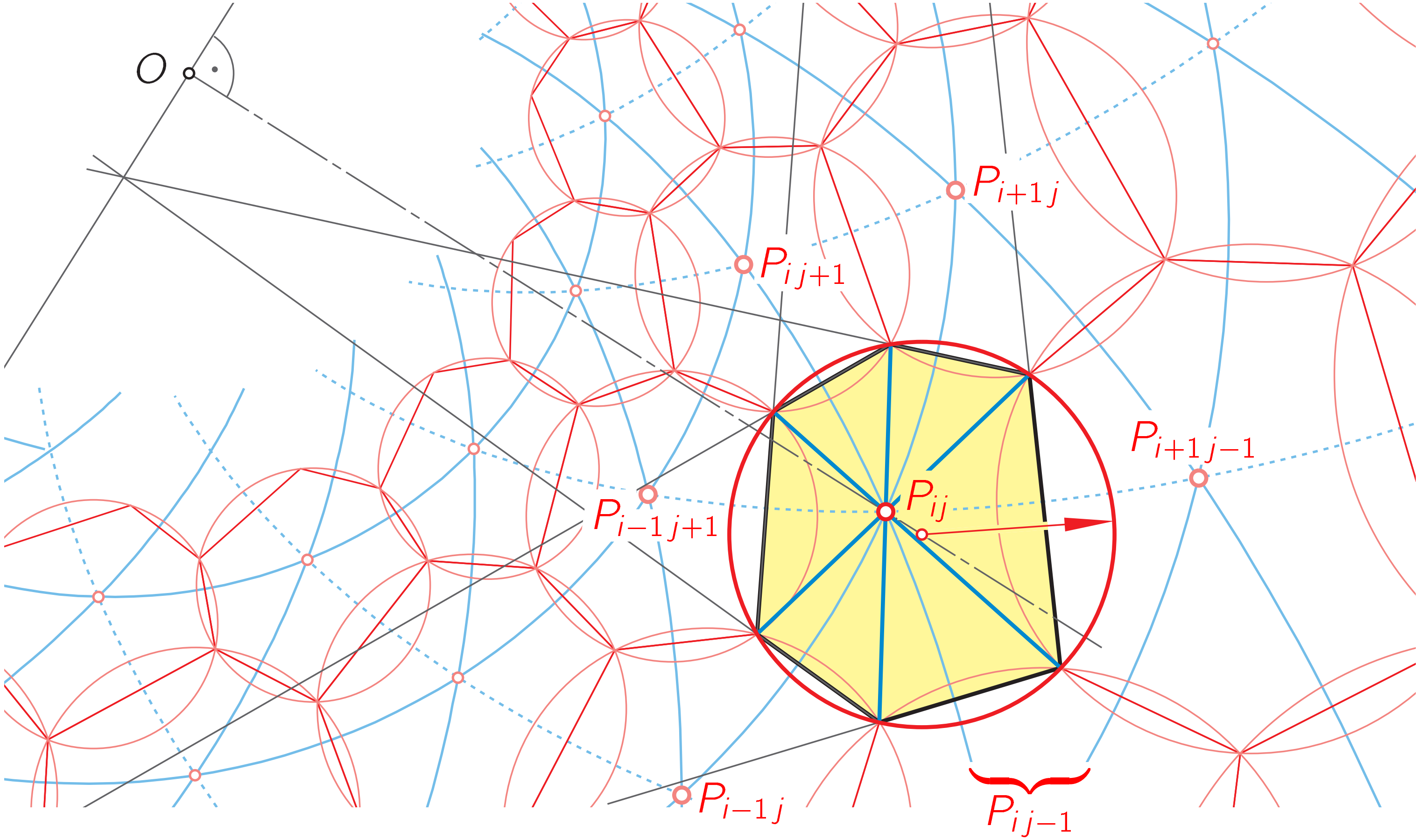
The grid approximates the position of leaves.

1. Spiral grids and spiral polyhedra



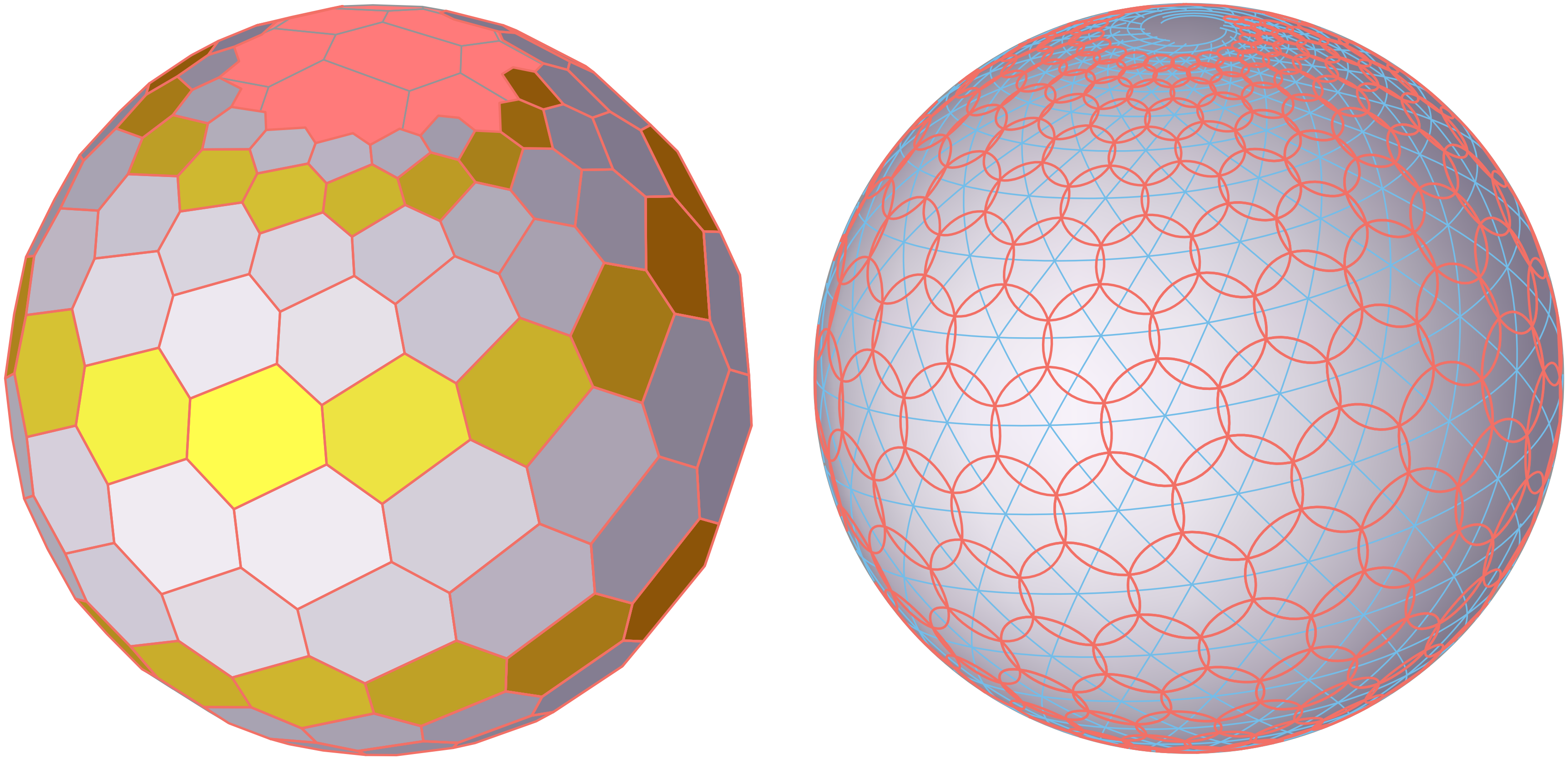
Closed **spiral grid** ($P_{i+17\,j} = P_{i\,j+3}$) and its mutually similar **Voronoi cells**

1. Spiral grids and spiral polyhedra

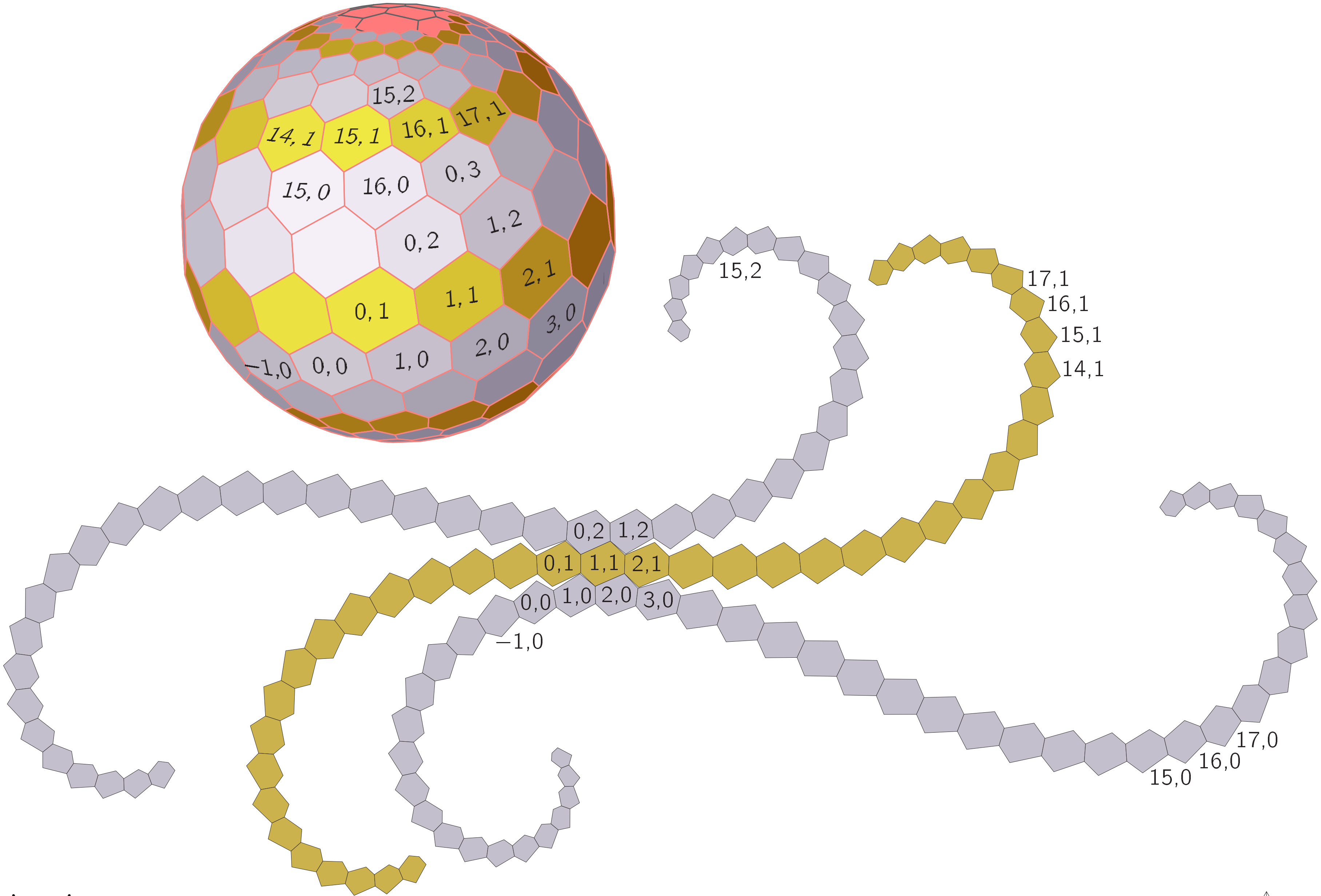


K. Myrianthis (2017): In general, the Voronoi cells are concircular hexagons with concurrent main diagonals . . . we apply an appropriate stereographic projection:

1. Spiral grids and spiral polyhedra



spherical **spiral polyhedron** and a net of circumcircles of the hexagonal faces


15,2
14, 1
15, 1
16, 1
17, 1
15, 0
16, 0
0, 3
0, 2
1, 2
2, 1
0, 1
1, 1
3, 0
2, 0
−1,0
0, 0
1, 0
15,2
17,1
16,1
15,1
14,1
0,2
1,2
0,1
1,1
2,1
0,0
1,0
2,0
3,0
−1,0
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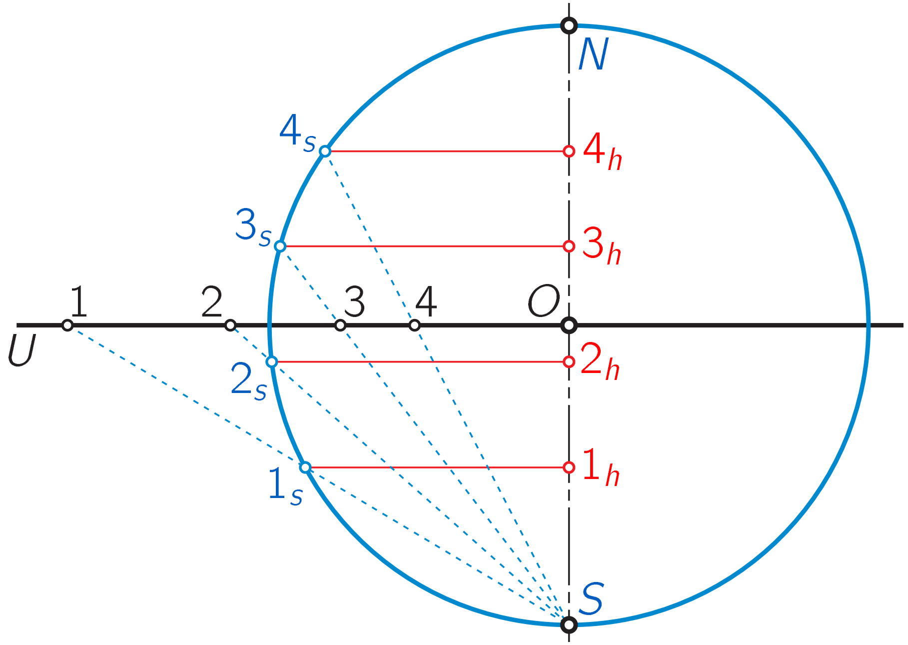
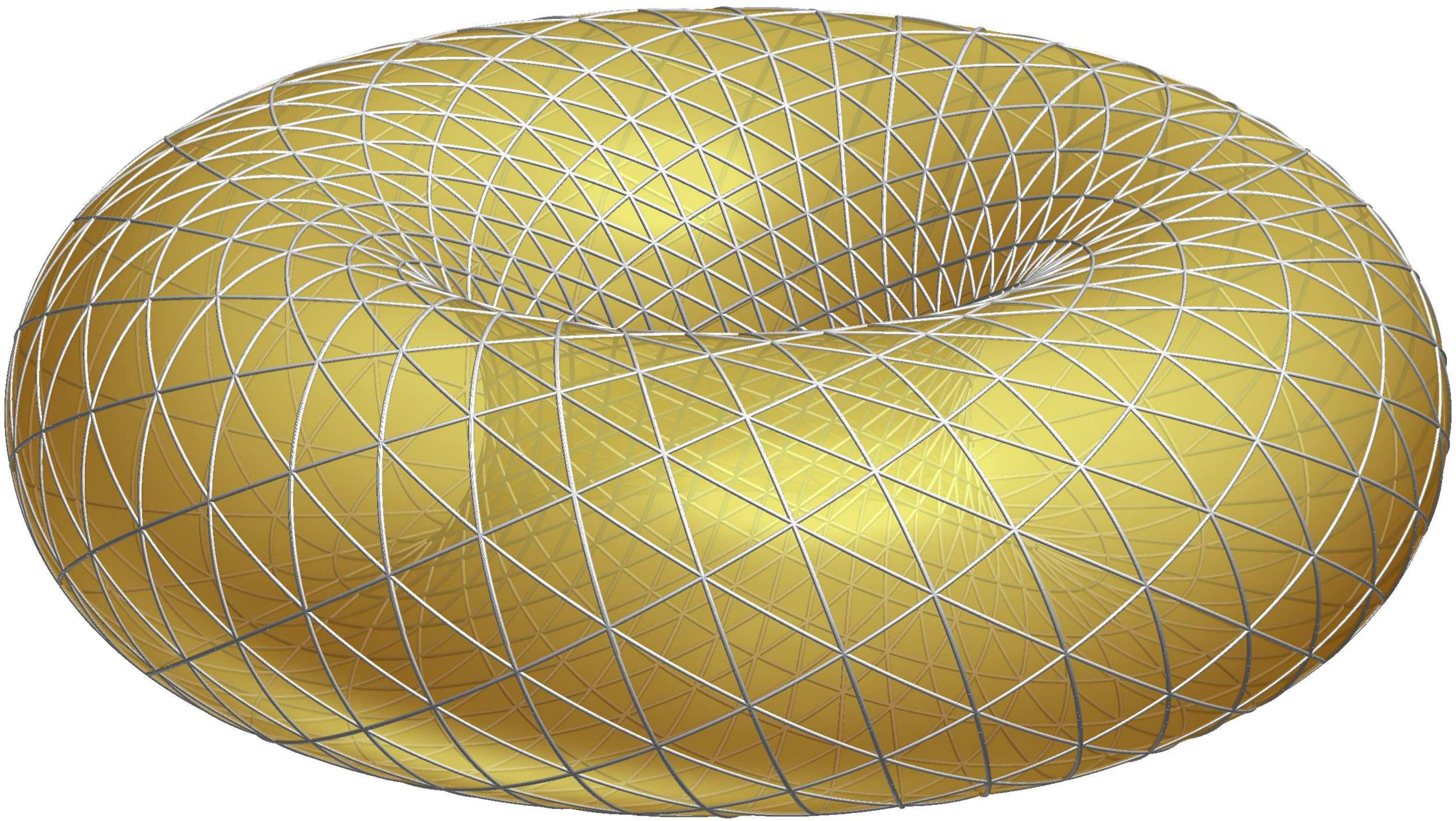
16,0
15,0

1. Spiral grids and spiral polyhedra



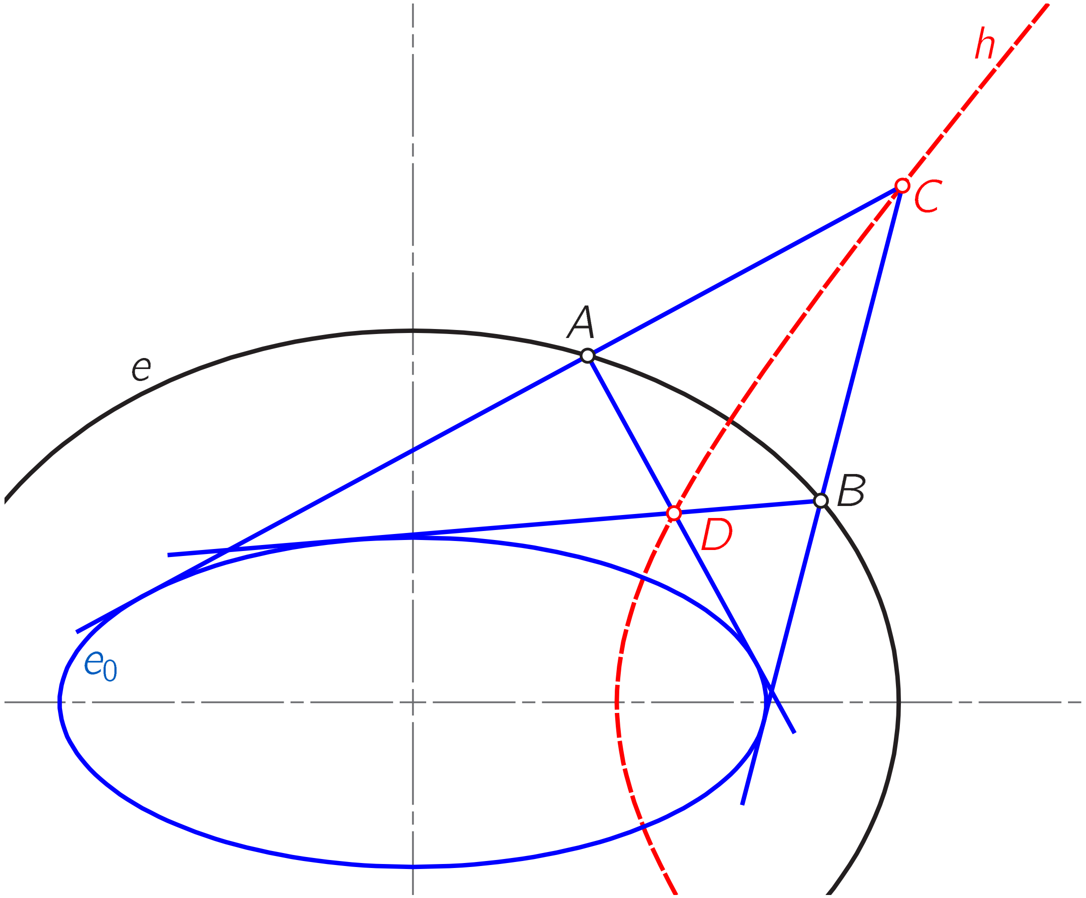
Stereographic projection transforms the central dilation with center O onto a hyperbolic translation.

generated by hyperbolic screw motion



. . . a research drawn from *“die Freude an der Gestalt”*, i.e., by an appreciation for aesthetics: A **3-web** of coaxial screws in the conformal model of the elliptic 3-space. The screws are **loxodromes** (by courtesy of **Georg Glaeser**).

2. A spherical incircular net

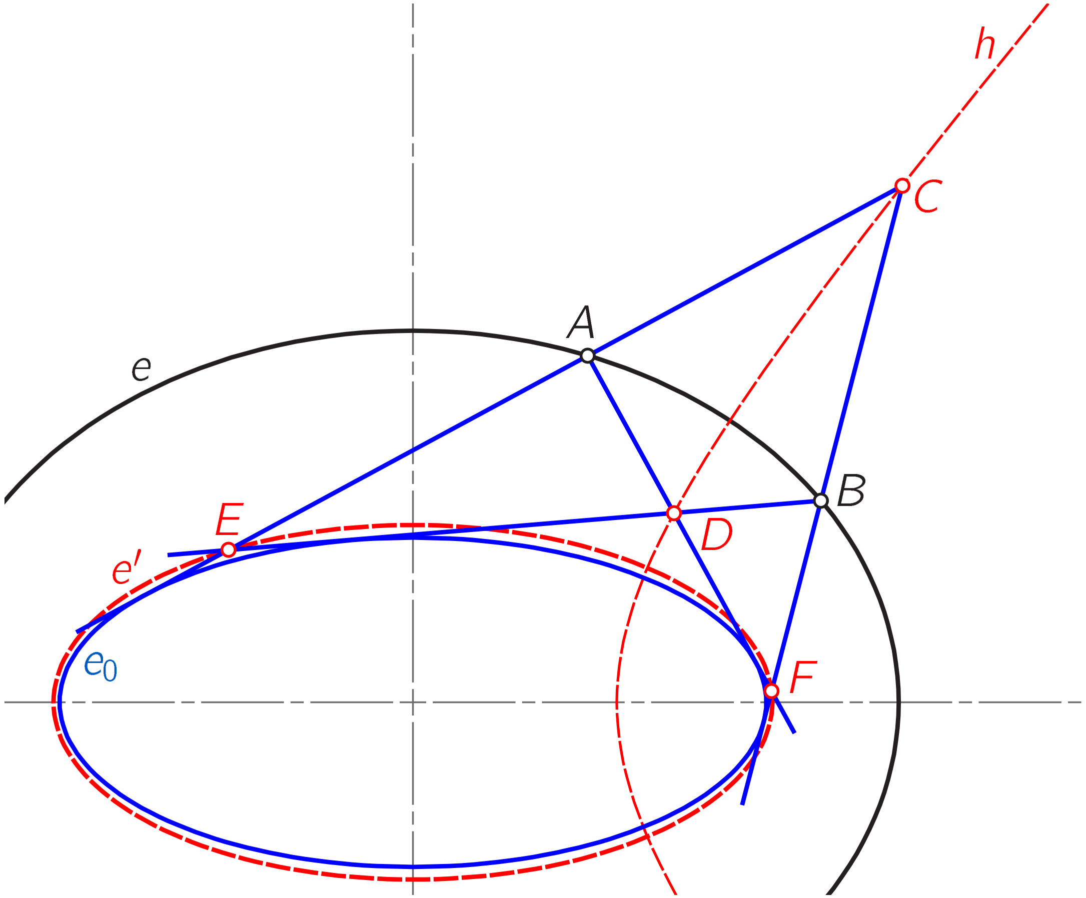


I. Izmestiev, S. Tabachnikov (2016):

Theorem:
For any two points A, B on the ellipse e the diagonal points C, D of the quadrilateral circumscribed to the confocal ellipse e_0 belong to the same confocal hyperbola h.

Chasles 1843, W. Böhm 1961

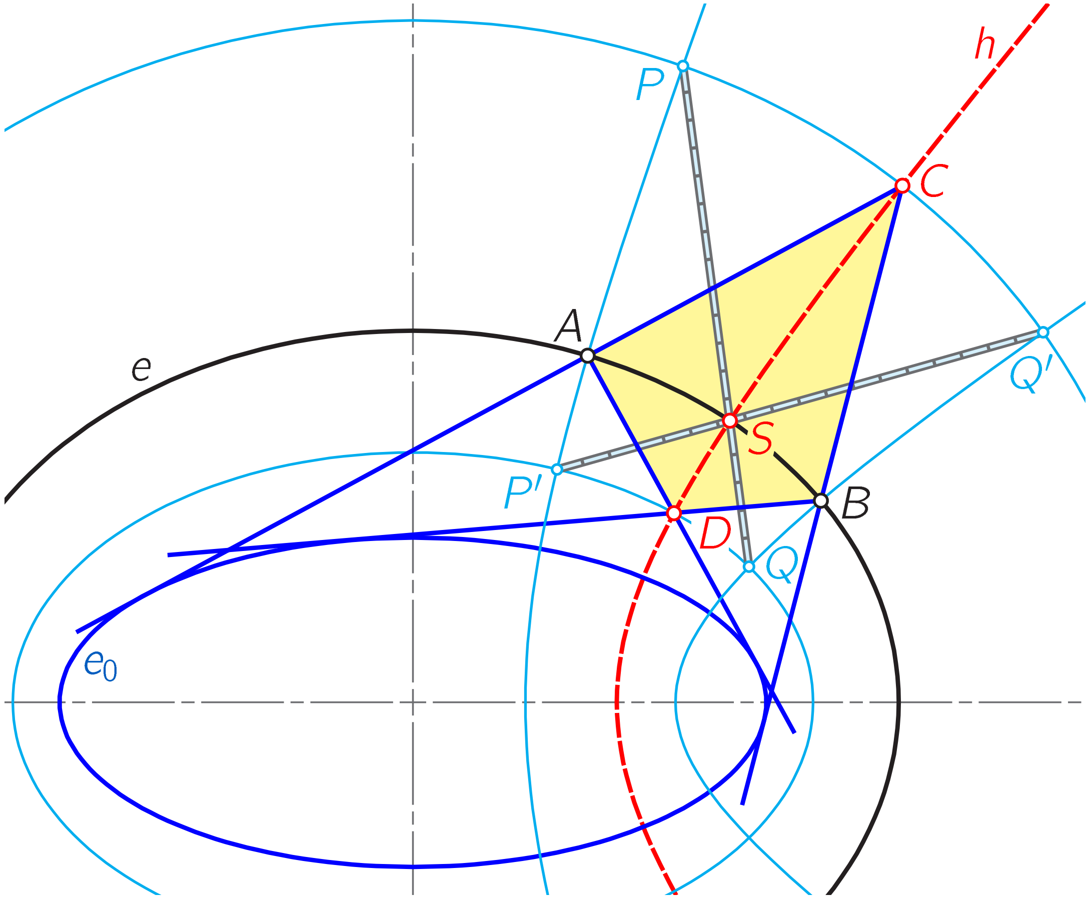
2. A spherical incircular net



Izmestiev and Tabachnikov use differential geometry in their proof. But the background of the problem is of projective nature.

The remaining diagonal points E, F of the circumscribed quadrilateral are again located on a conic e' confocal with e_0 and e.

2. A spherical incircular net

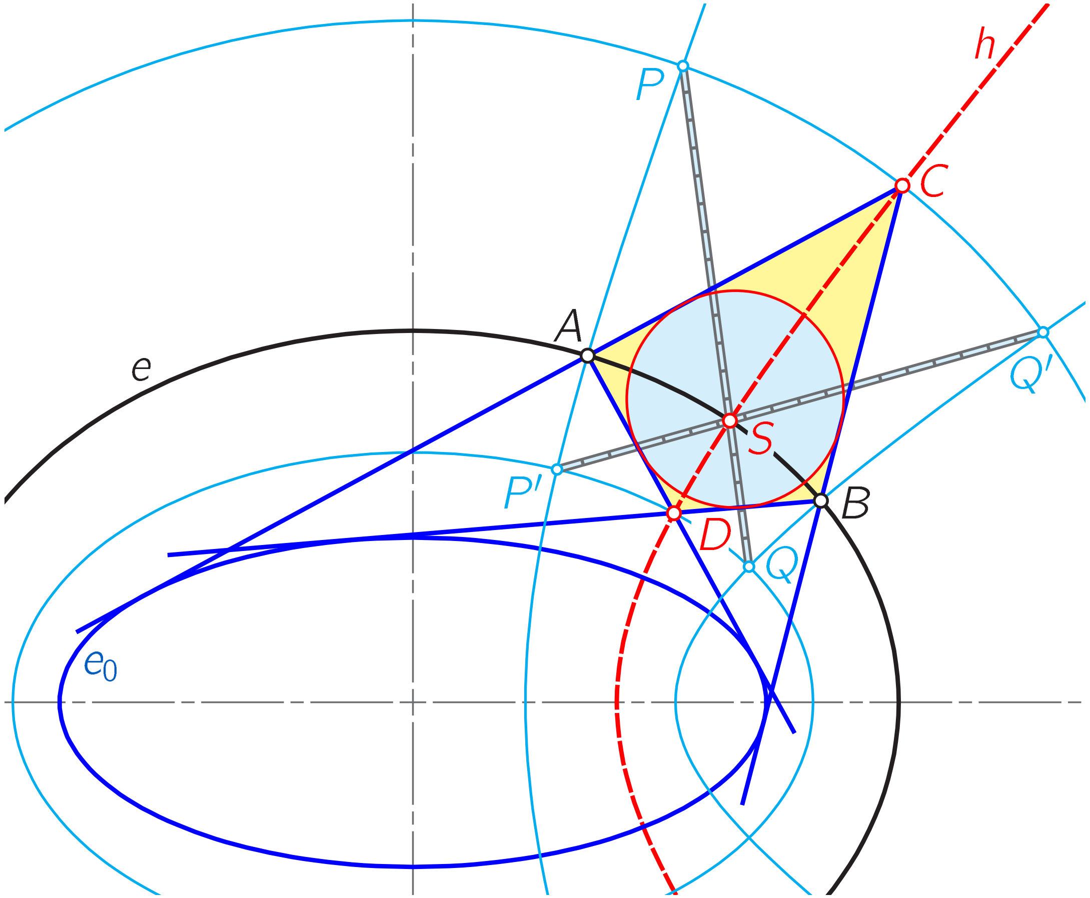


Proof with Ivory's Theorem:

There is a curvilinear quadrangle $PQ'QP'$ such that e and h are confocal conics passing through the 'midpoint' S, the point of intersection $PQ \cap P'Q'$.

$$\overline{AC} = \overline{PS},\ \overline{BD} = \overline{QS},\ \ldots \implies$$
$$\overline{AC} + \overline{BD} = \overline{PQ} = \overline{AD} + \overline{BC}$$

2. A spherical incircular net



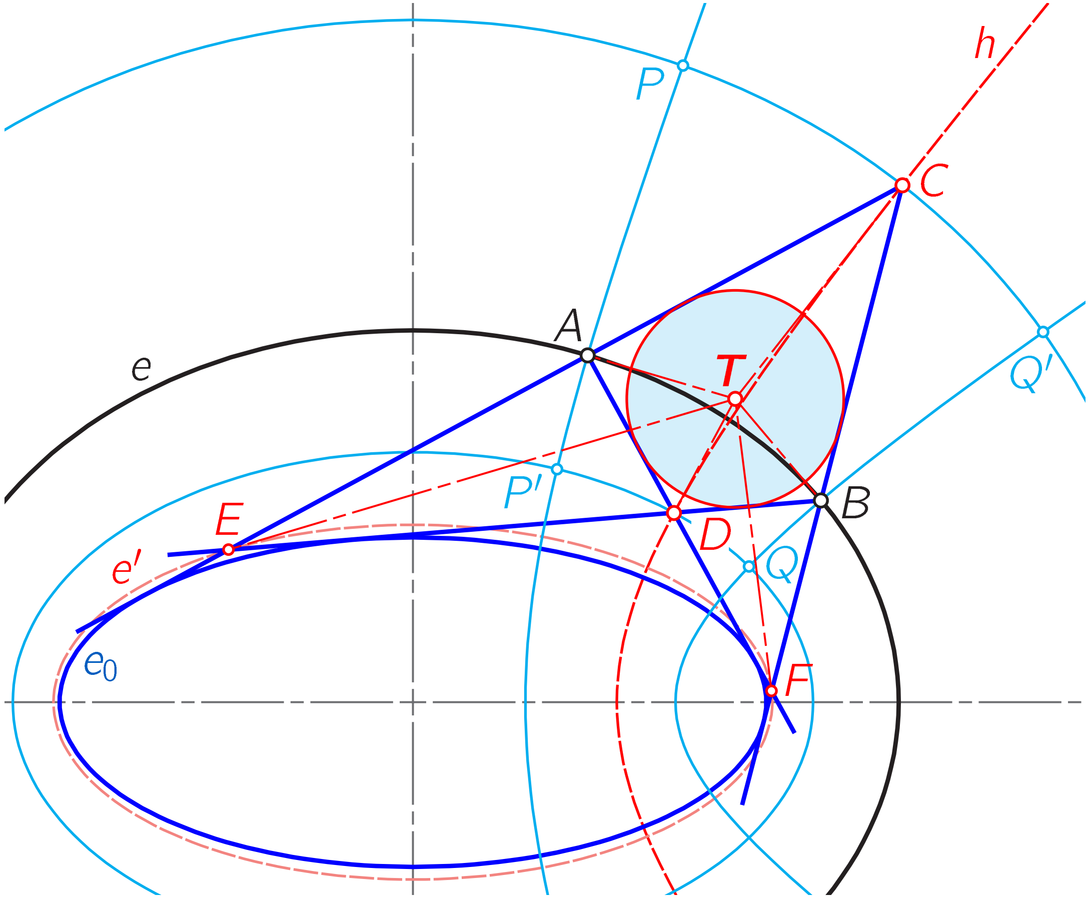
$$\overline{AC} + \overline{BD} = \overline{AD} + \overline{BC} = \overline{PQ}$$

In the quadrangle $ACBD$ opposite sides lengths give the same sum $\overline{PQ}$. The quadrangle is 'incircular'.

A.W. Akopyan, A.I. Bobenko: *Incircular nets and confocal conics*, Trans. Amer. Math. Soc., Nov. 16, 2017

Chasles 1843, W. Böhm 1961

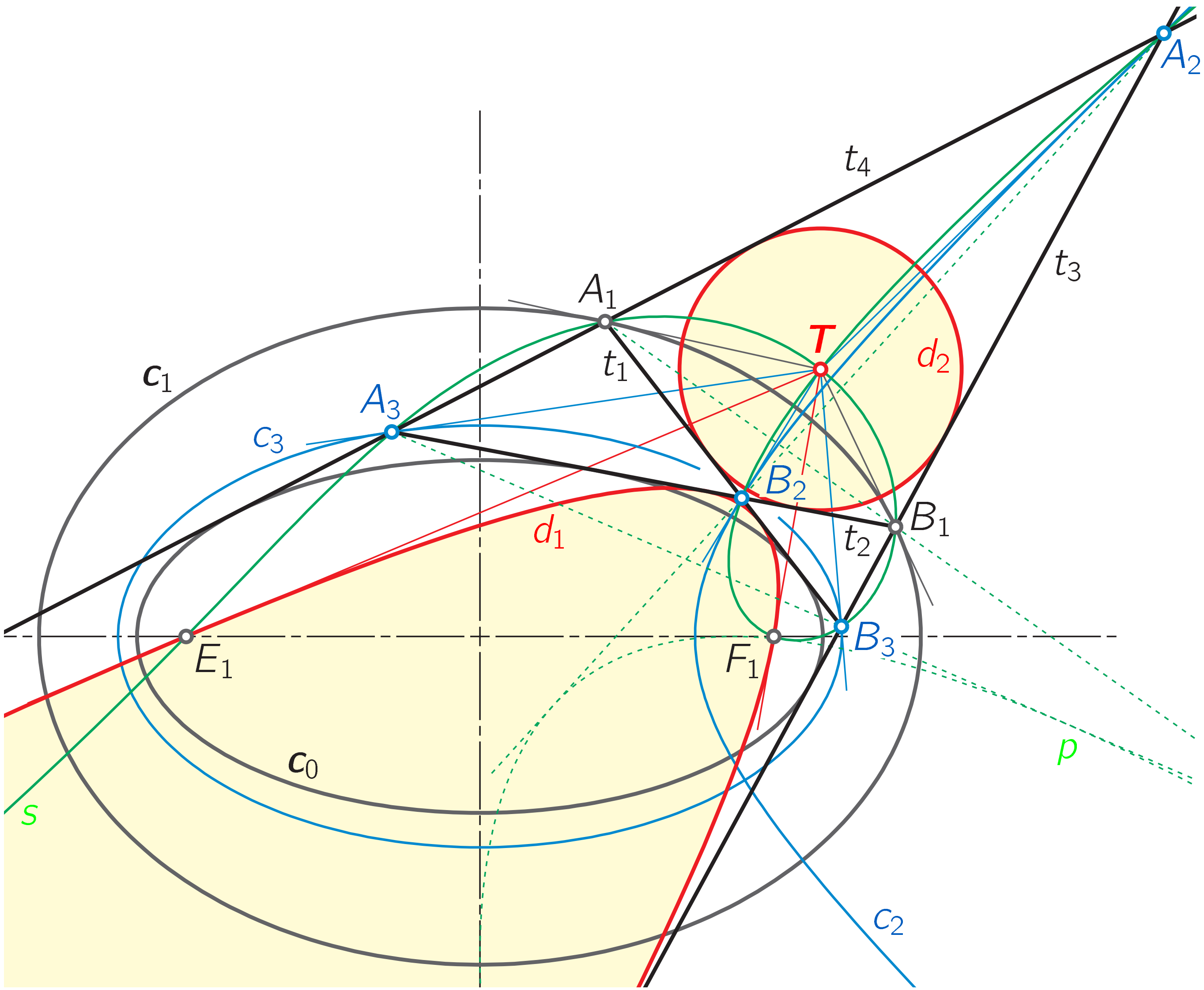
2. A spherical incircular net



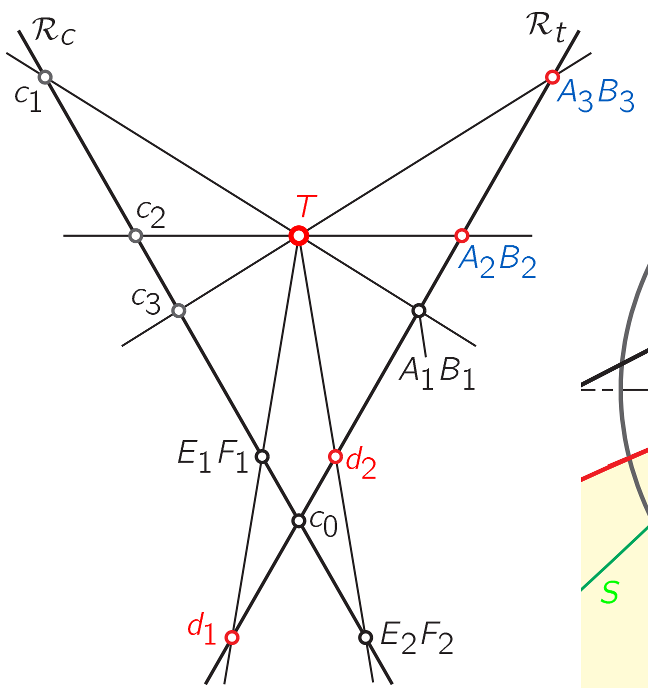
The tangents at A, B to e, at E, F to e' and at C, D to h pass through point T.

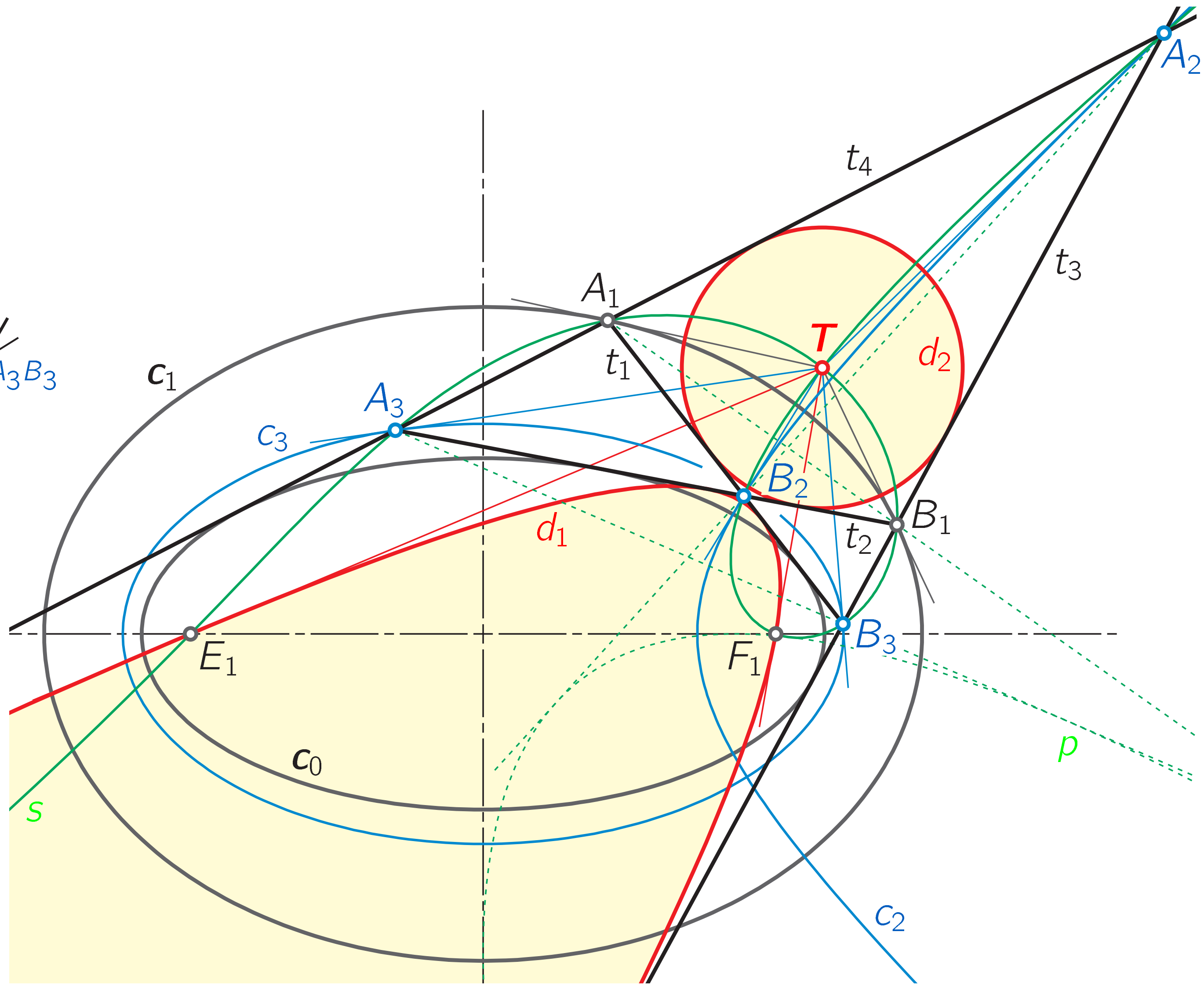
Generalising theorems of Michel Chasles (1843) and Wolfgang Böhm (1961) concerning two-parametric linear systems of conics:

Theorem: There is a net of conics spanned by c_0, c_1 and the pair of line pencil with carriers (A_1, B_2).
If the pair (X_1, X_2) is included in this net then there is a **conic** tangent to $t_1, \ldots, t_4$ and passing through X_1 and X_2.

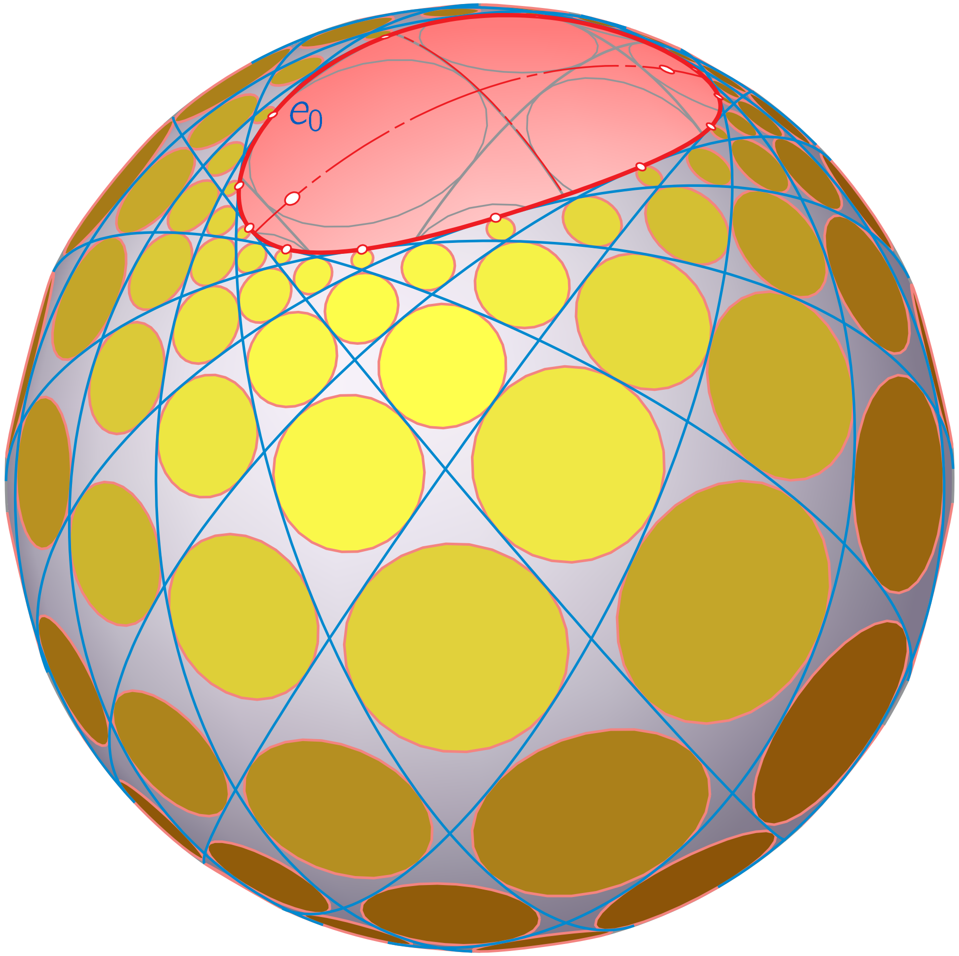


The net is a projective plane spanned by two ranges $\mathcal{R}_c$ and $\mathcal{R}_t$ through c_0:





2. A spherical incircular net

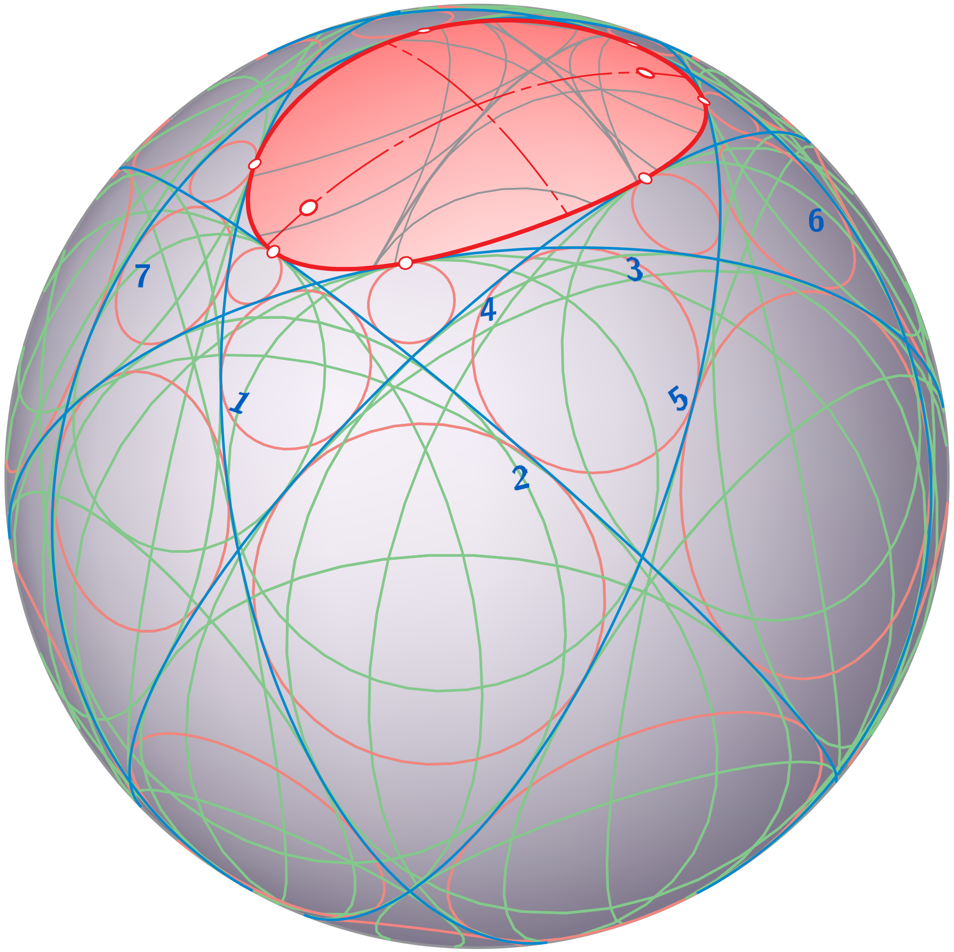


All presented results are also valid on the sphere.

Left:
An incircular net of 13 great circles, each tangent to the spherical conic e_0. The great circles extend the sides of a closed billiard.

This net has **no** rotational symmetry!

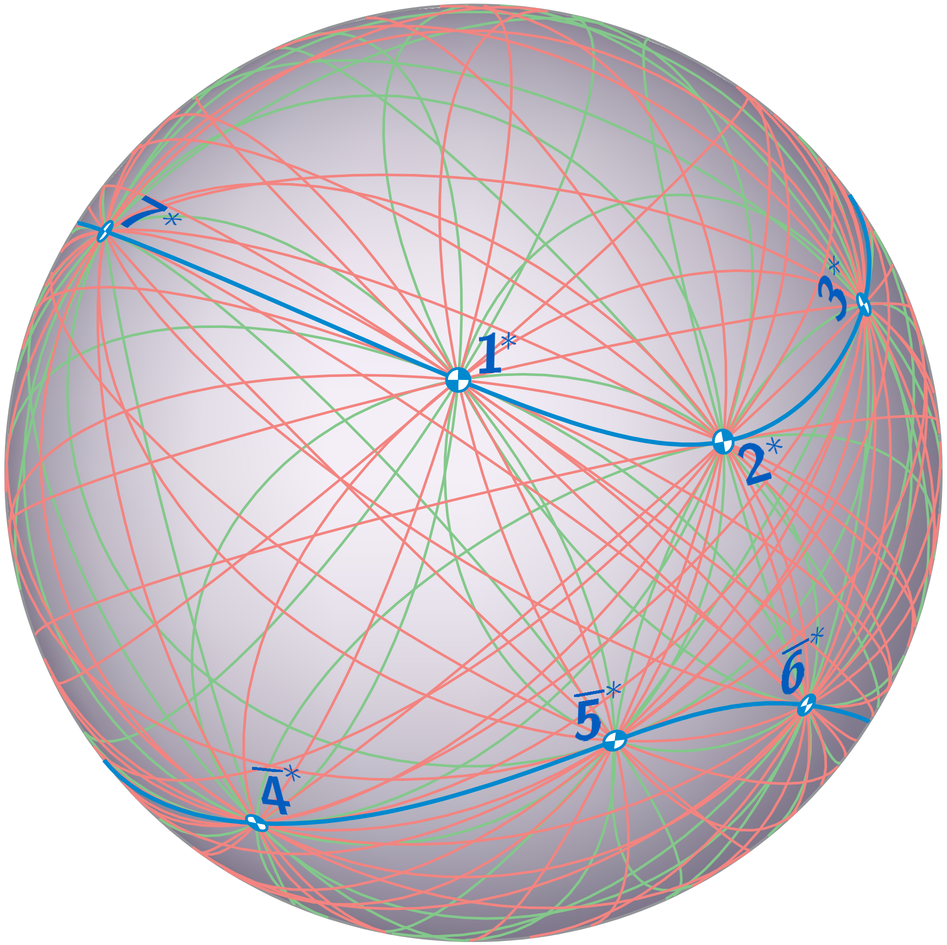
2. A spherical incircular net



Left: A spherical net of 7 great circles, each tangent to the spherical conic, and 42 enclosed incircles.

Instead of pairs of consecutive great circles, one can also select every second circle (or every third . . .) in order to obtain incircular quadrangles.

2. A spherical incircular net



dual version: all quadruples of admissible points out of $\{1^*, \ldots, \overline{7}^*\}$ have a circumcircle.

Thank you for your attention!

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