Between Rigidity and Flexibility

Hellmuth Stachel

TECHNISCHE UNIVERSITÄT WIEN

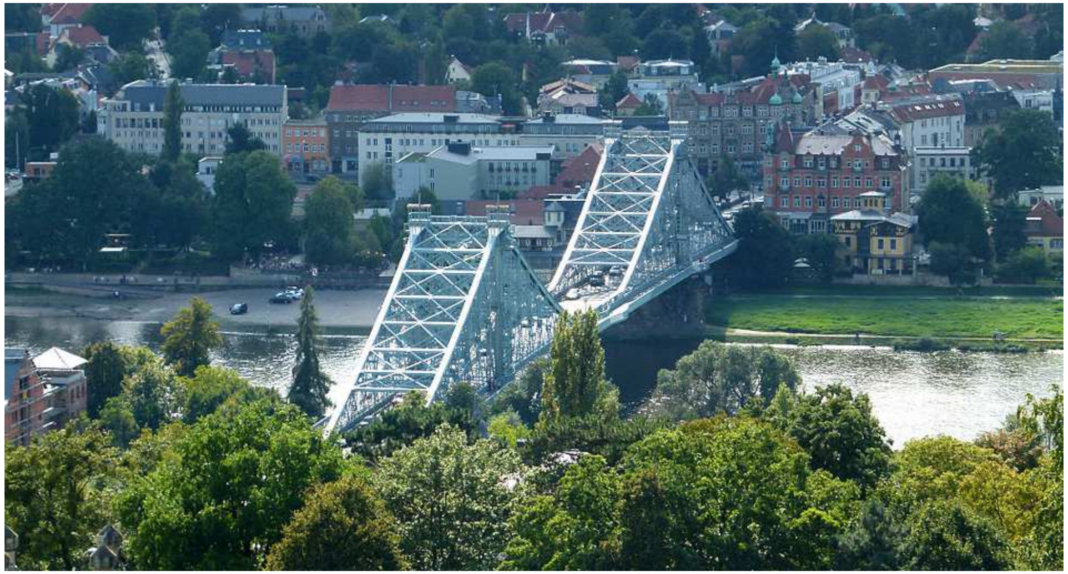
stachel@dmg.tuwien.ac.at — http://www.geometrie.tuwien.ac.at/stachel

Workshop on “Rigidity and Flexibility of Geometric Structures”, Sept. 24–28, 2018
Erwin Schrödinger International Institute for Mathematics and Physics, Vienna/Austria

Table of contents

Acknowledgement: Georg **Nawratil**, TU Wien

1. Introduction



“Blue Wonder” bridge in Dresden (1893),

without supporting river piers; the name reflects also a sceptic view of contemporary commentators

We expect that a bridge is **rigid**, i.e., that the framework admits no self-motion.

However, we concentrate only on the geometry of the framework.

We don’t study the influence of clearances at the joints, of bendings of the material or of *vibrations*.

Each bar is understood as a rigid body, each knot is a revolute joint.

1. Introduction

"Morandi Bridge" on the motorway A10 in Genoa/Italy after Aug. 14, 2018

One can't explain this deplorable catastrophe with a bad geometry,

but conversely, for any structure a correct underlying geometry is an inevitable prerequisite for its utilization.

1. Introduction

Definition: A geometric structure (e.g., framework or polyhedron) is called **flexible**, if its shape can continuously vary without changing its inner metric (combinatorial struction and lengths of edges or metric of faces). Otherwise it is called **rigid**.

It turns out that the borderline between flexibility and rigidity is not as strict as one might conjecture. There are different types of flexibility to distinguish.

1. Introduction

- The structure is **globally rigid** when its shape in space is uniquely defined by its inner metric — apart from movements in space.

- The structure is called **locally rigid**, when it is not flexible; but it admits mutually incongruent realizations. If two realizations are sufficiently close they can be forced to change from one realization into the other (**flipping structure**).

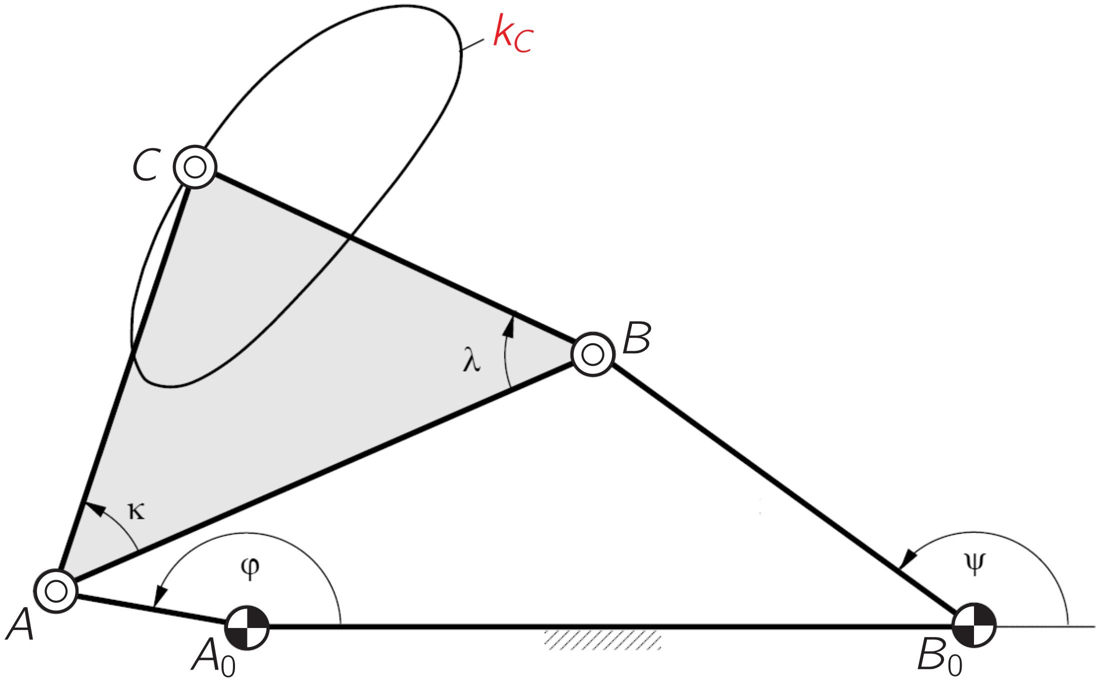
- The structure is called **generically rigid** if its combinatorial structure admits only rigid poses — independently of its metric.

1. Introduction

The computation of realizations is an algebraic problem. It has been recently proved that a combinatorial property characterizes those generically rigid structures for which a metric exists which makes them flexible.

G. Grasegger, J. Legerský, J. Schicho: *Graphs with Flexible Labelings.*
Discrete & Computational Geometry 2018

2. Four-bar linkages



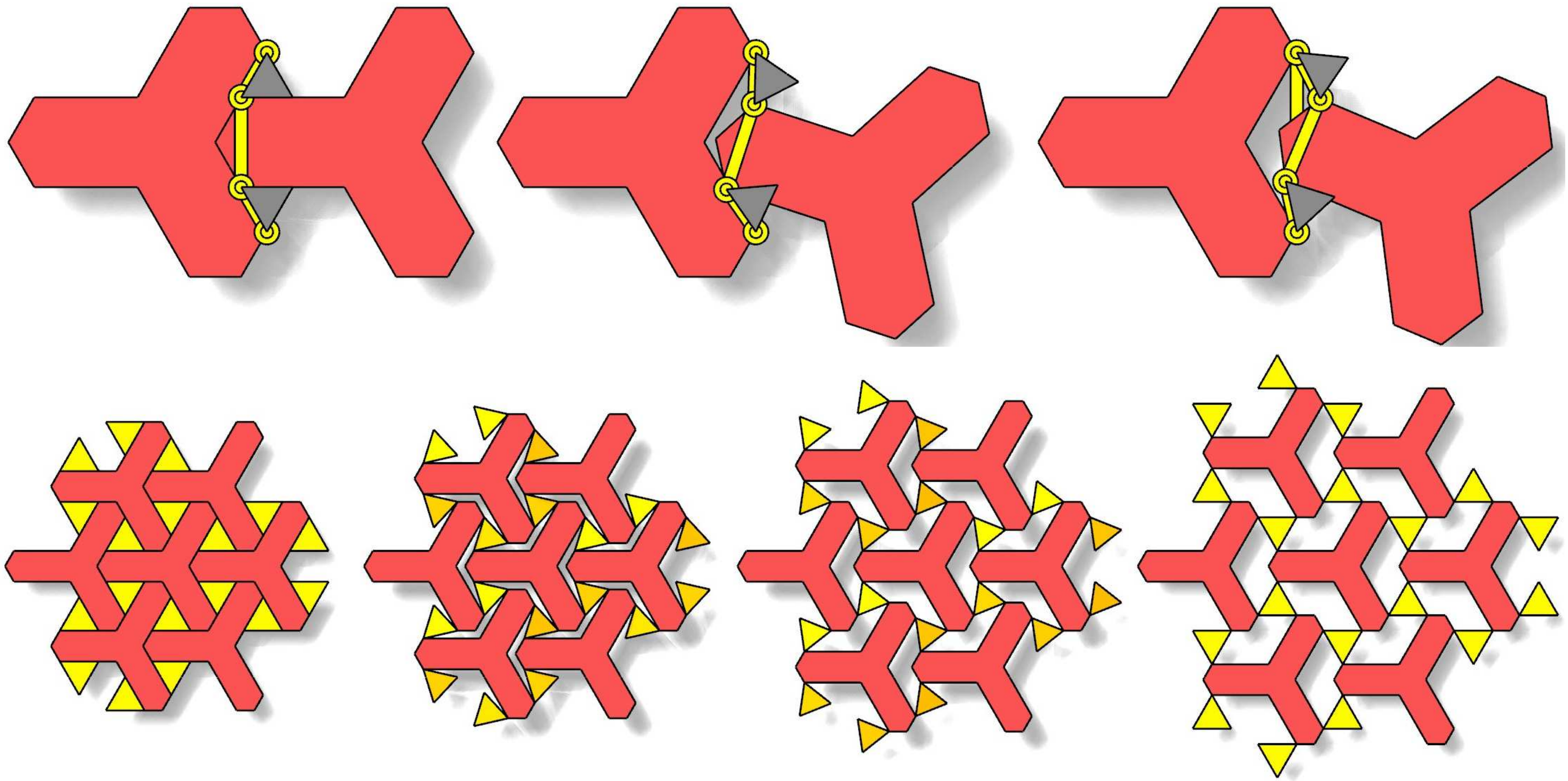
A four-bar mechanism consists of a flexible quadrangle with one side A_0B_0 fixed.

The opposite side AB is called coupler; points C attached to this side trace coupler curves k_C of various shapes.

2. Four-bar linkages

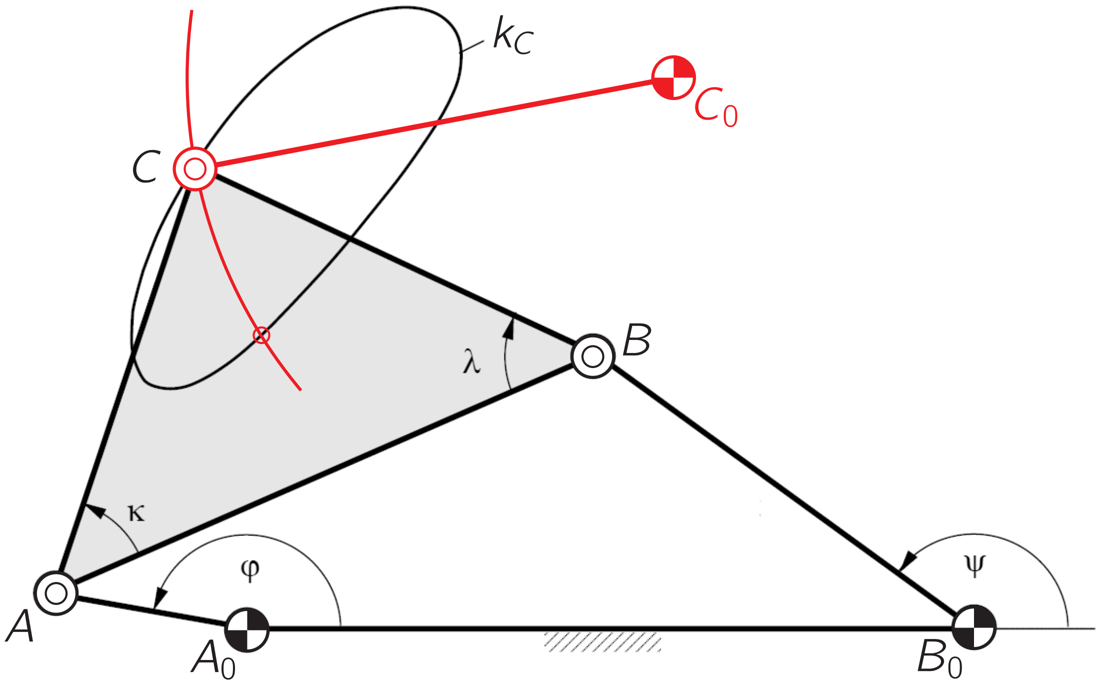
Four-bar linkages show up at cranes and at links

2. Four-bar linkages



Four-bars can even be used for designing auxetic structures
(courtesy M. Stavric and A. Wiltsche, TU Graz)

2. Four-bar linkages

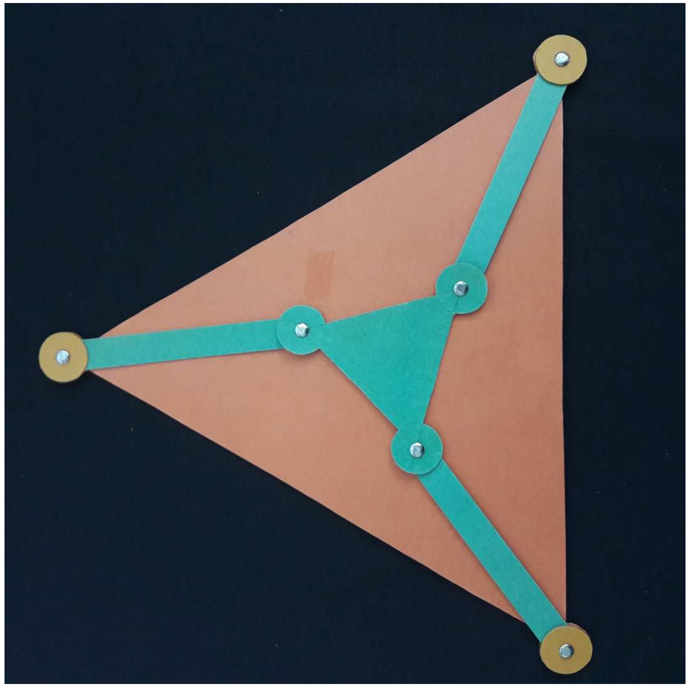


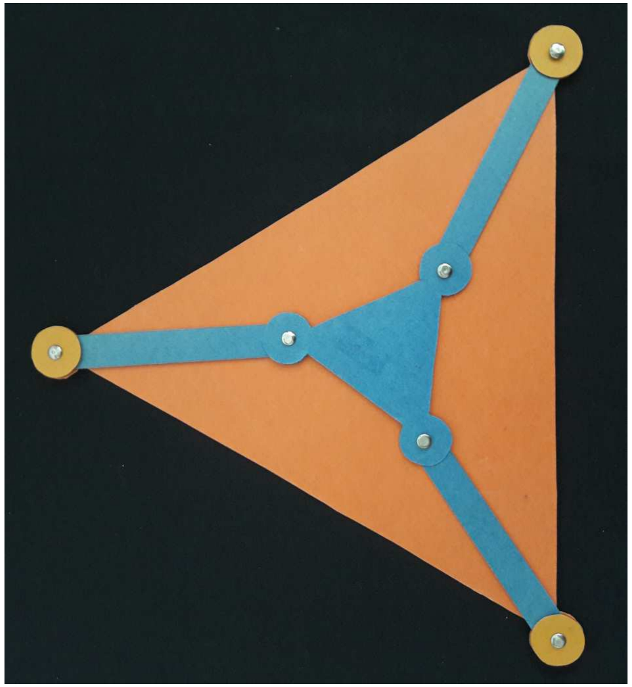
Let an additional bar connect point C with a fixed point C_0. Then the framework is rigid.

But it admits a second realization with C at the second point of intersection between k_C and the circle. The framework can **flip** between two poses.

When the circle touches the coupler curve k_C at C, the mechanism is **infinitesimally flexible**.

2. Four-bar linkages





A flipping framework left and an infinitesimally flexible framework (right)

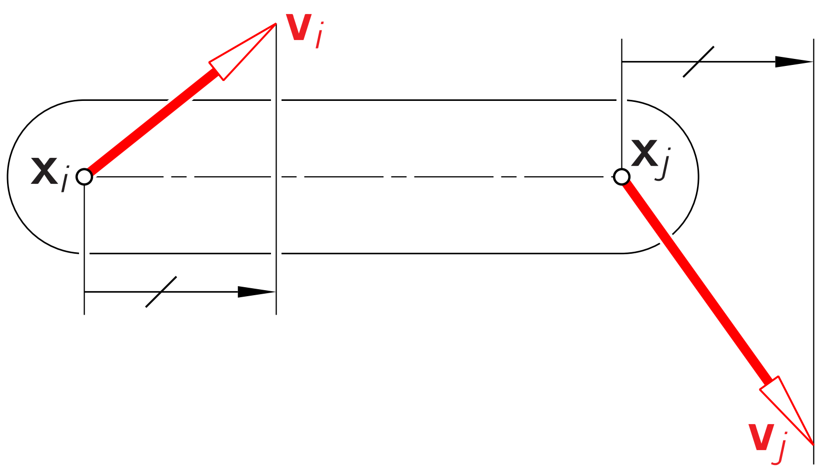
2.1 Projection Theorem

Suppose, the lengths of edges are only infinitesimally constant.

Definition: A polyhedron is called **infinitesimally flexible** $\Longleftrightarrow$

to each vertex $\mathbf{x}_i$ a velocity vector $\mathbf{v}_i$ can be assigned such that

- for any edge $\mathbf{x}_i\mathbf{x}_j$ the projection theorem holds, and
- the assignment is nontrivial, i.e., the velocity vectors do not originate from a motion of the framework as a rigid body.



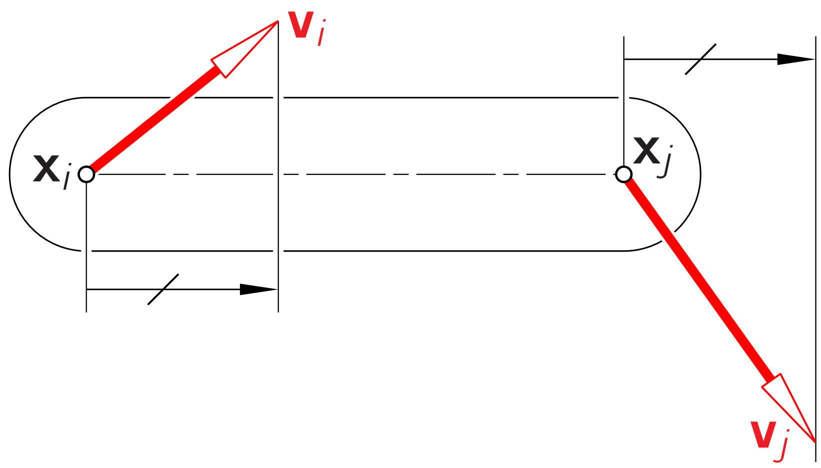
Projection theorem

2.1 Projection Theorem

$$(\mathbf{x}_i - \mathbf{x}_j) \cdot (\mathbf{x}_i - \mathbf{x}) = \text{const.} \Longrightarrow$$

$$(\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) = 0 \iff$$

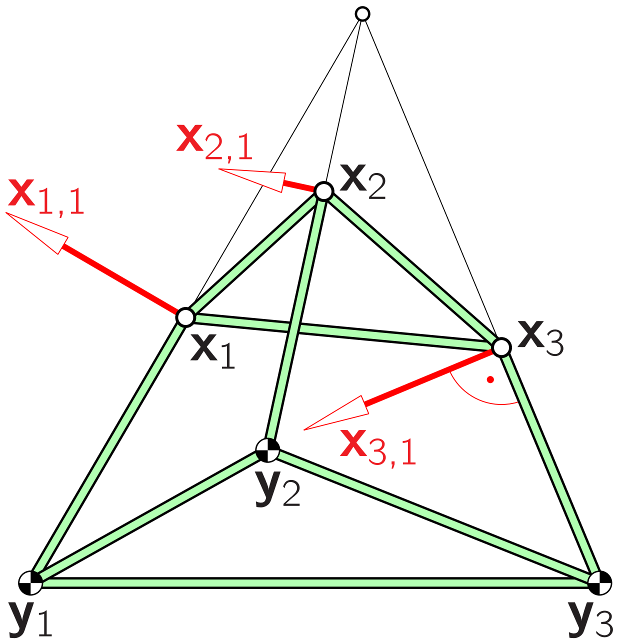
$$(\mathbf{x}_i - \mathbf{x}_j) \cdot \underbrace{\dot{\mathbf{x}}_i}_{\mathbf{v}_i} = (\mathbf{x}_i - \mathbf{x}_j) \cdot \underbrace{\dot{\mathbf{x}}_j}_{\mathbf{v}_j}$$

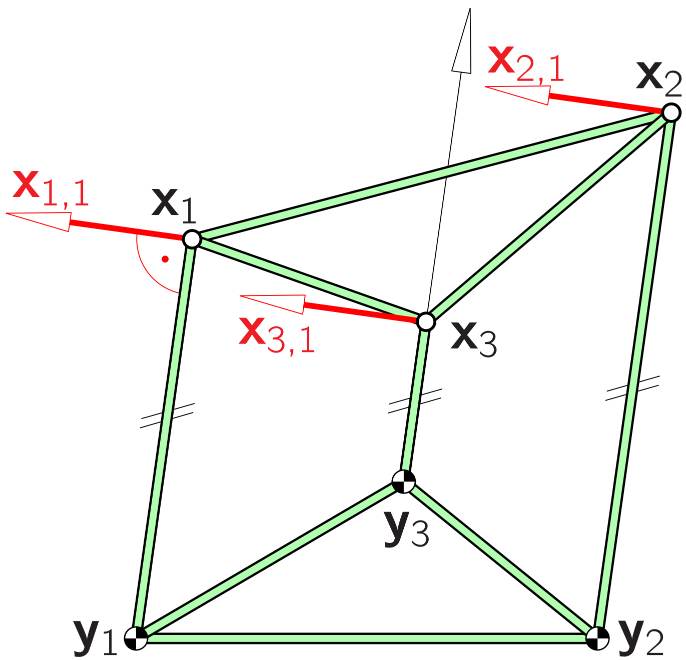


Projection theorem

A physical model of an infinitesimally flexible polyhedron or framework is really slightly flexible due to bendings of the faces and clearances at the vertices and edges.

2.1 Projection Theorem





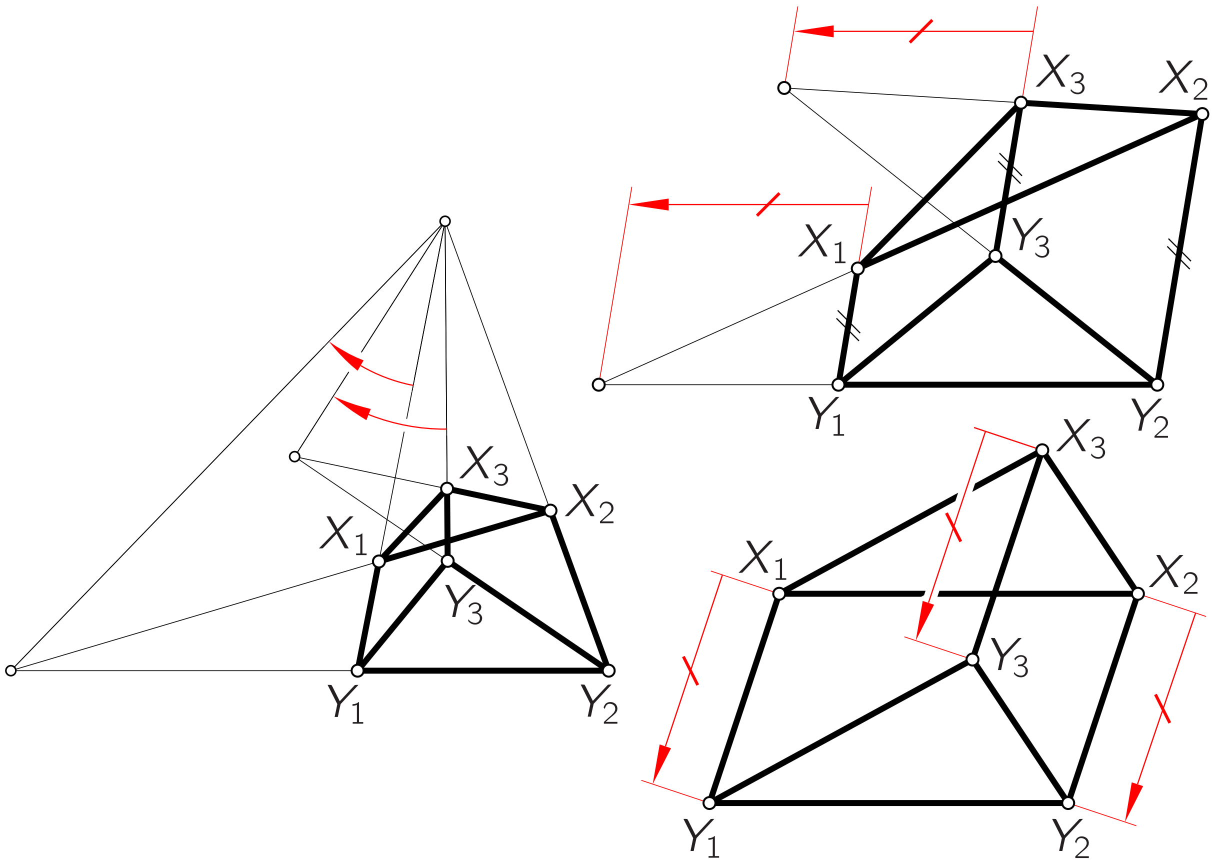
The assignment of velocity vectors to an infinitesimally flexible framework F is not unique. Apart from a scaling we can impose an infinitesimal motion, i.e., we can add at each vertex $\mathbf{x}_i$ the vector $\mathbf{s} + S\mathbf{x}_i$ where S is a skew-symmetric matrix.

2.2 Higher order infinitesimal flexibility

Definition: If there is a one-parameter family of frameworks F_t with vertices $\mathbf{x}_1(t), \ldots, \mathbf{x}_v(t)$ and $F_0 = F$ such that the function

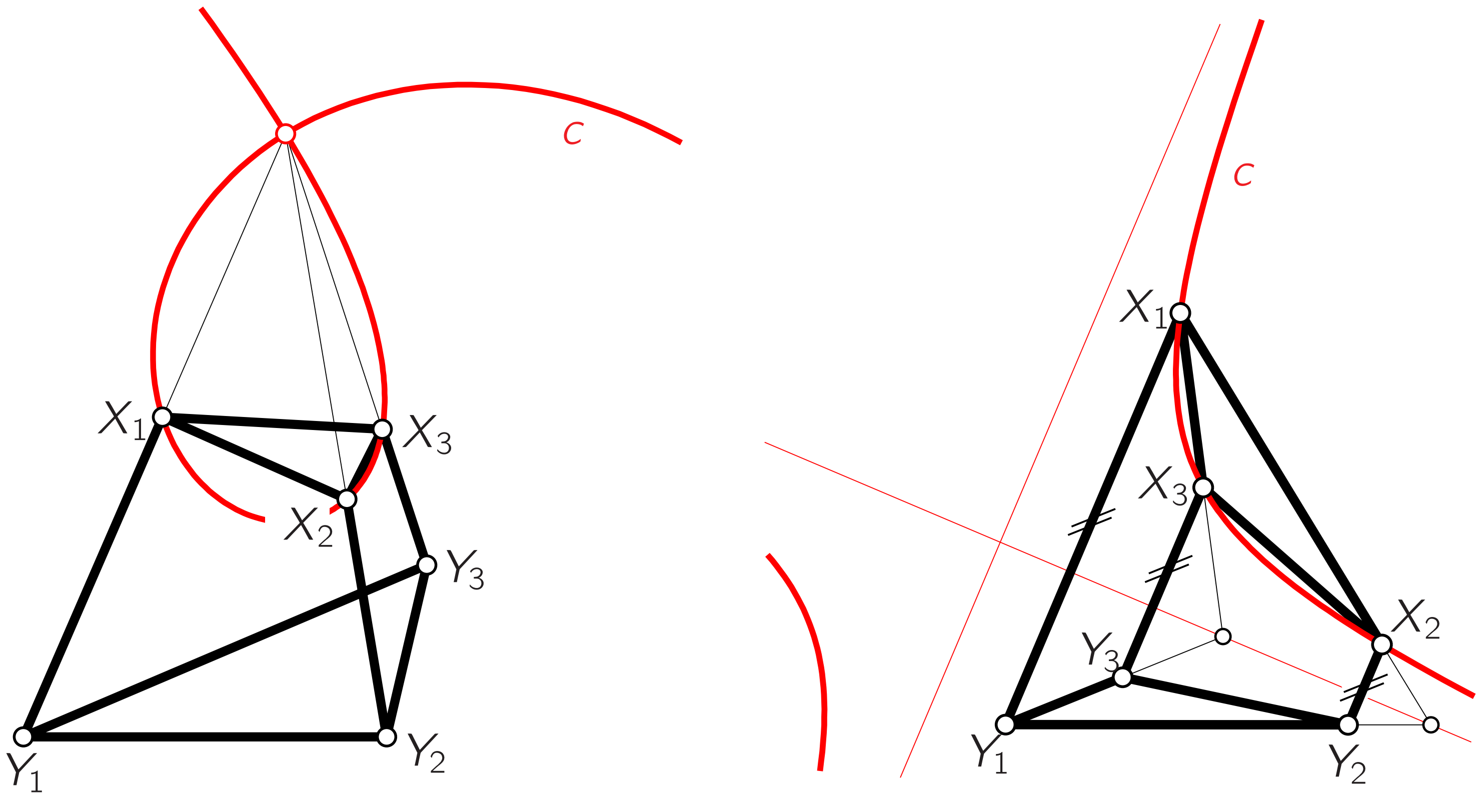
$$f(t) := \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| - l_{ij}$$

has a zero or order k at $t = 0$ for all bars, then F is called **infinitesimally flexible of order k**.

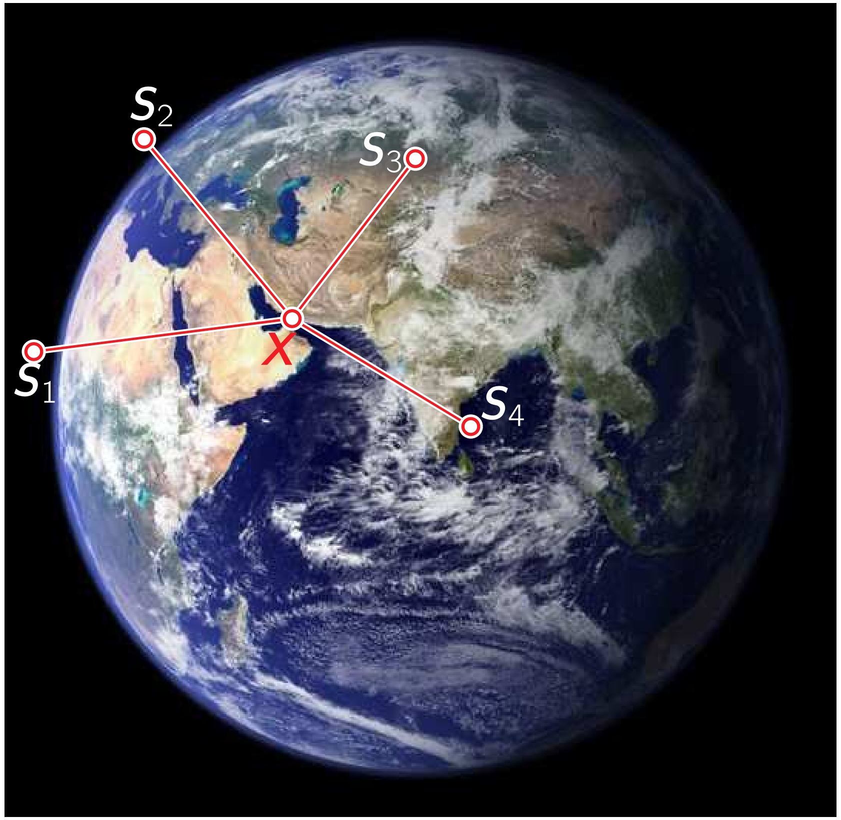


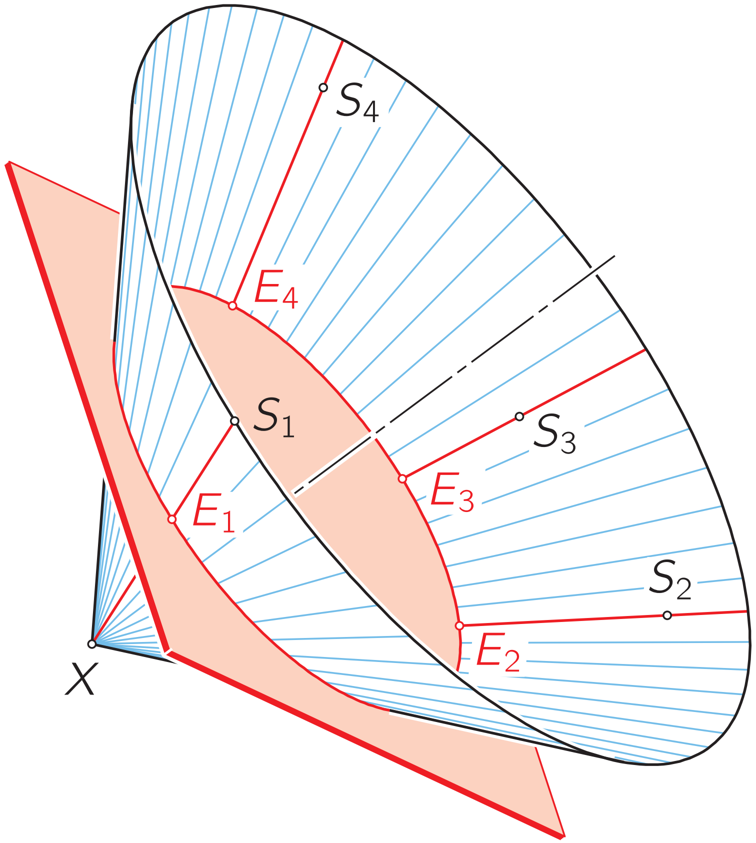
Above the condition for 2nd-order flexibility, which is no more projectively invariant.

2.2 Higher order infinitesimal flexibility



Two examples for 3rd-order flexible frameworks





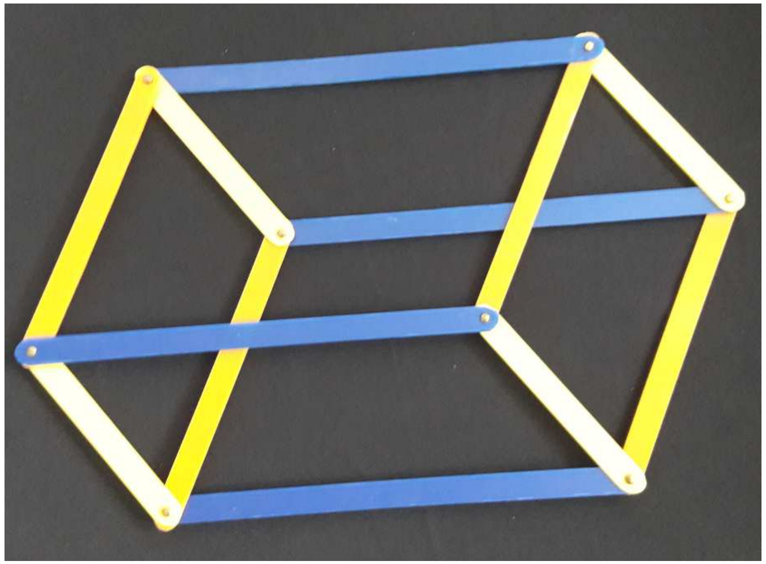
Global positioning: the position of $\mathbf{p}$ is computed from measured distances to 4 satellites $\mathbf{s}_1, \dots, \mathbf{s}_4$. These measures are determined up to a common additive error.

The configuration is critical (high Geometric Dilution of Precision), if the satellites $\mathbf{s}_1, \dots, \mathbf{s}_4$ are instantly located on a right cone with the apex at $\mathbf{p}$.

2.2 Related overconstrained mechanisms

If F is flexible of sufficiently high order then F is continuously flexible (overconstrained mechanism).

For frameworks derived from a four-bar the only continuously flexible version arises in the case of a parallelogram.

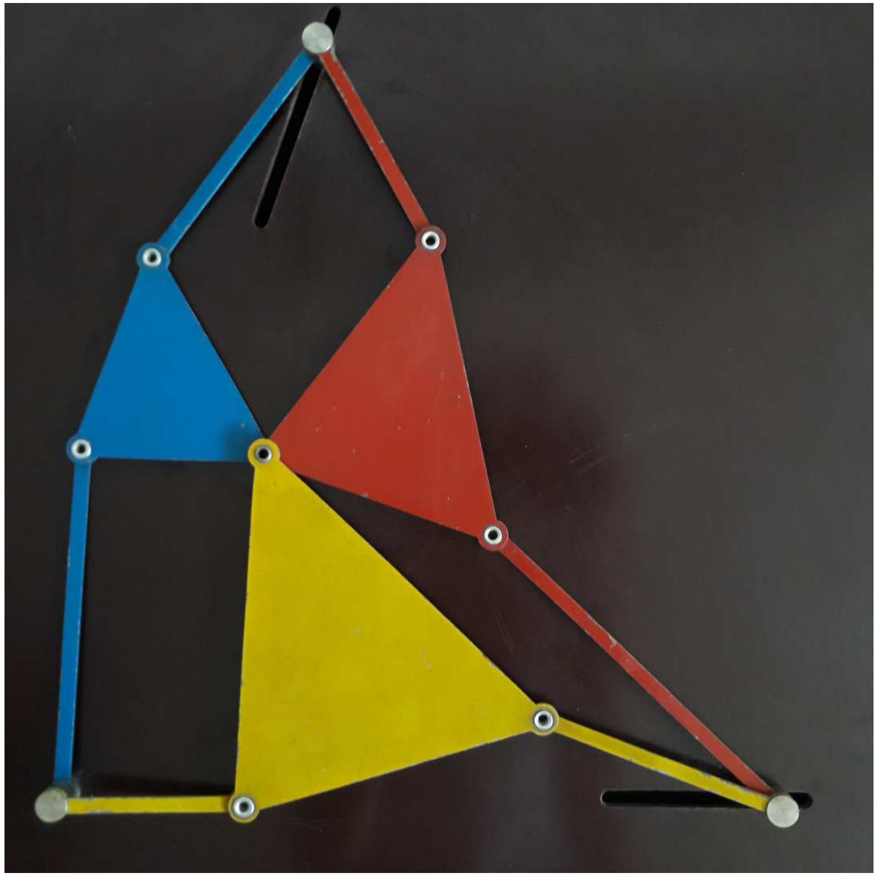


The aligned position admits bifurcations:

The degree of freedom is still two when four parallelograms change to antiparallelograms.

We obtain $\mathrm{dof}=1$ if only two opposite parallelograms become antiparallelograms.

2.2 Related overconstrained mechanisms



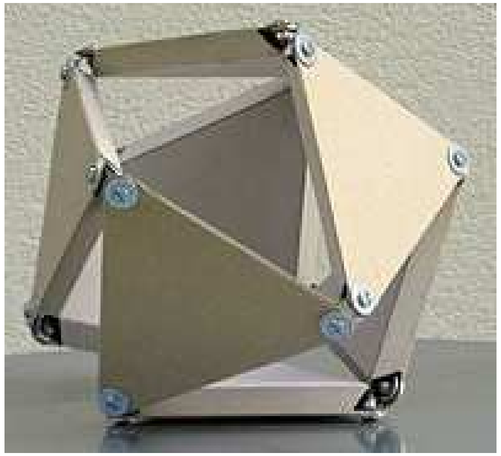
Coupler curves are curves of degree 6 with triple points at the absolute circle points.

The points of intersection between corresponding complex conjugate tangents are the base points of three four-bar mechanisms which share single coupler curve (= **Roberts' Theorem**).

2.2 Related overconstrained mechanisms

Overconstrained mechanisms are sensitive against imprecisions.

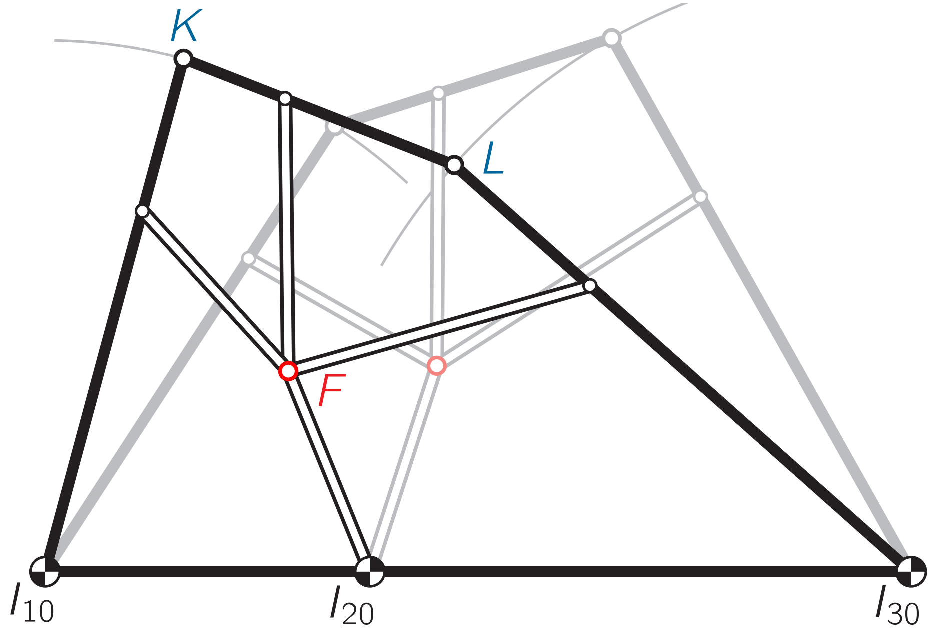
At the Science Exposition 1991 in Zürich the plates at this **Heureka-Polyhedron** (6 m side lengths) broke several times.



2.2 Related overconstrained mechanisms

Burmester's mechanism:

For each four-bar there are points F such that additional bars connecting F with appropriate points on the sides do not restrict the flexibility.



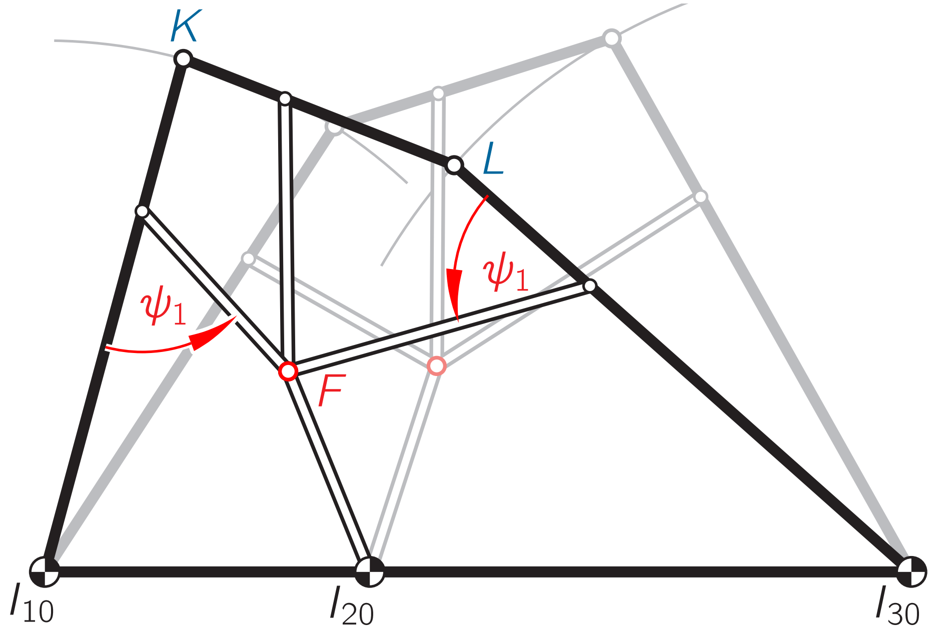
F is a focal point of any conic tangent to the four sides (L. Burmester 1893).

2.2 Related overconstrained mechanisms

Due to A. C. Dixon (1900), the angle ψ_1 shows up two times.

The angles at F are congruent to the interior angles of the quadrangle. Hence they sum up to $360^\circ \Longrightarrow$

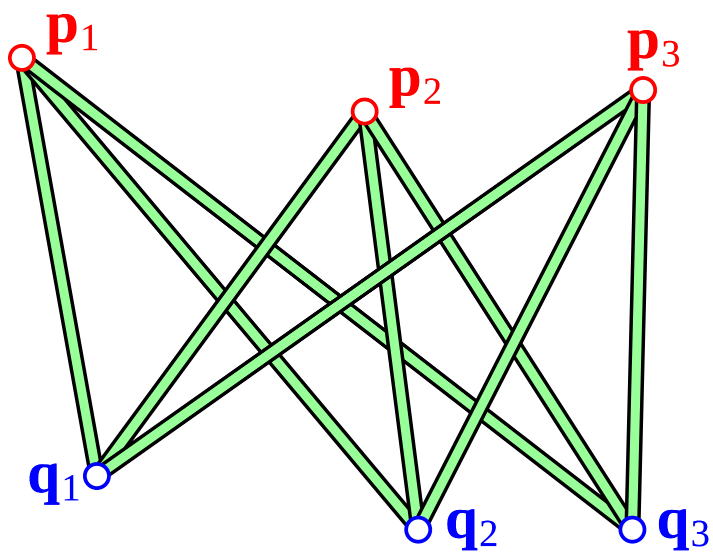
there is no spherical analogue!



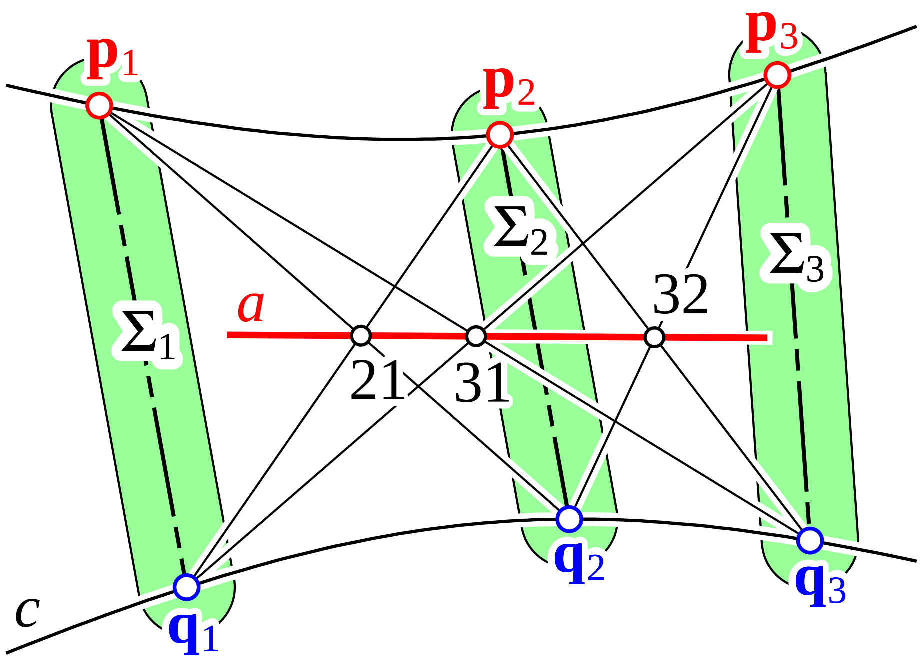
3. Bipartite frameworks

Three versions of a bipartite framework, rigid (left), infinitesimally flexible of order 1 (middle) and of order 2 (right).

3. Bipartite frameworks



bipartite graph $K_{3,3}$



Characterization of infinitesimal flexibility

3.1 Projective invariance

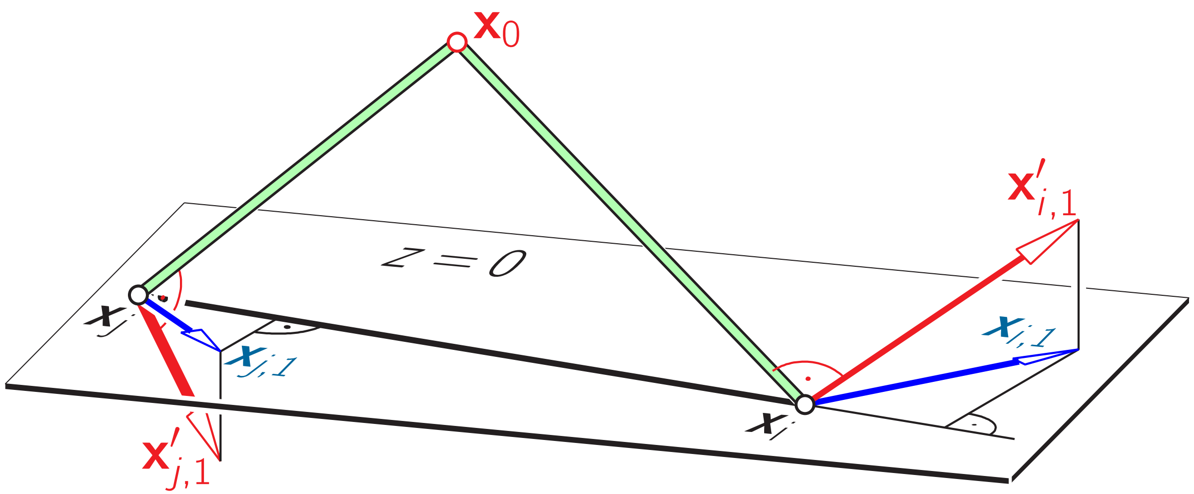
Theorem: Liebmann (1920)

Infinitesimal flexibility is projectively invariant.

Proof: (B. Wegner 1984)

The planar framework F with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_v$ is located in the plane $z = 0$.

We extend F to a conical framework F' in $\mathbb{R}^3$ by adding vertex $\mathbf{x}_0$ outside $z = 0$ and by inserting the v edges $\mathbf{x}_0\mathbf{x}_i$.

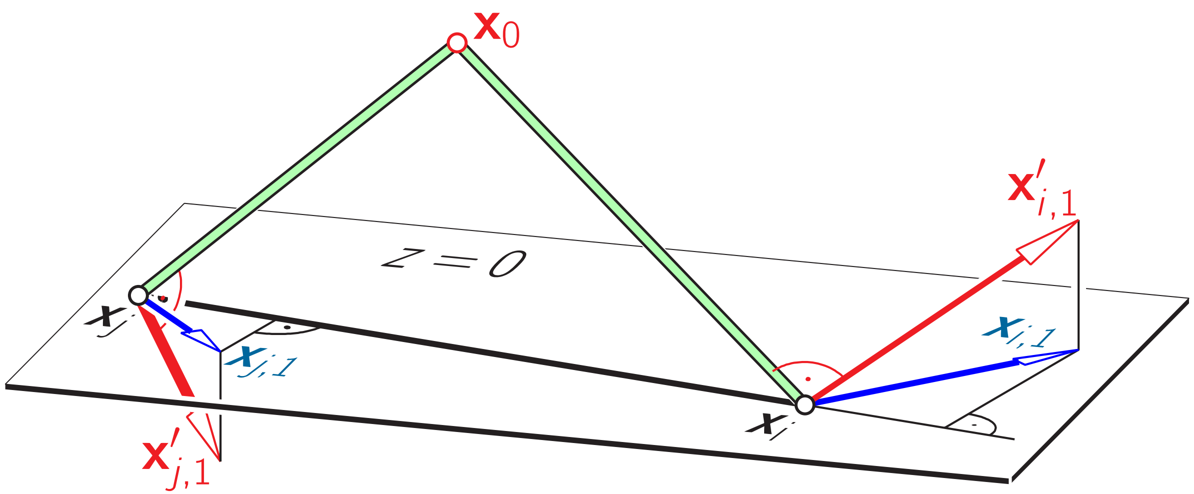


The extended framework F' actually consists of triangular plates $\mathbf{x}_0\mathbf{x}_i\mathbf{x}_j$.

3.1 Projective invariance

F' flexible $\iff$ F is flexible.

We set $\mathbf{x}'_{0,1} = 0$. And $\mathbf{x}'_{i,1}$ is specified perpendicular to $\mathbf{x}_0\mathbf{x}_i$ and its top view coincides with $\mathbf{x}_{i,1}$.
Thus all edges of F' the Projection Theorem is fulfilled.



The proof of the converse works similar. Hence each planar section of the conical framework F' is infinitesimally flexible, too.

3.2 Flipping bipartite frameworks

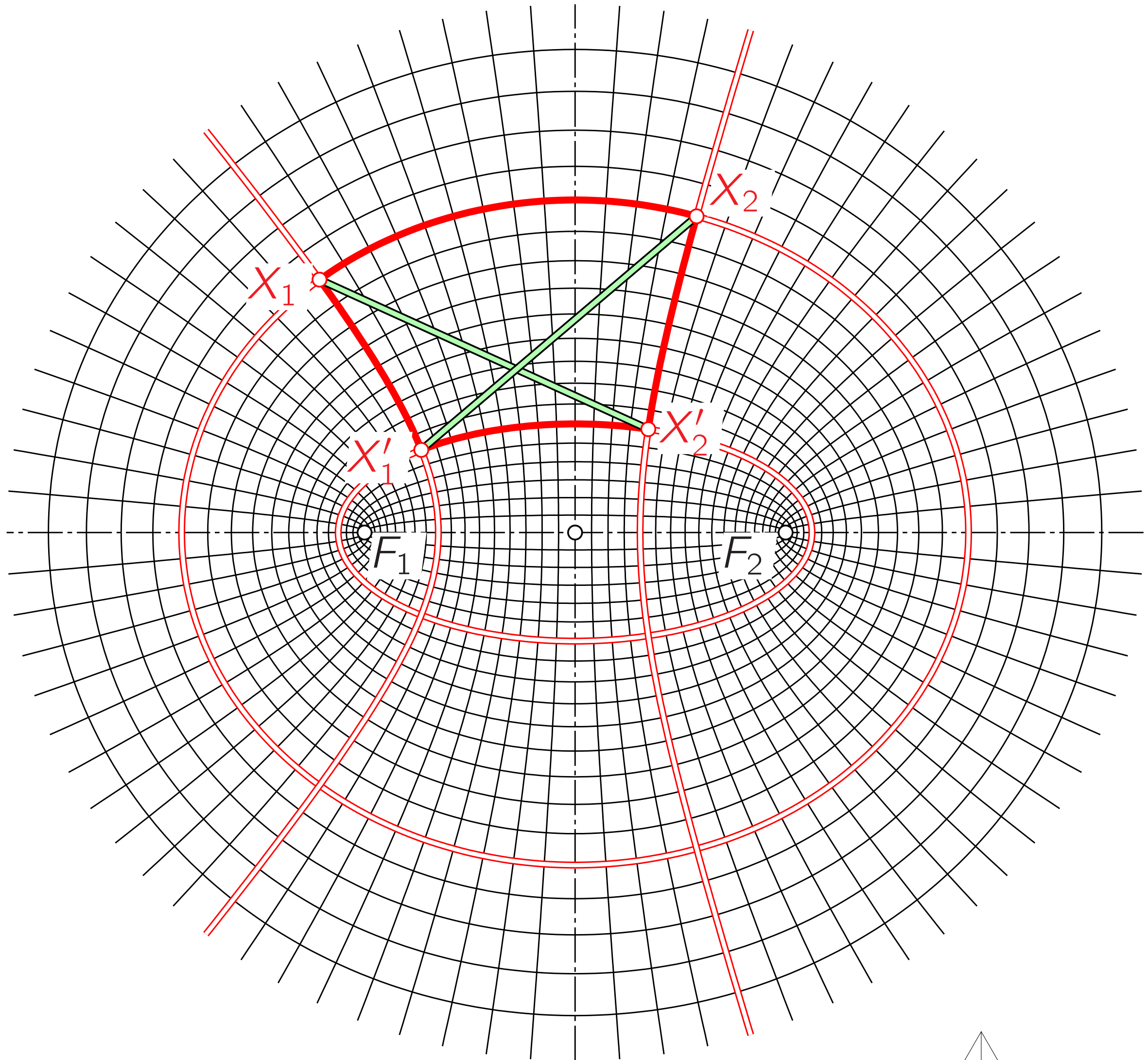
Given:

Net of *confocal conics* in the Euclidean plane $\mathbb{E}^2$:

Ivory's Theorem:[1]

$$\overline{X_1 X_2'} = \overline{X_1' X_2}$$

[1] J. Ivory, 1809



3.2 Flipping bipartite frameworks

Second explanation:

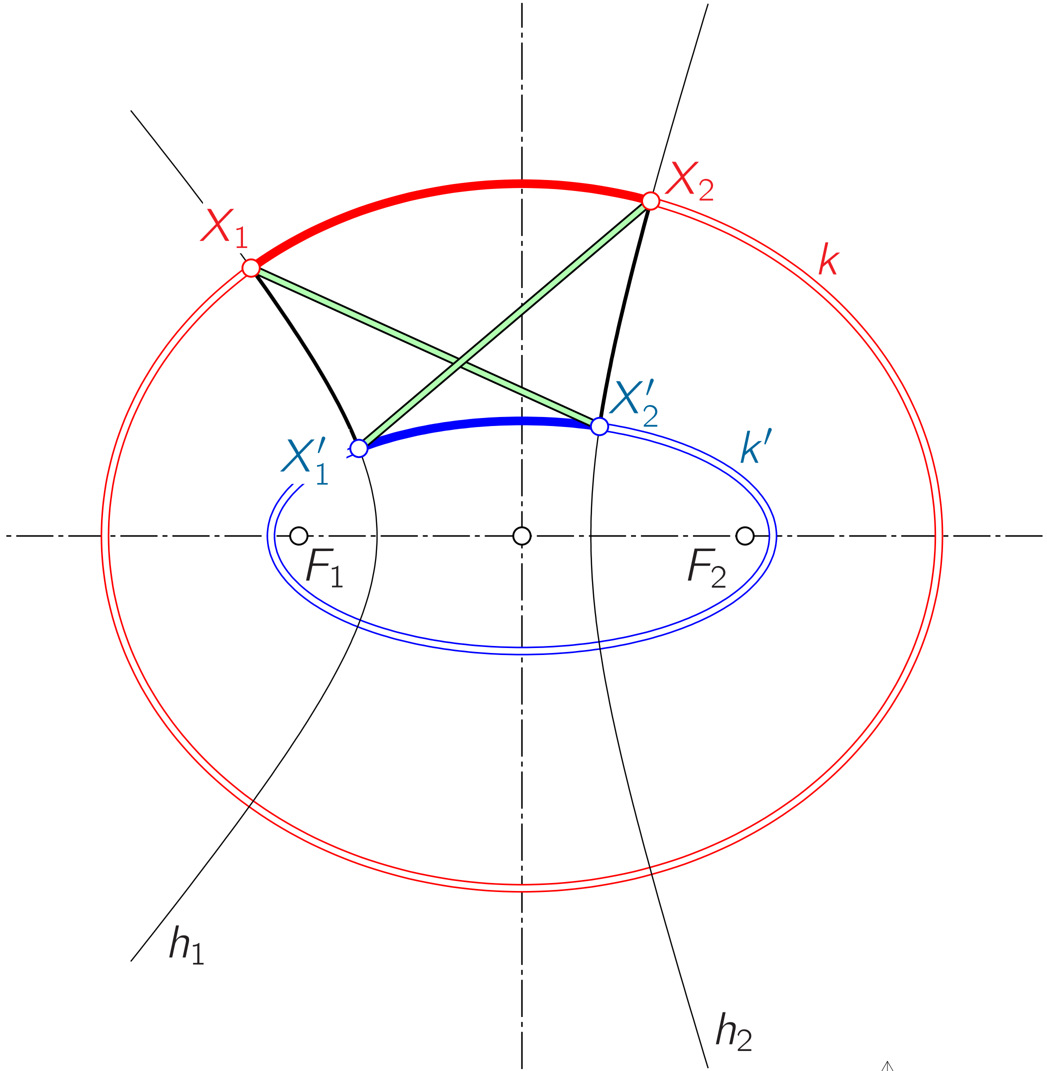
There is an *affine transformation* (scaling) between the ellipses

$$\alpha\colon\ k \mapsto k',\ X_i \mapsto X_i',\ i = 1, 2,$$

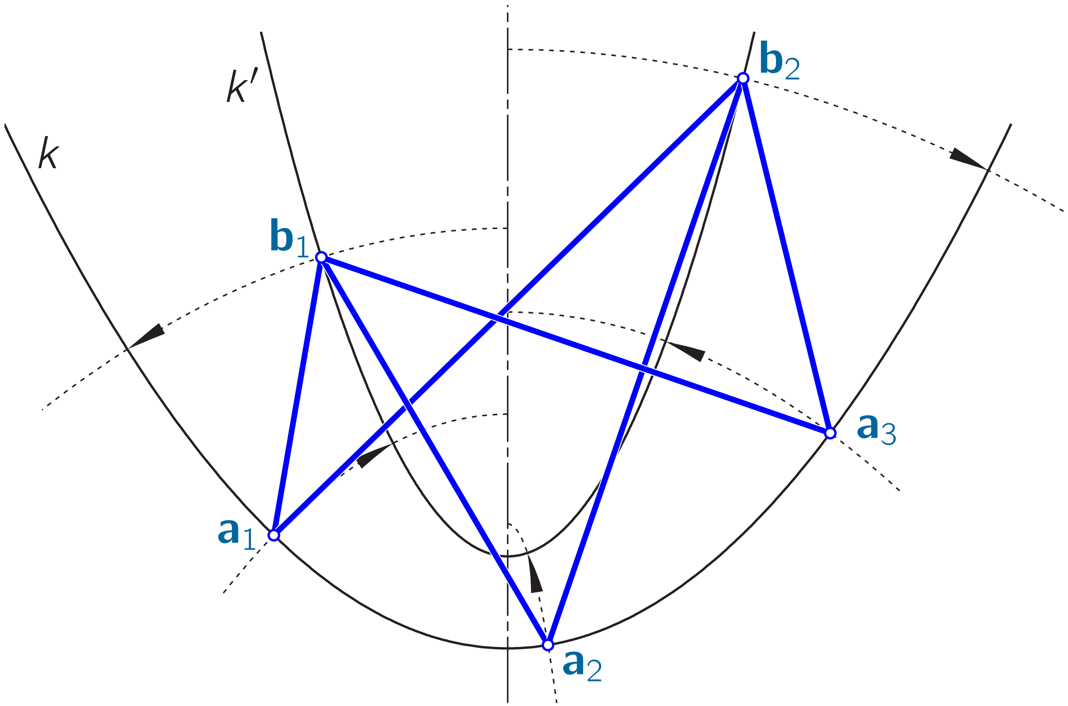
and

$$\overline{X_1\,\alpha(X_2)} = \overline{\alpha(X_1)\,X_2}.$$

X_i and $\alpha(X_i)$ are located on the same confocal hyperbola h_i, which intersects the ellipses orthogonally.



3.2 Flipping bipartite frameworks

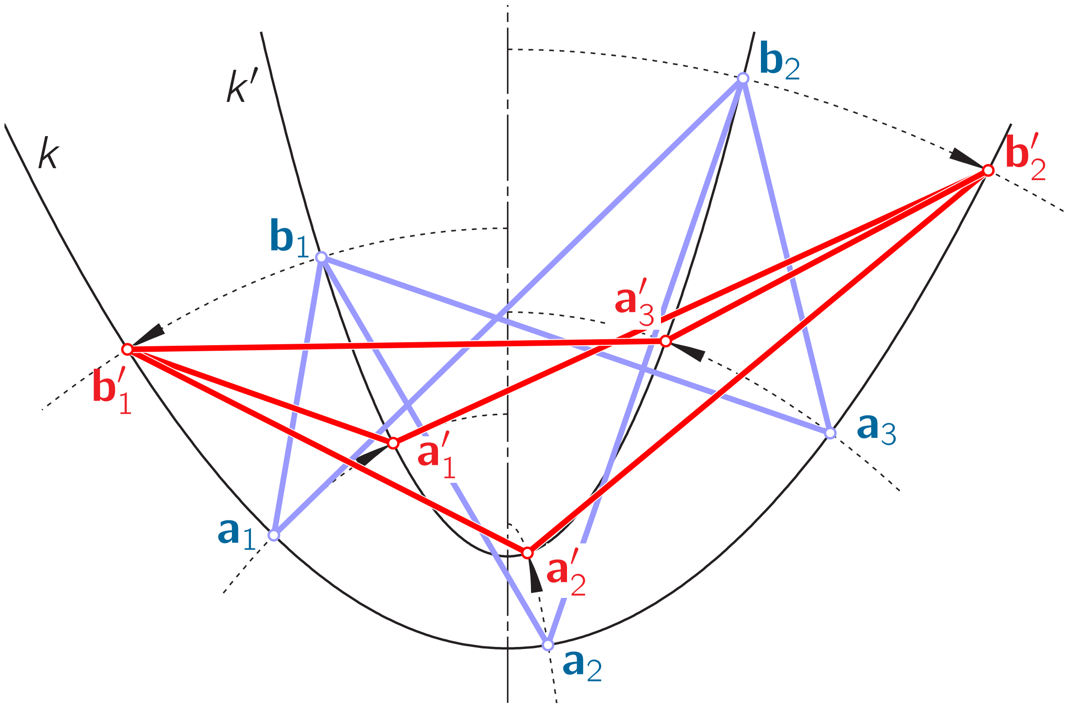


k and k' are confocal parabolas

Ivory's Theorem applied to bipartite frameworks:

Let the knots $\mathbf{a}_i$ and $\mathbf{b}_j$ of the two classes be placed on k and k', respectively.

3.2 Flipping bipartite frameworks



k and k' are confocal parabolas

Ivory's Theorem applied to bipartite frameworks:

Now we replace the knots $\mathbf{a}_i$ and $\mathbf{b}_j$ by their respectively conjugate knots $\mathbf{a}'_i$ and $\mathbf{b}'_j$ and obtain a second incongruent realization of the same framework.

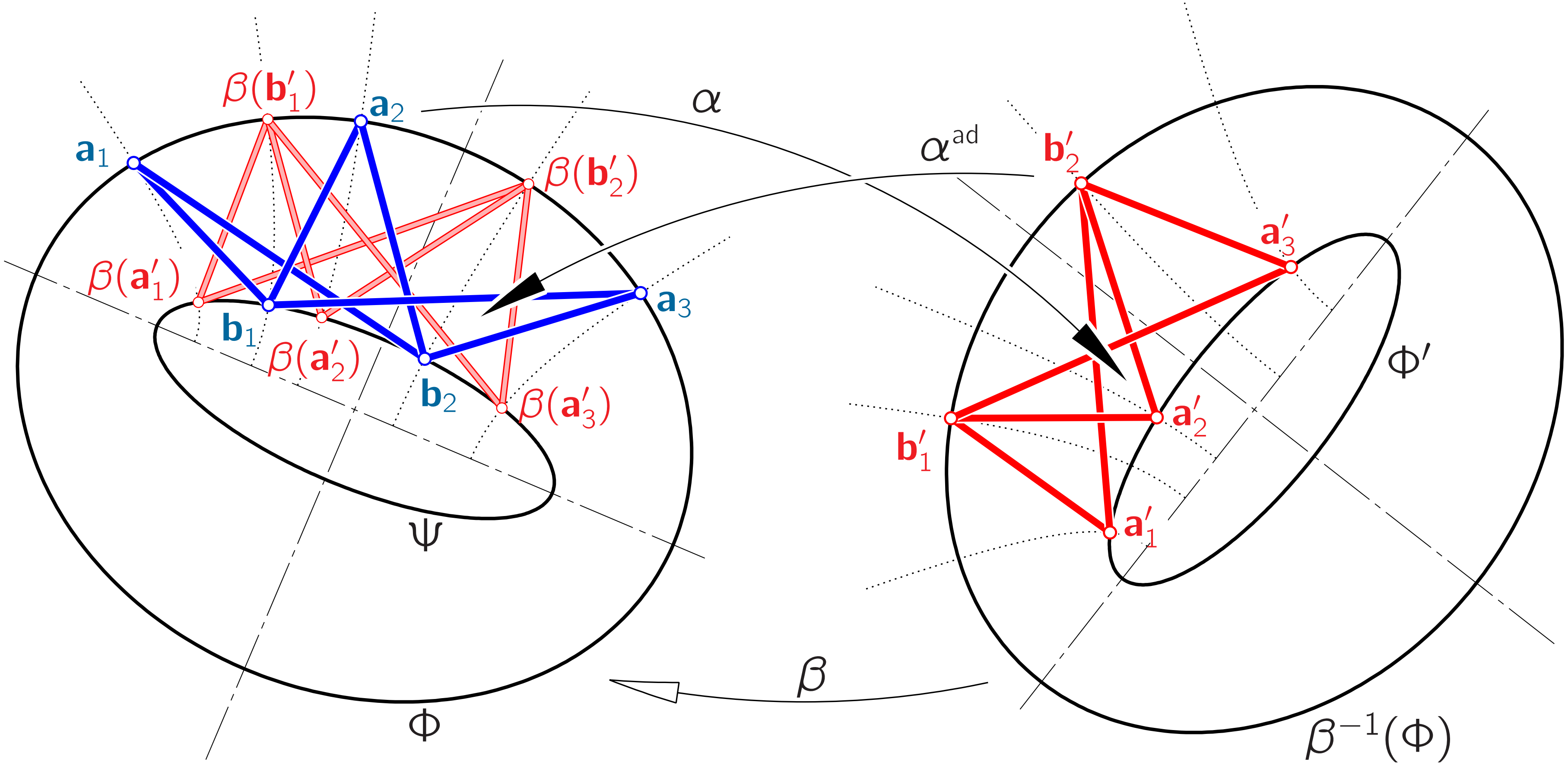
3.2 Flipping bipartite frameworks

Converse of Ivory's Theorem:
Let $\mathcal{F}$ and $\mathcal{F}'$ be two incongruent realizations of a complete bipartite framework in $\mathbb{E}^n$.

- There is an appropriate displacement $\beta : \mathbb{E}^n \to \mathbb{E}^n$ such that for $\mathcal{F}$ and the displaced $\beta(\mathcal{F}')$ are in Ivory position with respect to two confocal quadrics.

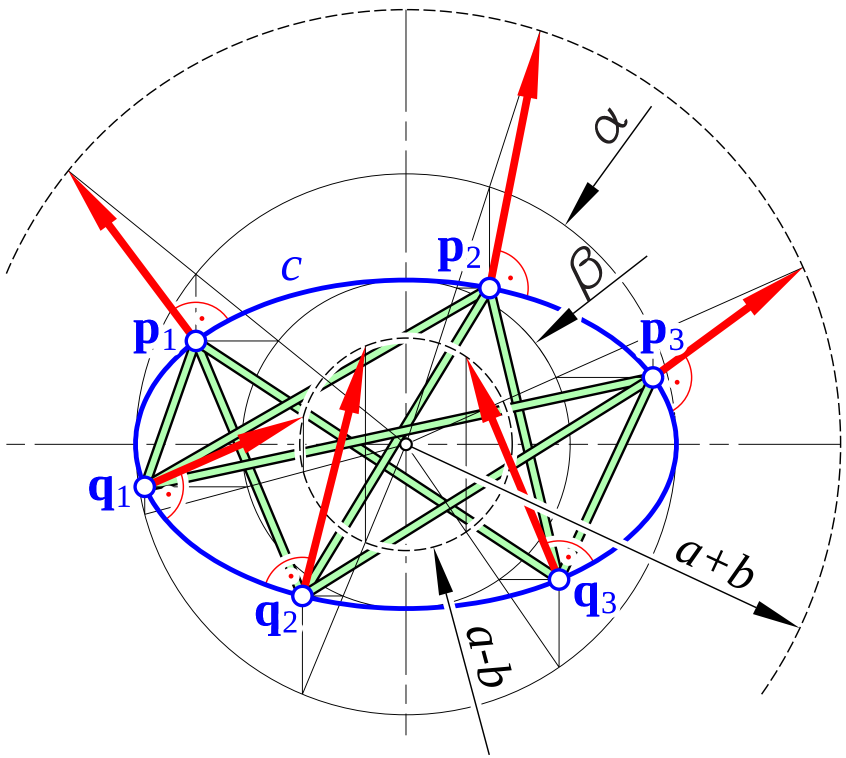
Confocal quadrics are characterized by confocal principal sections.

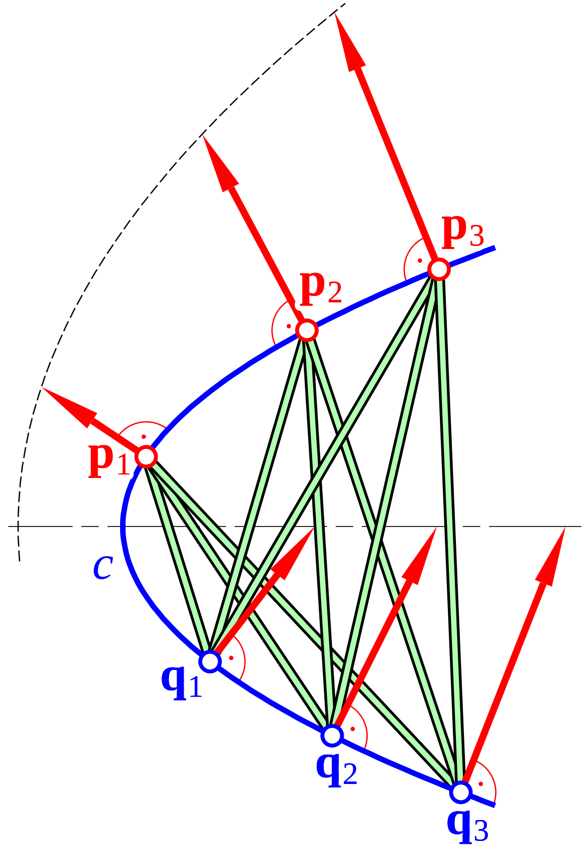
3.2 Flipping bipartite frameworks



$$\|\alpha(\mathbf{a}_i) - \mathbf{b}'_j\| = \|\mathbf{a}_i - \alpha^{\mathrm{ad}}(\mathbf{b}'_j)\|$$

3.3 Infinitesimal flexibility of order 1





two examples with infinitesimal mobility

The velocity vectors are orthogonal to the conic c

3.3 Infinitesimal flexibility of order 1

W. Whiteley, 1984:
Bipartite framework in $\mathbb{P}^n$ with vertices $\mathbf{p}_i$, $\mathbf{q}_j$.

Let $q(\mathbf{x}) = \mathbf{x}^T M\,\mathbf{x}$ be a quadratic form vanishing on all $\mathbf{p}_i$ and $\mathbf{q}_j$. Then the assignment of velocities

$$\mathbf{p} \mapsto \mathbf{v}_\mathbf{p} = M\,\mathbf{p}, \quad \mathbf{q} \mapsto \mathbf{v}_\mathbf{q} = -M\,\mathbf{q}$$

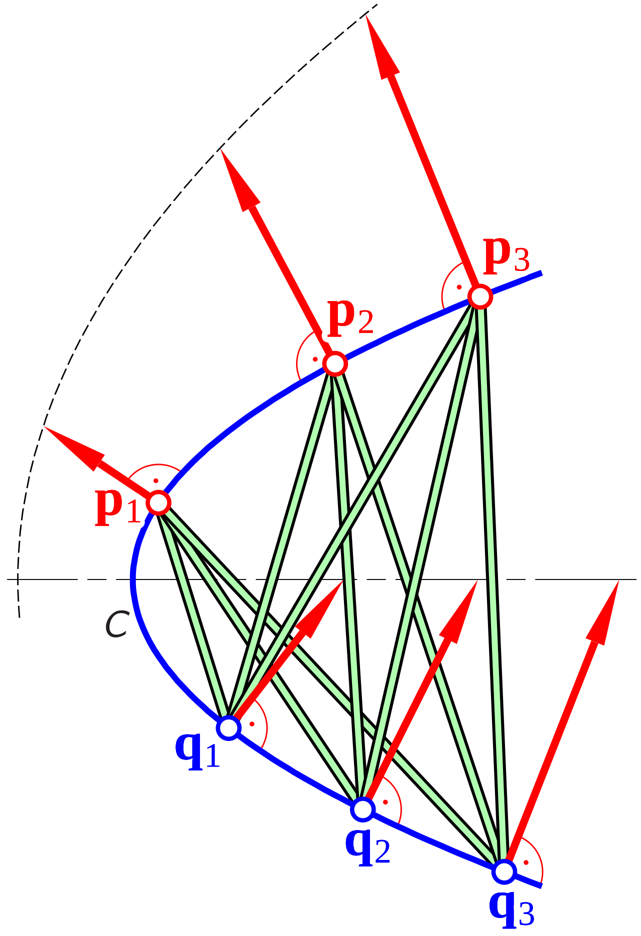
gives an infinitesimal flex.

Proof: $\mathbf{p}^T M\,\mathbf{p} = \mathbf{q}^T M\,\mathbf{q} = 0 \implies$ (Proj.Th.)

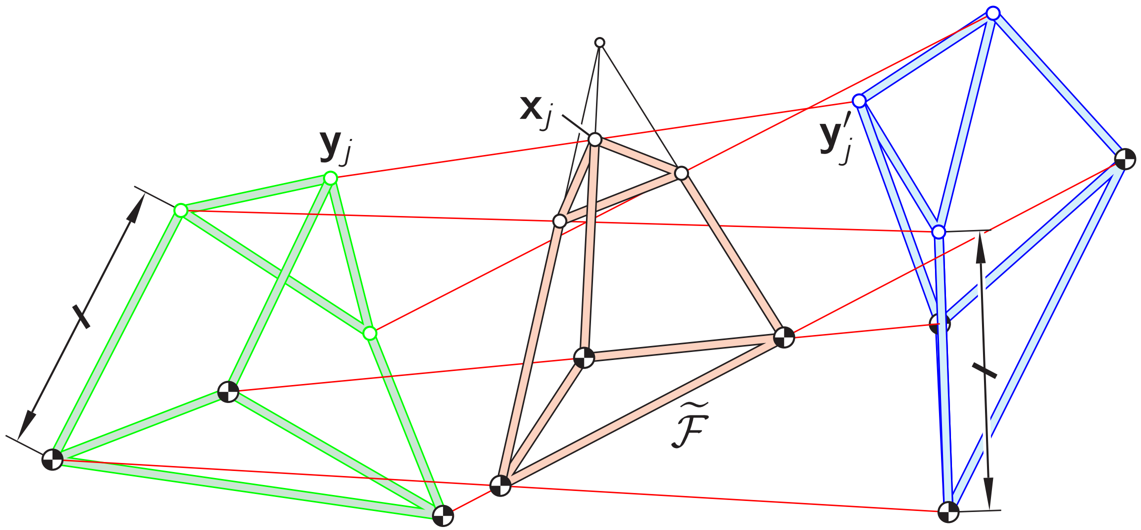
$$(\mathbf{p}-\mathbf{q})^T(\mathbf{v}_\mathbf{p}-\mathbf{v}_\mathbf{q}) = (\mathbf{p}-\mathbf{q})^T(M\,\mathbf{p}+M\,\mathbf{q}) =$$

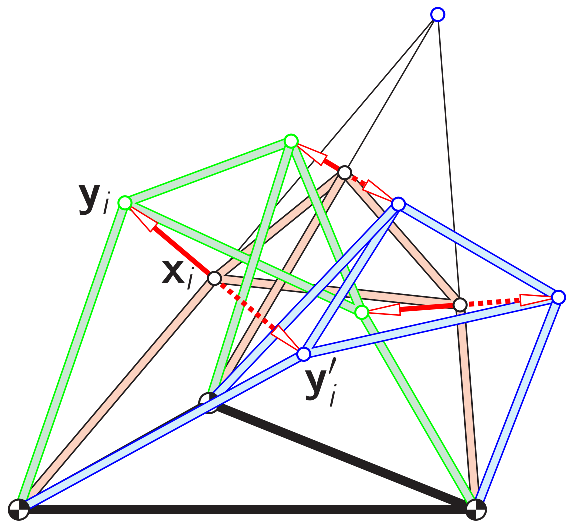
$$= \mathbf{p}^T M\,\mathbf{p} - \mathbf{q}^T M\,\mathbf{q} + \mathbf{p}^T M\,\mathbf{q} - \mathbf{q}^T M\,\mathbf{p} = 0.$$

Also the converse is true.



3.3 Infinitesimal flexibility of order 1

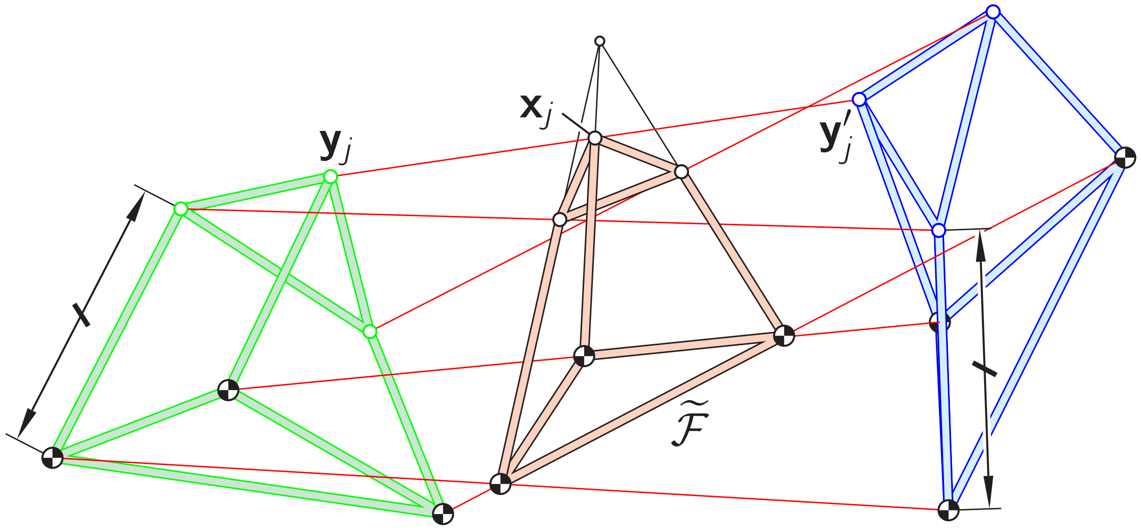


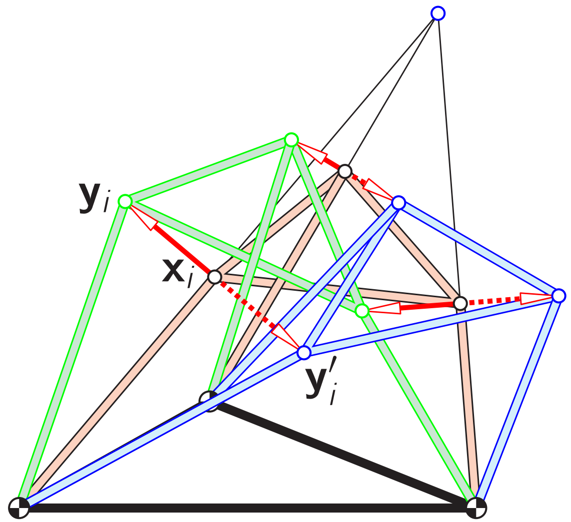


Theorem (W. Whiteley (1990)), *Principle of* ***averaging****:*

Let $\mathbf{y}_1, \ldots, \mathbf{y}_v$ and $\mathbf{y}'_1, \ldots, \mathbf{y}'_v$ be vertices of two incongruent realizations of a framework $\mathcal{F}$. Then the midpoints $\mathbf{x}_i = \frac{1}{2}(\mathbf{y}_i + \mathbf{y}'_i)$ constitute an infinitesimally flexible framework $\widetilde{\mathcal{F}}$ of the same combinatorial structure with velocity vectors $\mathbf{x}_{i,1} = \frac{1}{2}(\mathbf{y}_i - \mathbf{y}'_i)$, and vice versa — provided ...

3.3 Infinitesimal flexibility of order 1





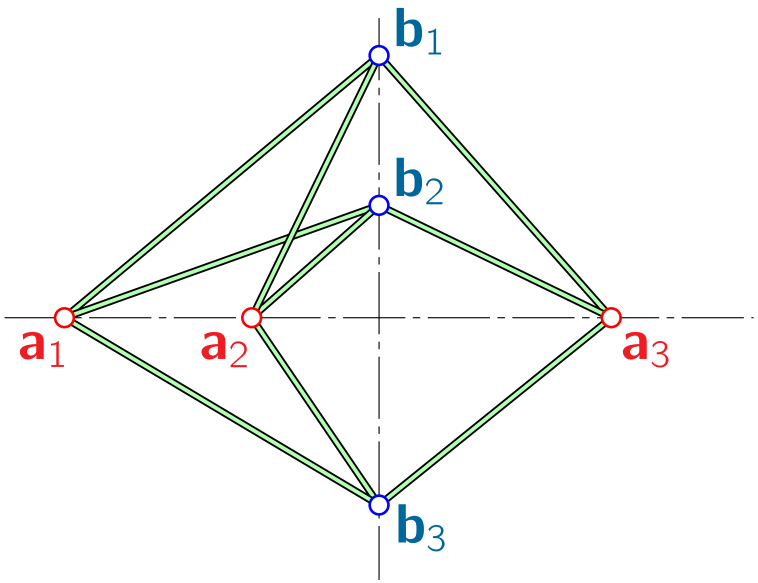
Proof: The condition $(\mathbf{y}_i - \mathbf{y}_j)^2 - (\mathbf{y}'_i - \mathbf{y}'_j)^2 = 0$ can be rewritten as

$$(\mathbf{y}_i - \mathbf{y}_j + \mathbf{y}'_i - \mathbf{y}'_j) \cdot (\mathbf{y}_i - \mathbf{y}_j - \mathbf{y}'_i + \mathbf{y}'_j) = 0$$

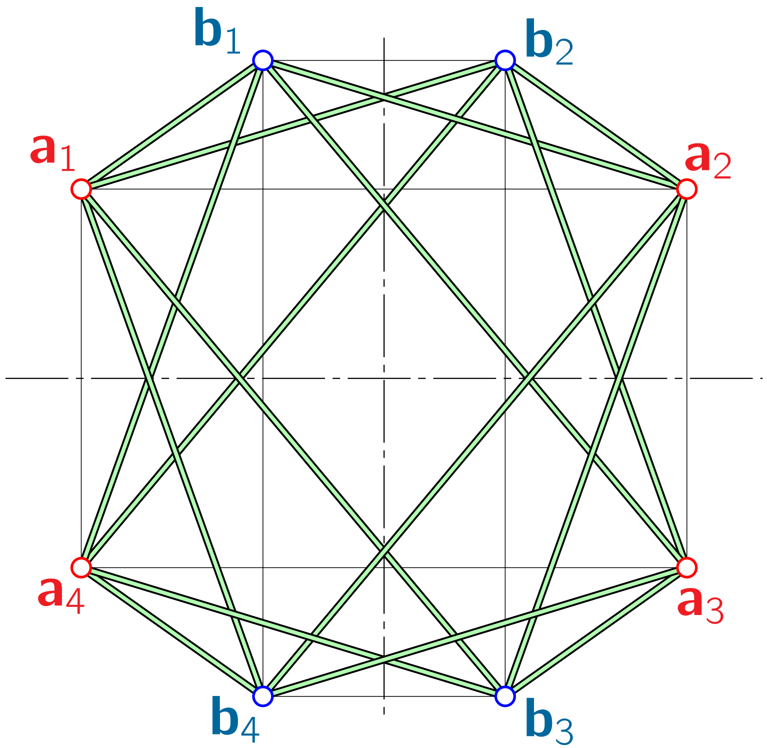
$$\Big(\underbrace{(\mathbf{y}_i + \mathbf{y}'_i)}_{2\mathbf{x}_i} - \underbrace{(\mathbf{y}_j + \mathbf{y}'_j)}_{2\mathbf{x}_j}\Big) \cdot \Big(\underbrace{(\mathbf{y}_i - \mathbf{y}'_i)}_{2\,\mathbf{v}_i} - \underbrace{(\mathbf{y}_j - \mathbf{y}'_j)}_{2\,\mathbf{v}_j}\Big) = 0 \; \ldots \; \text{Projection Thm.}$$

3.4 Dixon's flexible bipartite frameworks

Due to A.C. Dixon (1899) there are two continuously flexible bipartite frameworks in $\mathbb{E}^2$:



Dixon I (unsymmetric):
$$x'^2 = x^2 + c, \quad y'^2 = y^2 - c$$



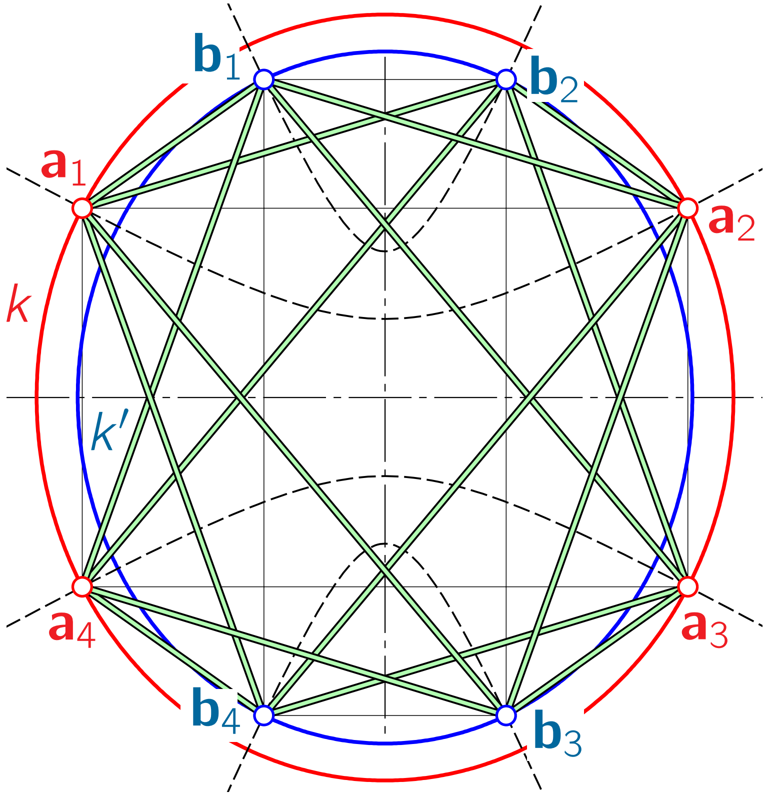
Dixon II (symmetric)

3.4 Dixon's flexible bipartite frameworks

We prove the flexibility of the Dixon II framework with the help of Ivory's Theorem:

There is a one-parameter set of conics k passing through $\mathbf{a}_1, \ldots, \mathbf{a}_4$. They have the same axes.

For each k there is a confocal conic k' through $\mathbf{b}_1, \ldots, \mathbf{b}_4$. Hence, by Ivory's Theorem we can switch to conjugate points thus obtaining a one-parameter set of incongruent realizations of the same framework.

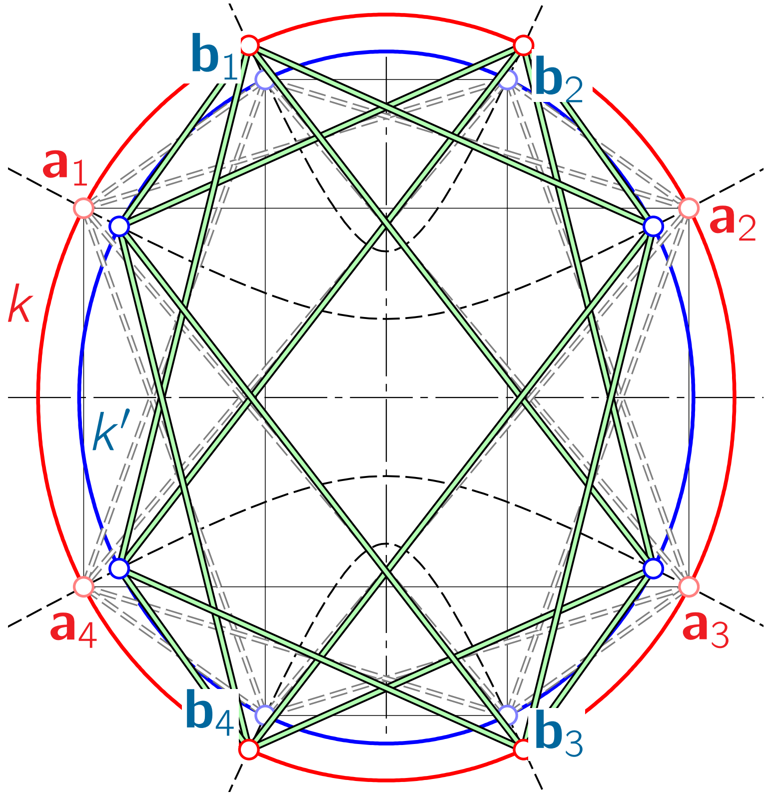


3.4 Dixon's flexible bipartite frameworks

We prove the flexibility of the Dixon II framework with the help of Ivory's Theorem:

There is a one-parameter set of conics k passing through $\mathbf{a}_1, \dots, \mathbf{a}_4$. They have the same axes.

For each k there is a confocal conic k' through $\mathbf{b}_1, \dots, \mathbf{b}_4$. Hence, by Ivory's Theorem we can switch to conjugate points thus obtaining a one-parameter set of incongruent realizations of the same framework.



3.4 Dixon's flexible bipartite frameworks

Spherical case:
Ivory's Theorem is true on the sphere $\mathbb{S}^2$ ($\sim$ elliptic plane).

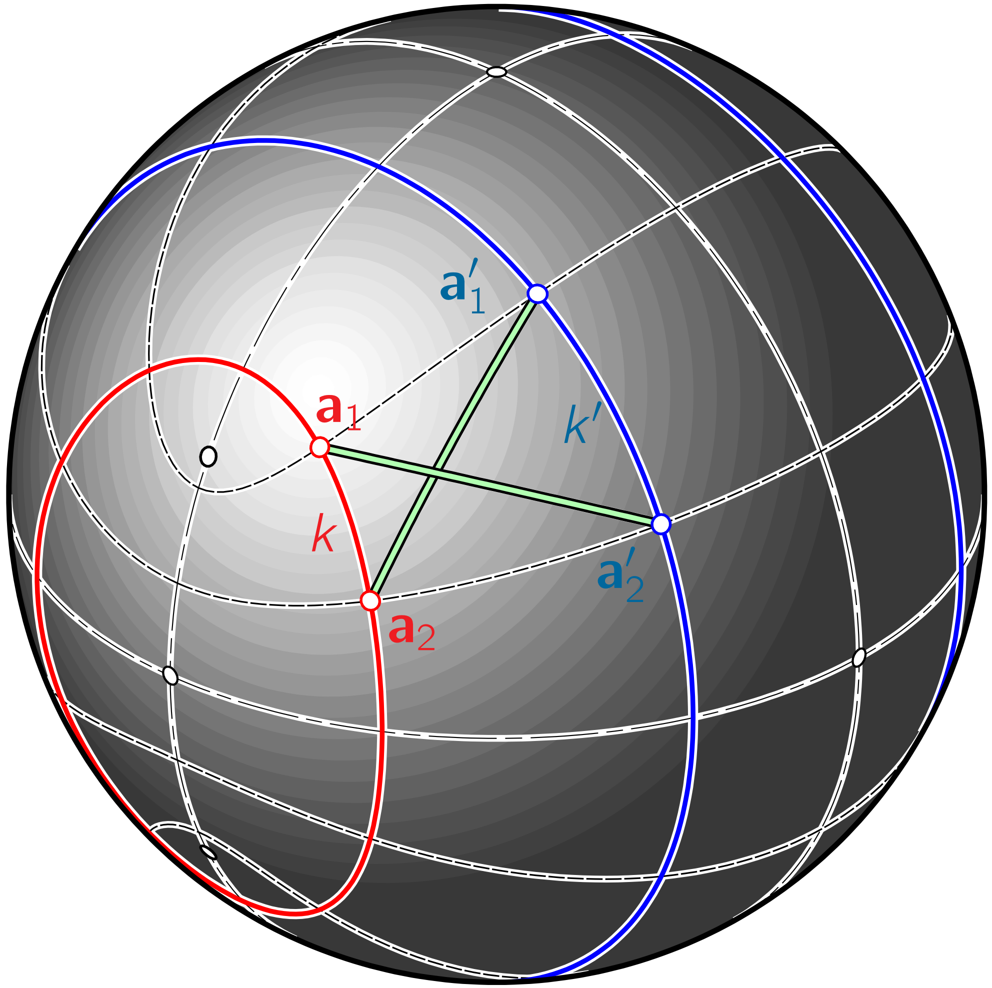
There is a linear map

$$\alpha:\ k \mapsto k',\ \mathbf{a}_i \mapsto \mathbf{a}_i',\ \ i = 1, 2,$$

and

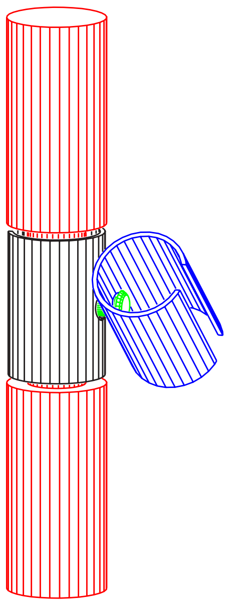
$$\alpha(\mathbf{a}_1) \cdot \mathbf{a}_2' = \mathbf{a}_1 \cdot \alpha^{\mathrm{ad}}(\mathbf{a}_2').$$

The spherical version of the Dixon-II framework is called O. Bottema's 16-bar framework

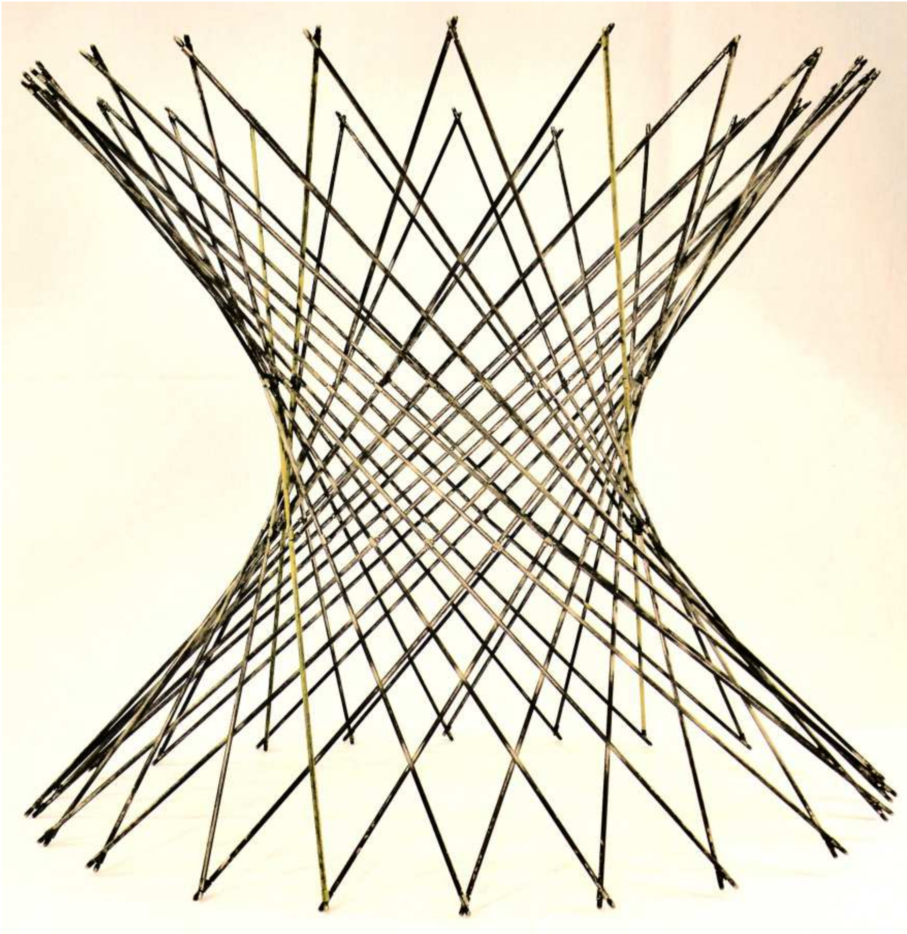


3.5 Confocal ruled quadrics

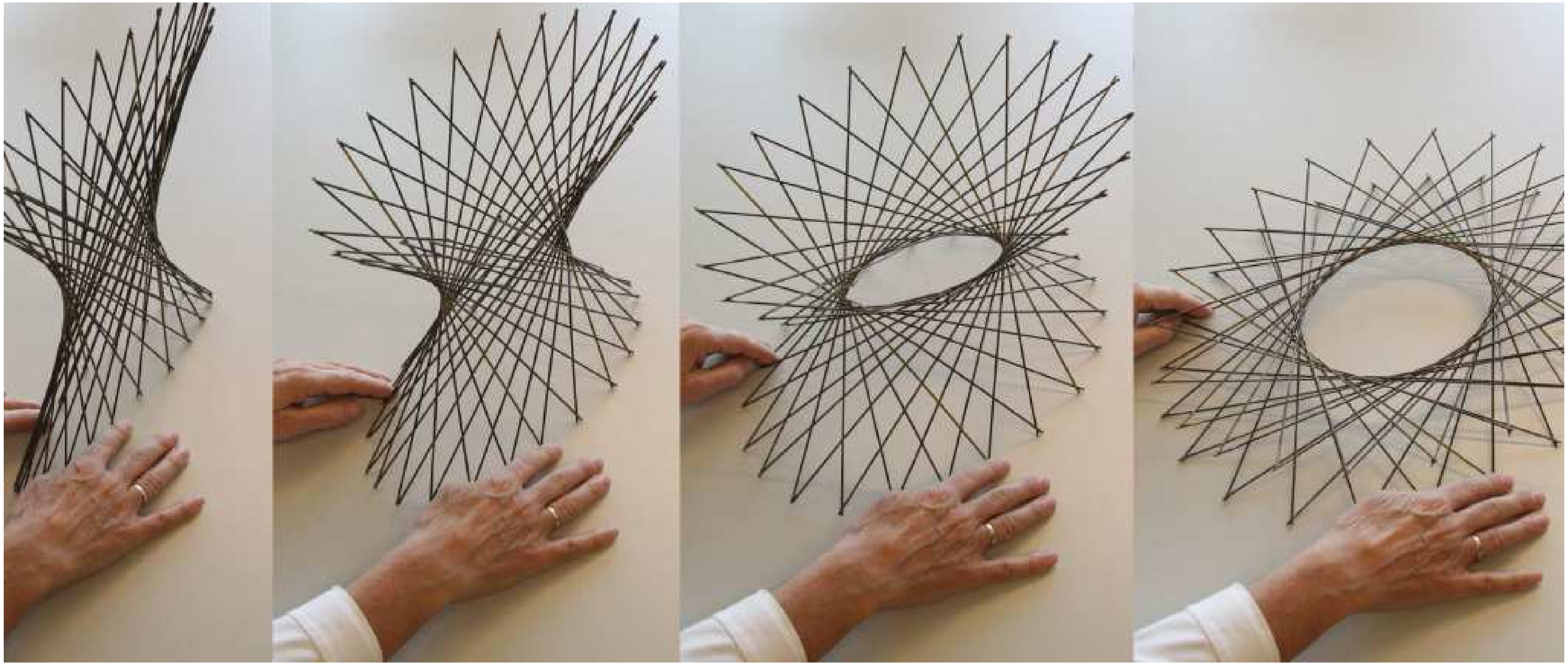
For two confocal one-sheet hyperboloids the affinity according to Ivory's Theorem preserves distances along the generators.



This is the basis for O. Henrici's flexible hyperboloid (1874).



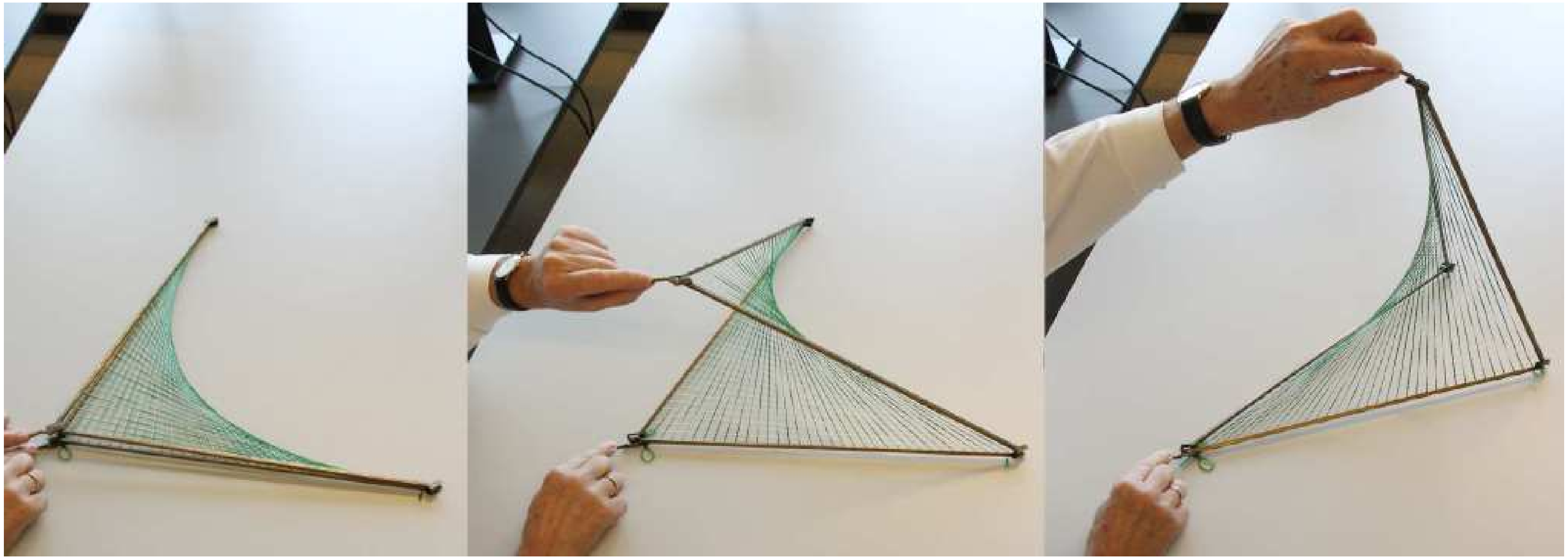
3.5 Confocal ruled quadrics



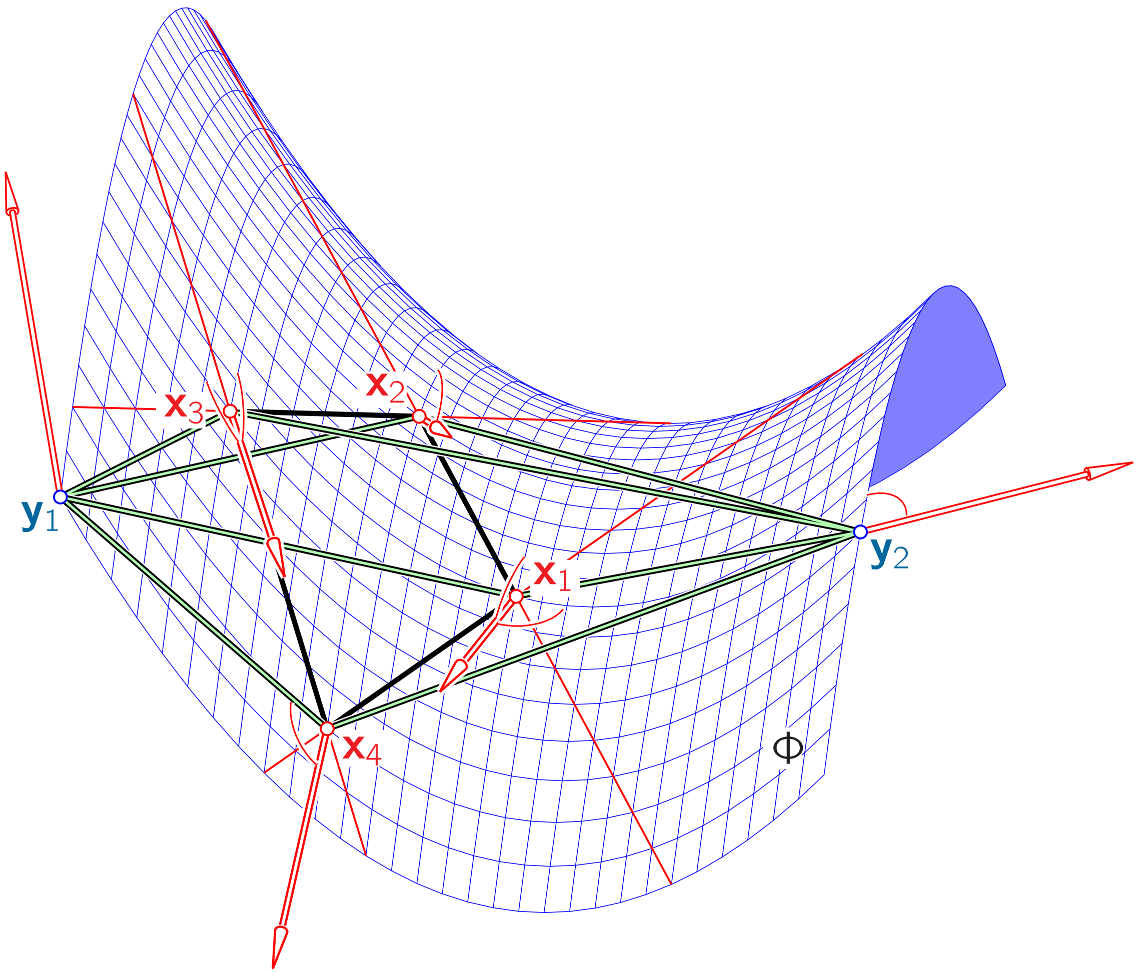
The flat limiting poses give tangent lines of the focal hyperbola and focal ellipse, the singular surfaces in the range of confocal hyperboloids.

3.5 Confocal ruled quadrics

The length-preserving property along the generators holds also for confocal hyperbolic paraboloids. This is used at a flexible model of hyperbolic paraboloids. All strings remain under tension during the flex.

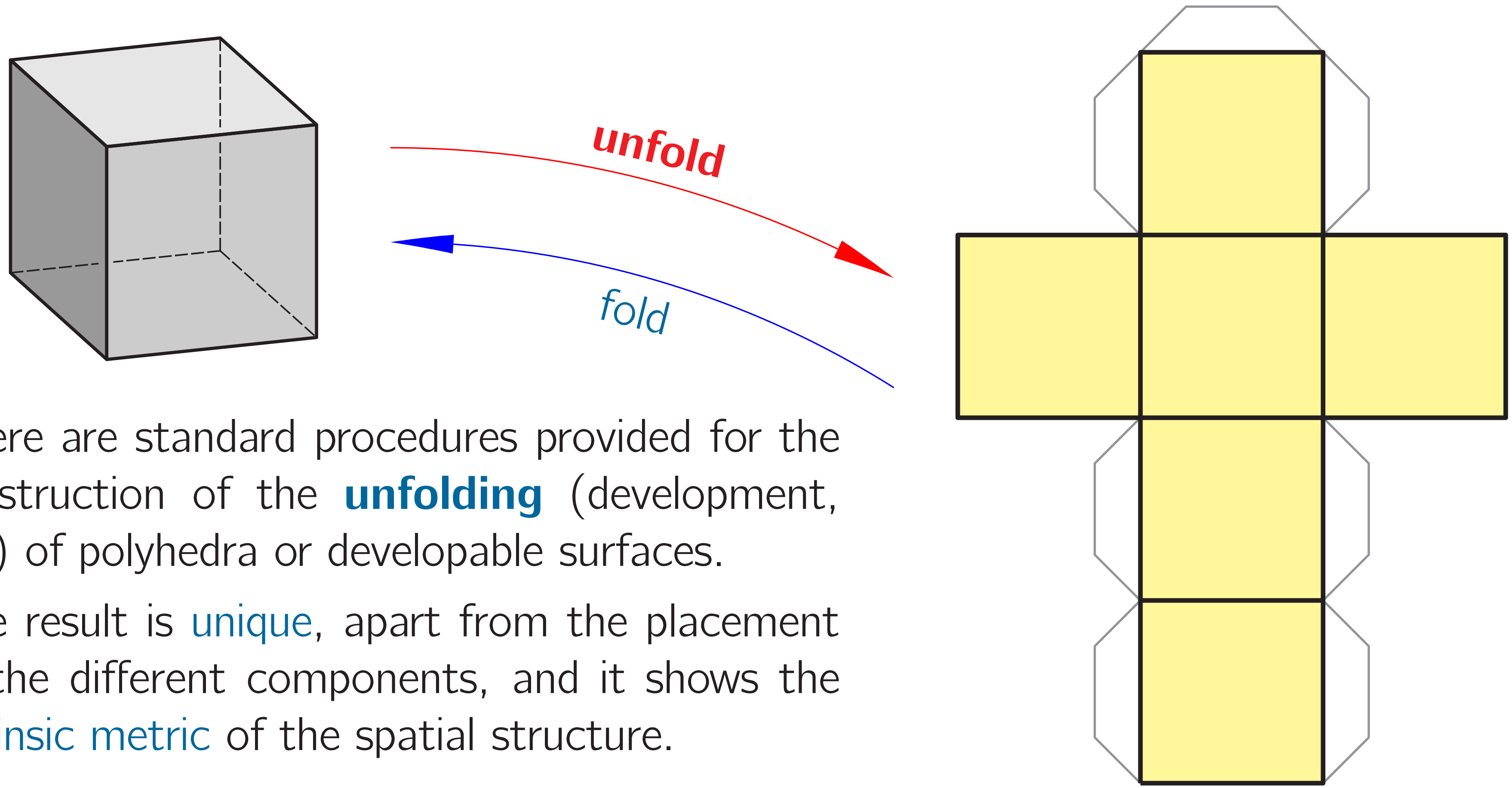


3.5 Confocal ruled quadrics



An octahedron, i.e., a four-sided double-pyramid, is infinitesimally flexible $\iff$ there is a quadric containing the sides of the basis $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4$ and the two apices $\mathbf{y}_1\mathbf{y}_2$.

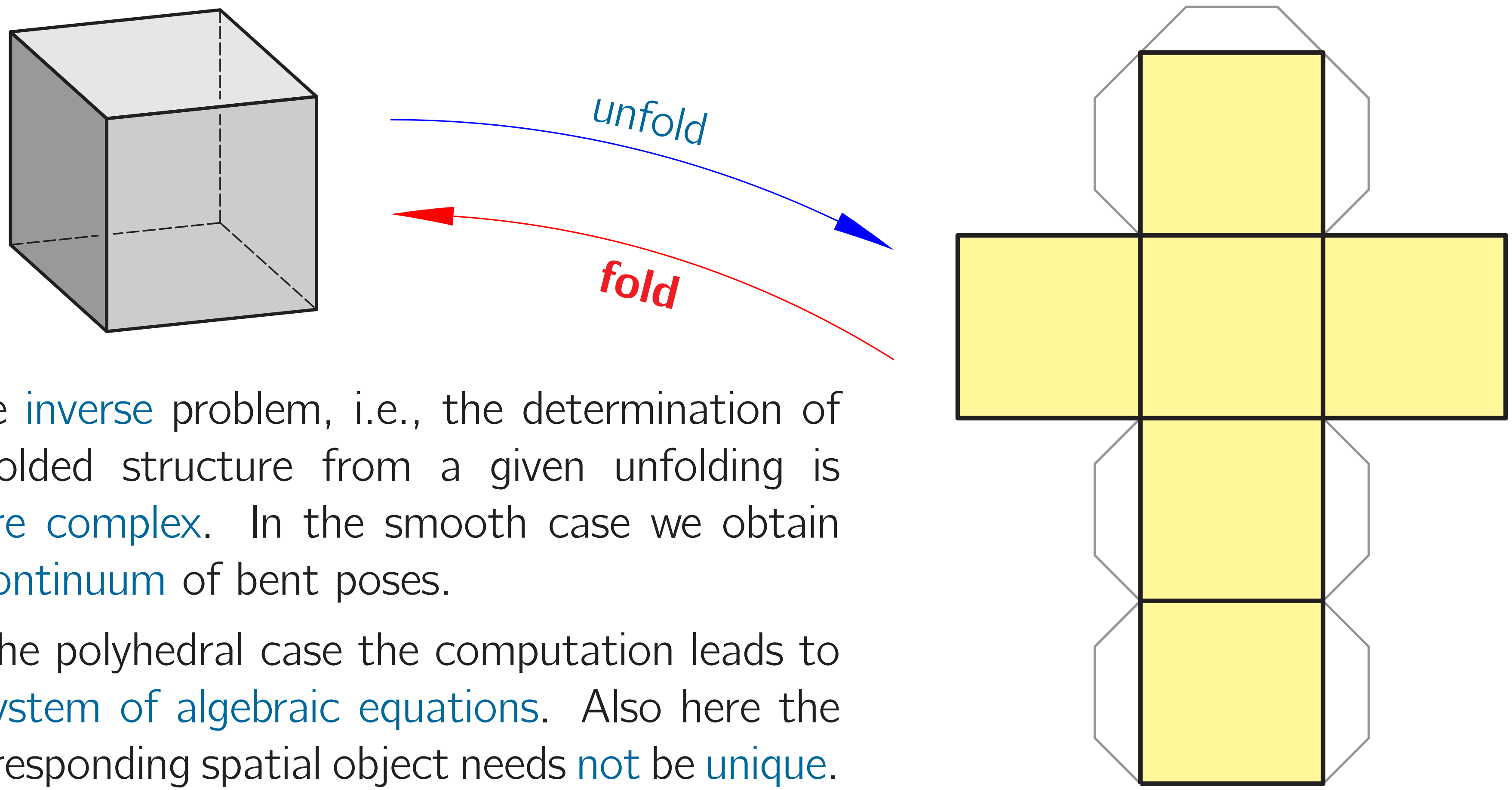
4.1 Unfolding and folding



There are standard procedures provided for the construction of the **unfolding** (development, net) of polyhedra or developable surfaces.

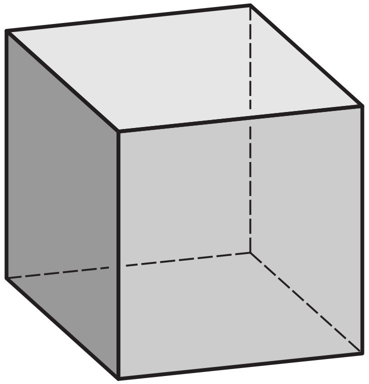
The result is unique, apart from the placement of the different components, and it shows the intrinsic metric of the spatial structure.

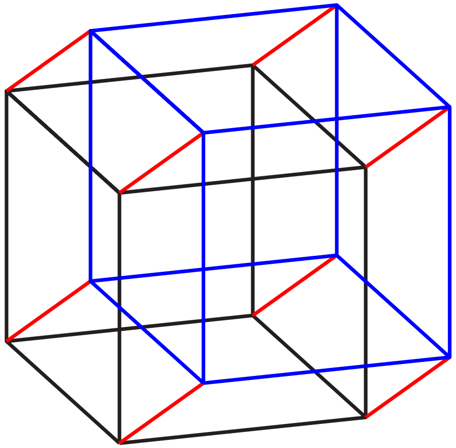
4.1 Unfolding and folding

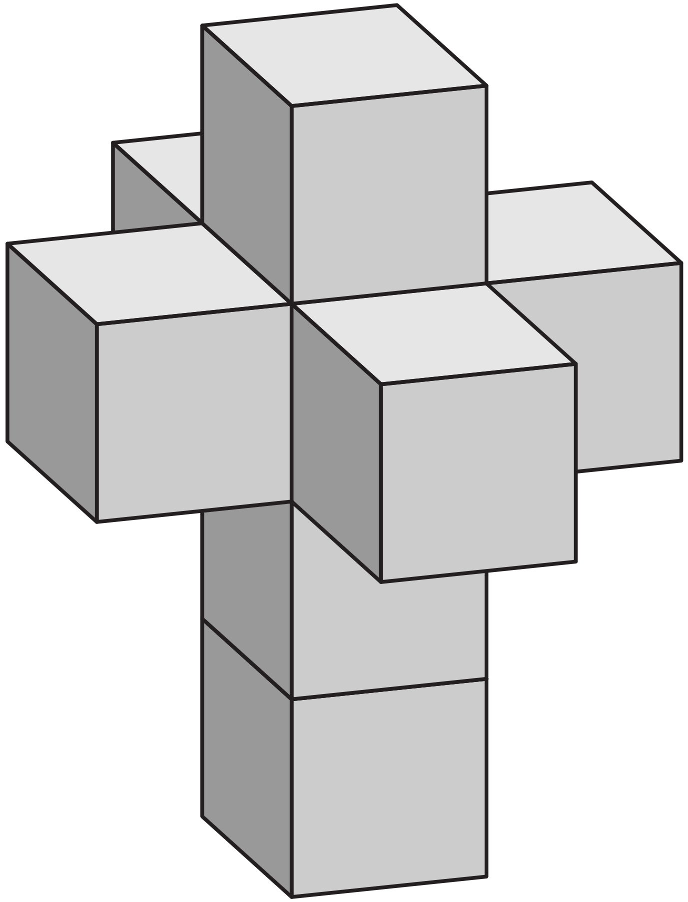


The inverse problem, i.e., the determination of a folded structure from a given unfolding is more complex. In the smooth case we obtain a continuum of bent poses.

In the polyhedral case the computation leads to a system of algebraic equations. Also here the corresponding spatial object needs not be unique.







A cube together with its translated copy (in blue) in the 4-space and the trajectories of the vertices (in red) form a **hypercube**.

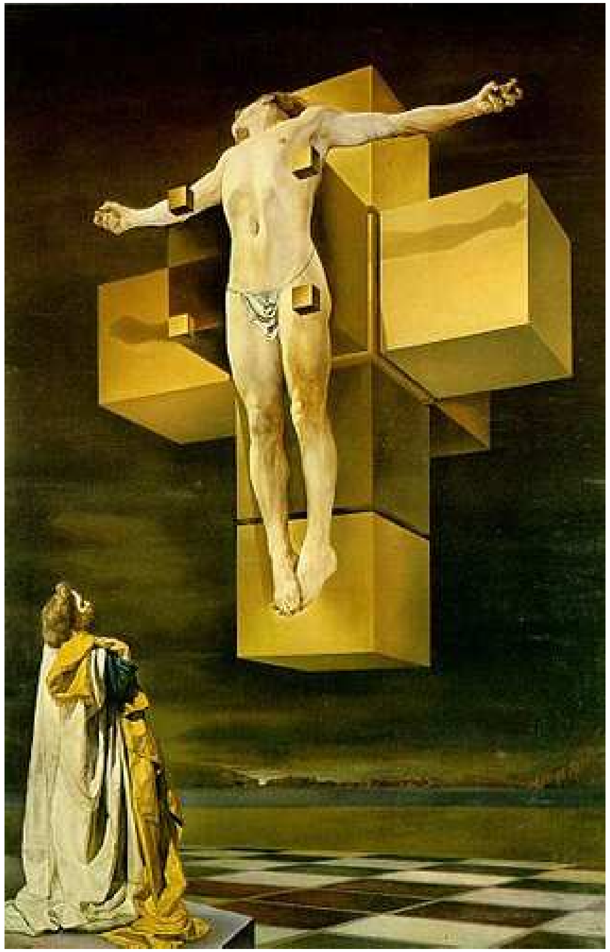
It has 8 cells (= 3-cubes). Each of the 24 faces is the meet of two cells.

Iterated rotations of cells about a face into the hyperplane of the neighboring cell results in a three-dimensional unfolding.

4.1 Unfolding and folding

Salvador Dalí: *Corpus Hypercubus*, 1954
194×124 cm, Metropolitan Museum of Art, New York

Instead of adhesive strips there are adhesive faces between different cells



4.1 Unfolding and folding

Only if the polyhedron bounds a **convex** solid then the result is unique, due to Aleksandr Danilovich **Alexandrov** (1941).

In this case, for each vertex the sum of intrinsic angles for all adjacent surfaces is $< 360°$ (= convex intrinsic metric).

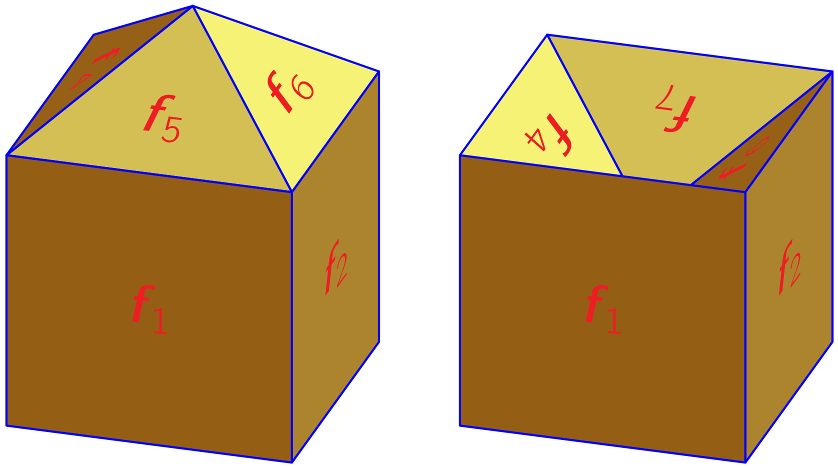
Theorem: [Uniqueness Theorem]
For any convex intrinsic metric there is a unique convex polyhedron.

A.I. **Bobenko** and I. **Izmestiev** (2006) developed an algorithm for constructing the convex polyhedron with given intrinsic metric.

4.1 Unfolding and folding

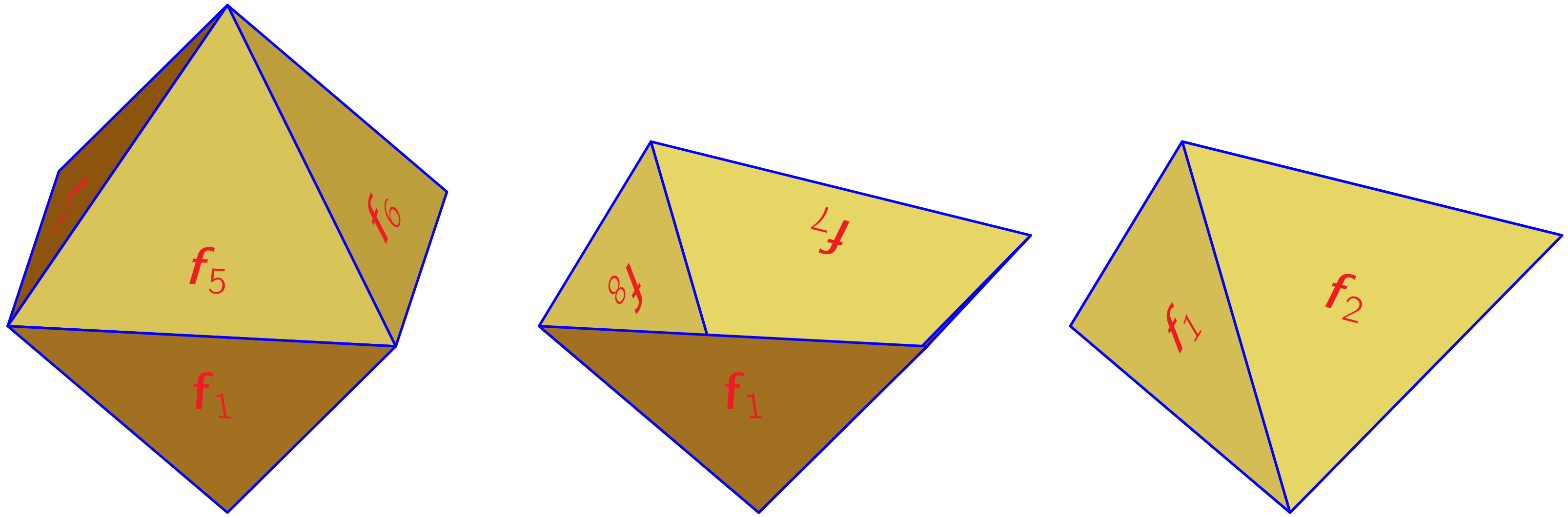
If convexity is not required the unfolding of a polyhedron needs not define its spatial shape uniquely !

A tetrahedron or compounds of tetrahedra are globally rigid.



A **flipping** (or snapping) polyhedron admits two sufficiently close realizations – by applying a slight force.

4.1 Unfolding and folding



Even a regular octahedron is flexible — after being re-assembled. The regular pose on the left hand side is called **locally rigid**.

4.1 Unfolding and folding

Milestones:

• A.L. Cauchy (1813): Each convex polyhedron is locally rigid.

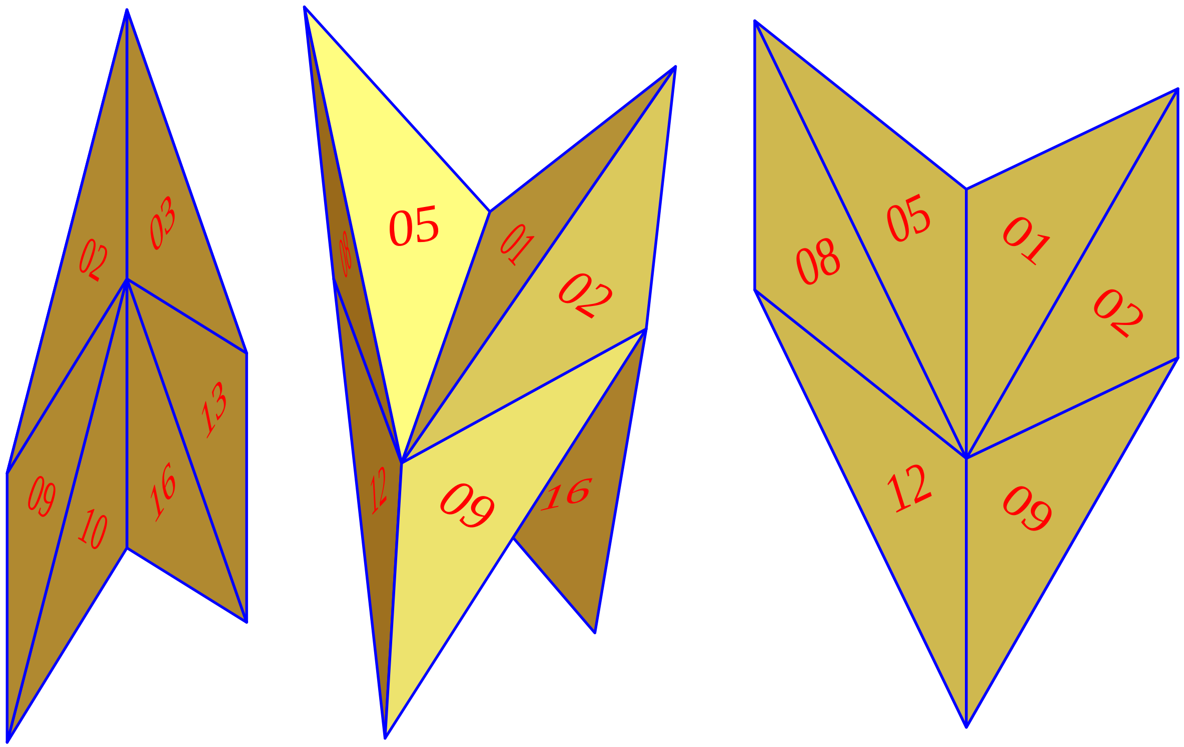
• A.D. Alexandrov (1941): For each convex polyhedral metric there exists exactly one convex polyhedron.

• R. Bricard (1897): There exist flexible octahedra (four-sided double pyramides) — however with self-intersections.

• R. Connelly (1977): There is a “flexing sphere”, d.h., a flexible polyhedron which is homeomorphic to a sphere.

• I.Kh. Sabitov (1996): The volume of a triangulated polyhedron is a root of a polynomial, whose coefficients depend only on the combinatorial structure and the edge length of the polyhedron.

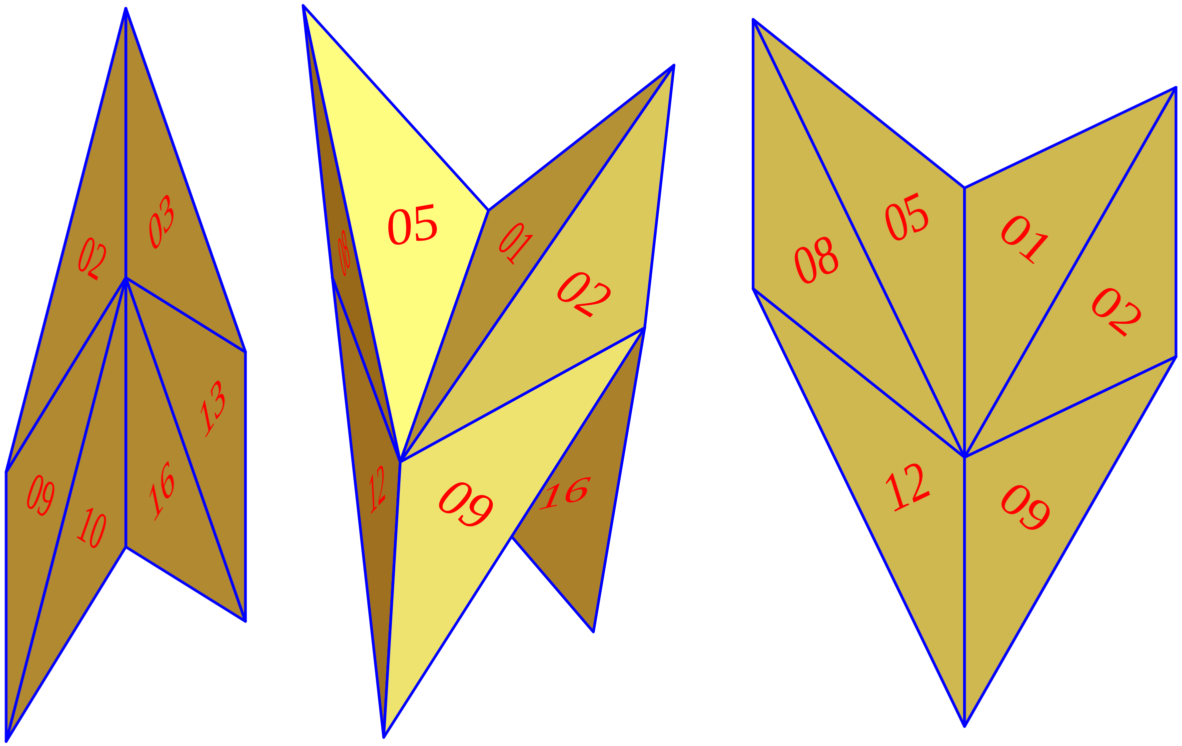
4.1 Unfolding and folding



This polyhedron called *"Vierhorn"* is locally rigid, but can flip between its spatial shape and two flat realizations in the planes of symmetry (W. Wunderlich, C. Schwabe).

At the science exposition *"Phänomena"* 1984 in Zürich this polyhedron was exposed and falsely stated that this polyhedron is flexible.

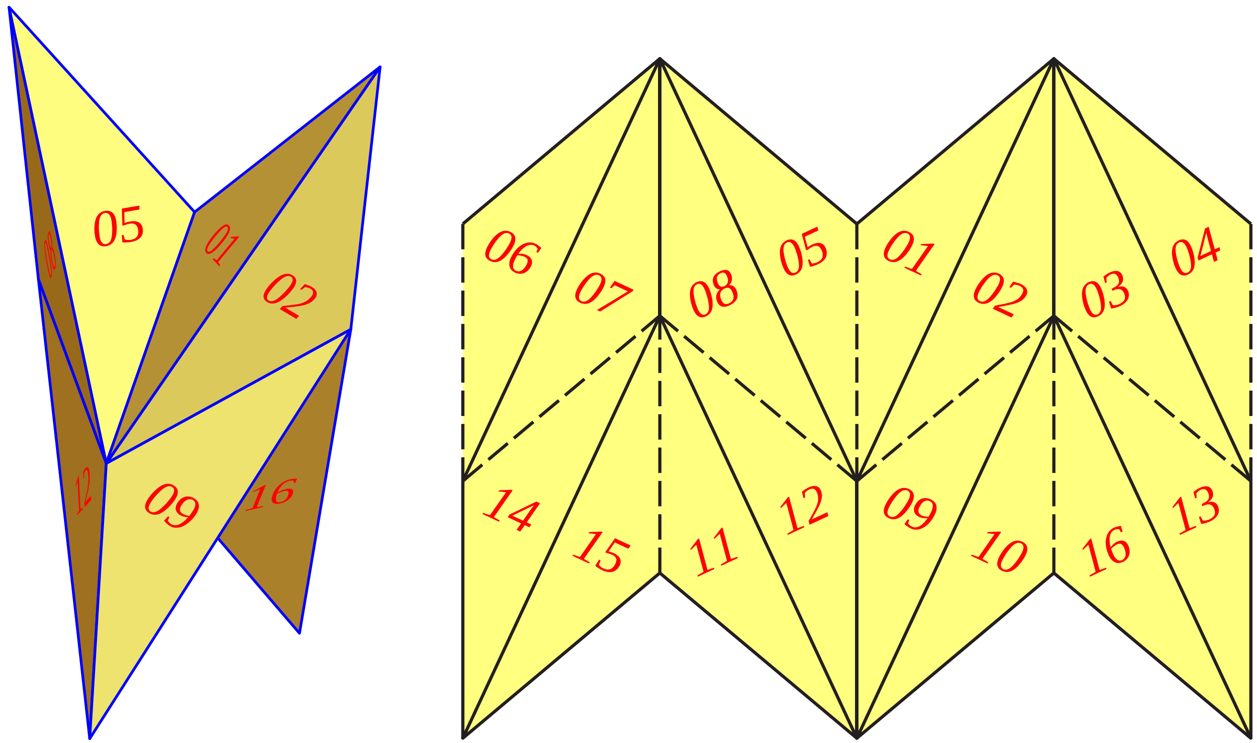
4.1 Unfolding and folding



The volume of the "Vierhorn" changes between the two poses. This already disproves continuous flexibility, because 1996 I. Kh. Sabitov proved the famous *Bellows Conjecture*

stating that for every flexible polyhedron in $\mathbb{E}^3$ the volume keeps constant during the flex.

4.1 Unfolding and folding



the “Vierhorn” and its unfolding

Wolfram MathWorld: A flexible polyhedron which flexes from one totally flat configuration to another, passing through intermediate configurations of positive volume.

4.1 Continuously flexible octahedra

R. Bricard (1897): Apart from trivial cases, there are three types of flexible octahedra in the Euclidean 3-space.

Typ 1: Octahedron with a plane of symmetry, passing through two opposite vertices;

Typ 2: Octahedron, where all pairs of opposite vertices are symmetric w.r.t. an axis;

Typ 3: Octahedron without any symmetry, but with two flat poses.

Bricard's octahedra are the basis of all known flexible polyhedra without self-intersections (R. Connelly 1978, K. Steffen 1980).

4.1 Continuously flexible octahedra

R. Bricard (1897): Apart from trivial cases, there are three types of flexible octahedra in the Euclidean 3-space.

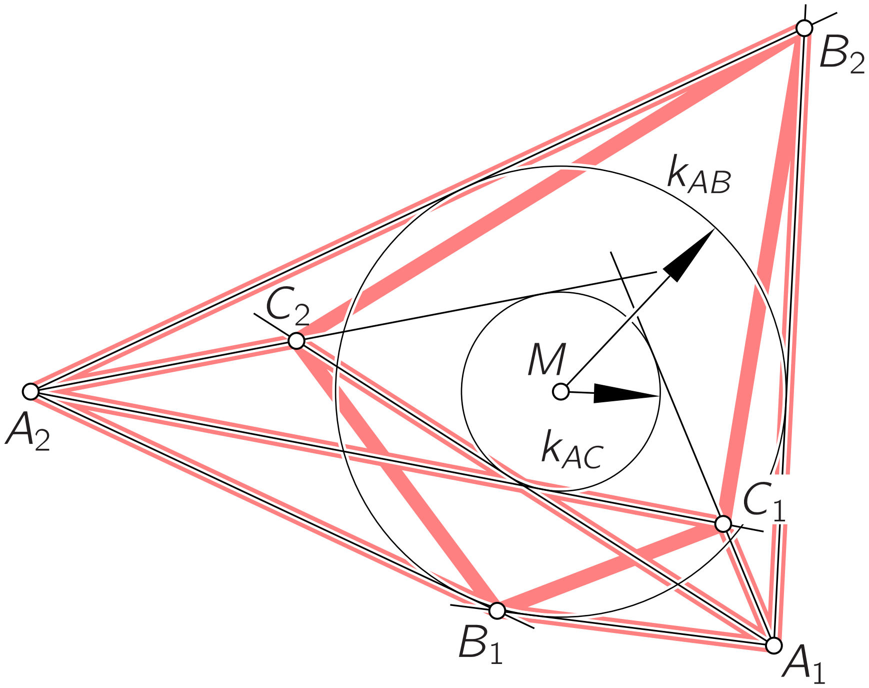
Typ 1: Octahedron with a plane of symmetry, passing through two opposite vertices;

Typ 2: Octahedron, where all pairs of opposite vertices are symmetric w.r.t. an axis;

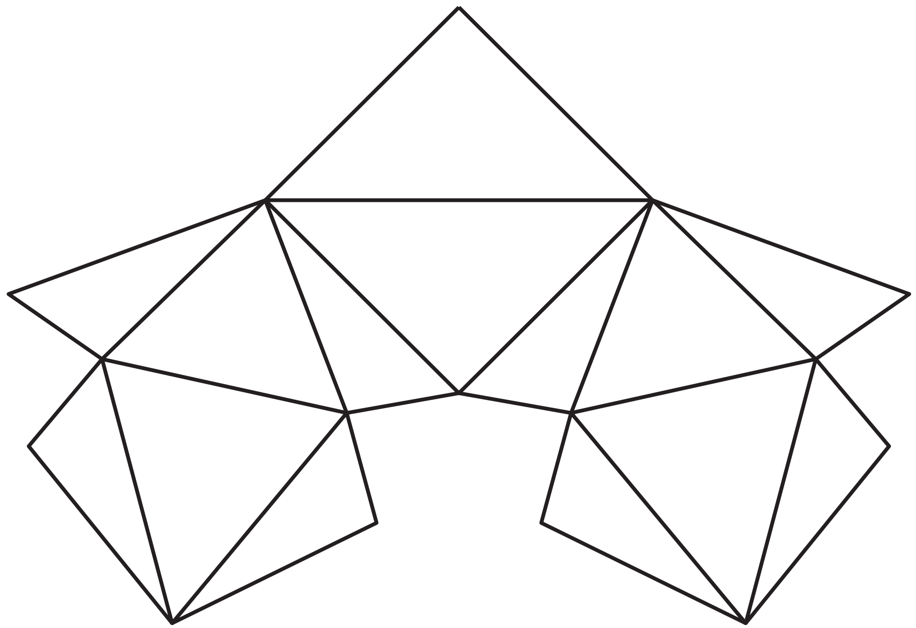
Typ 3: Octahedron without any symmetry, but with two flat poses.

Bricard's octahedra are the basis of all known flexible polyhedra without self-intersections (R. Connelly 1978, K. Steffen 1980).

4.1 Continuously flexible octahedra



Bricard octahedron of type 3
in a flat pose. Given: vertices A_1, A_2
and concentric circles k_{AB}, k_{AC}



Unfolding of Steffen's flexing sphere
(with 9 vertices) as a compound
of two Bricard octahedra

4.1 Continuously flexible octahedra

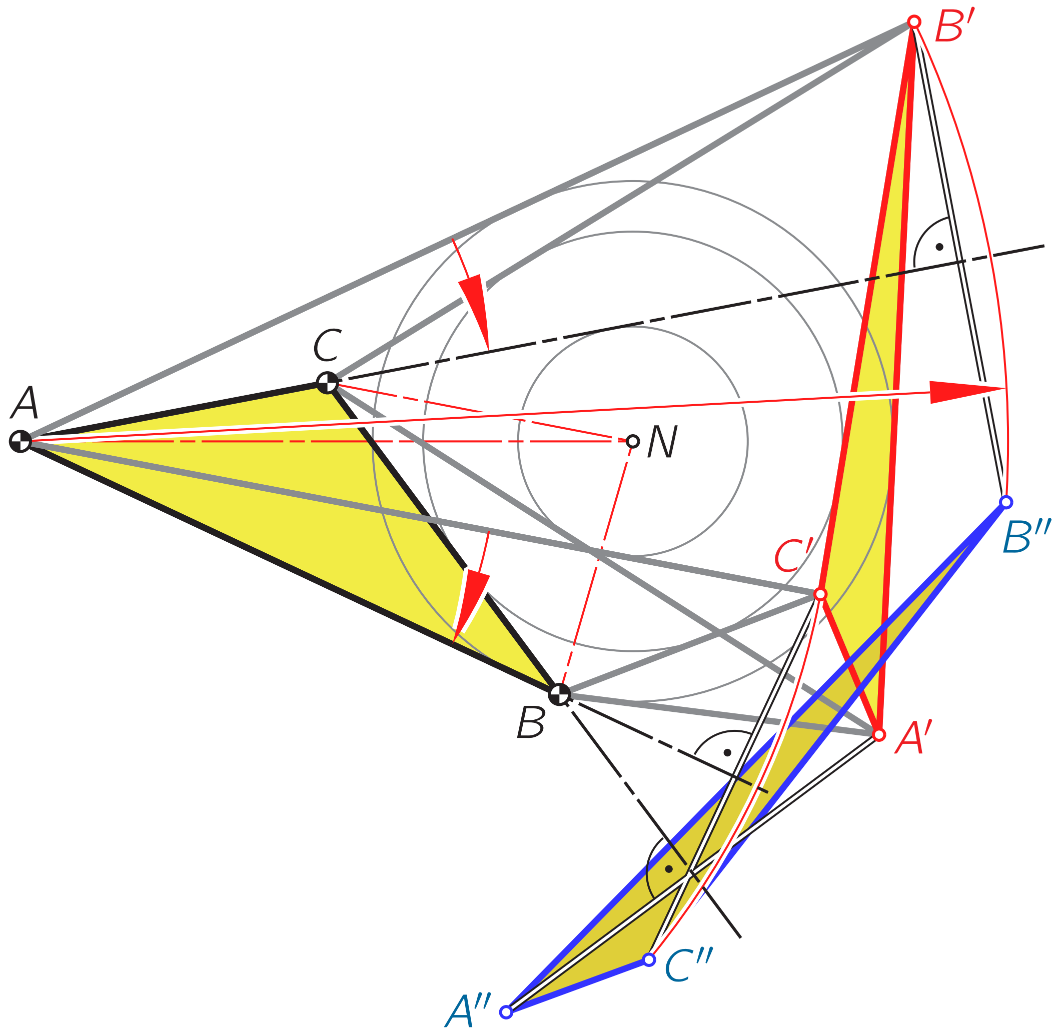
A comment in between:

Bricard octahedra are flexible closed 6R chains where consecutive axes of rotations intersect each other.

In the year 2012 G. Hegedüs, J. Schicho and H.-P. Schröcker could prove that there is a one-to-one correspondence between these chains and the factorizations of a polynomial dual quaternion $Q(t)$ of degree 6 into a product of linear factors.

G. Hegedüs, J. Schicho, H.-P. Schröcker: *Factorization of rational curves in the Study quadric and revolute linkages.* Mech. Mach. Theory **69**(1), 142–152 (2013)

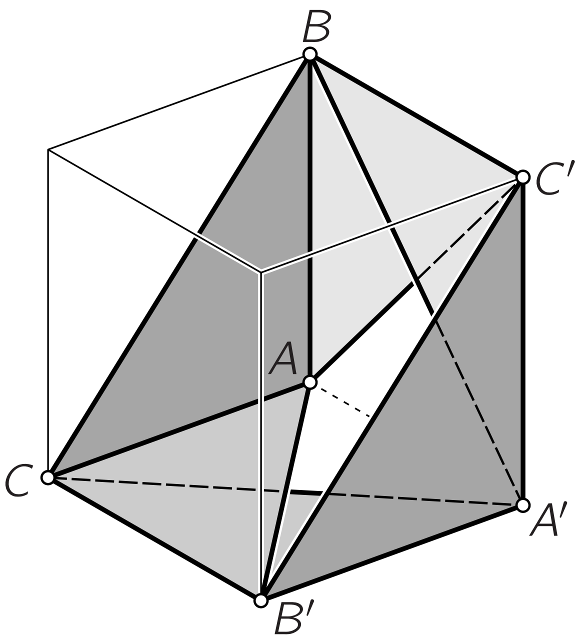
4.1 Continuously flexible octahedra

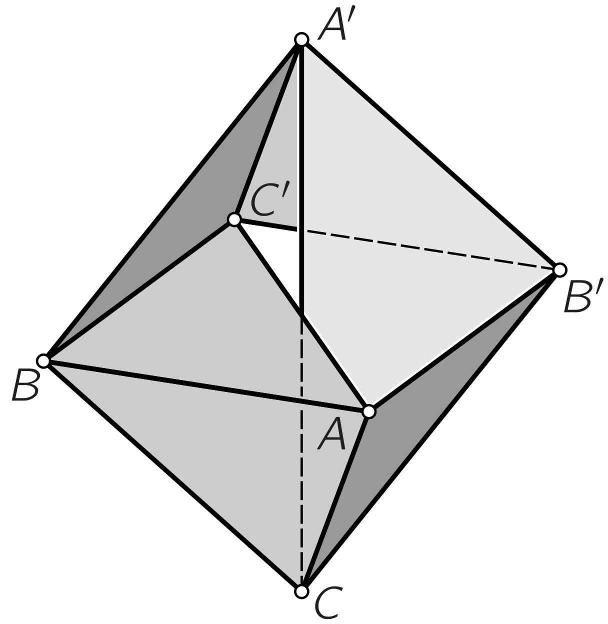


According to Bricard's construction, all bisectors must pass through the midpoint N of the concentric circles.

The two flat poses of a type-3 flexible octahedron, when ABC remains fixed.

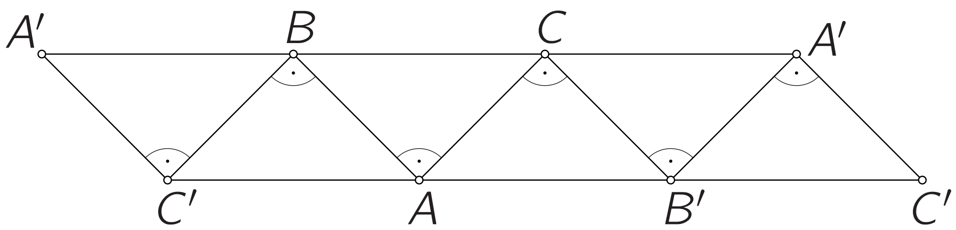
4.1 Continuously flexible octahedra

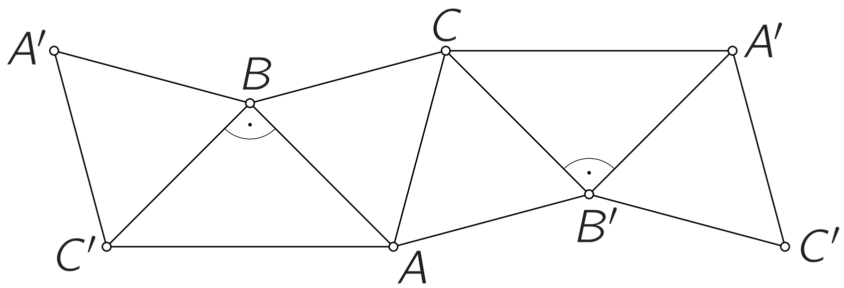




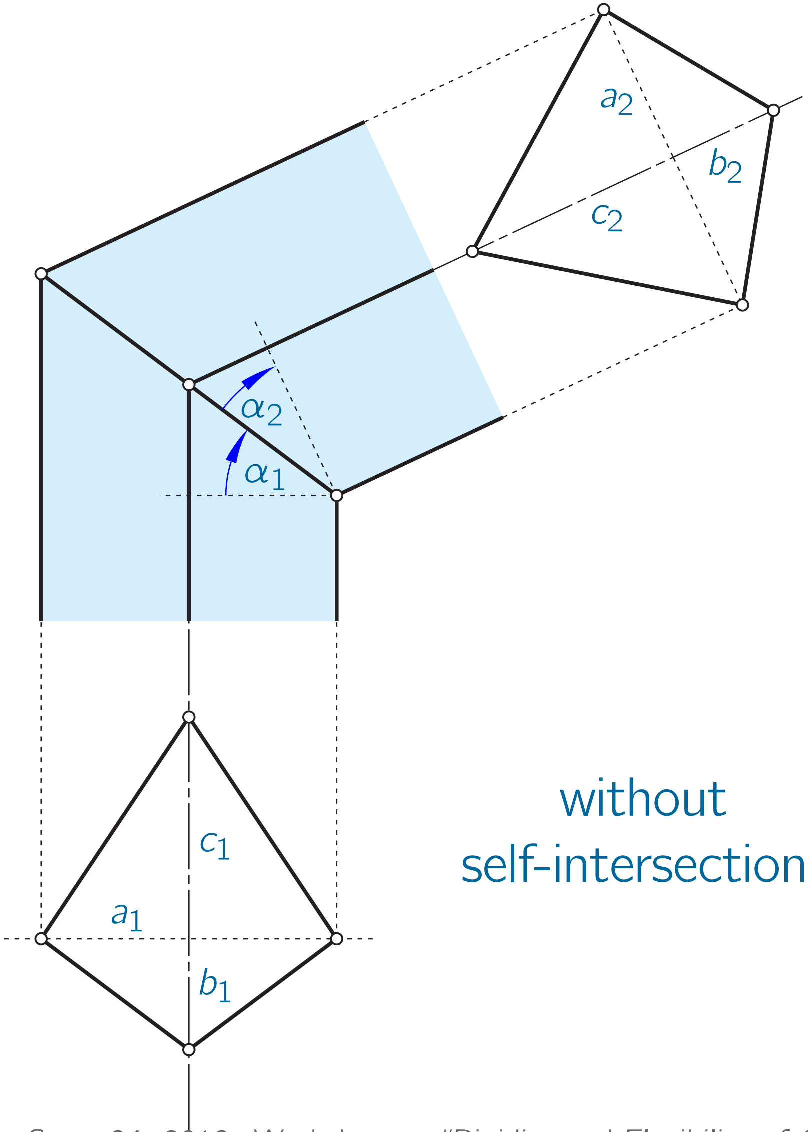
Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry (types 1 and 2)

Below: Nets of the two octahedra.





4.1 Continuously flexible octahedra



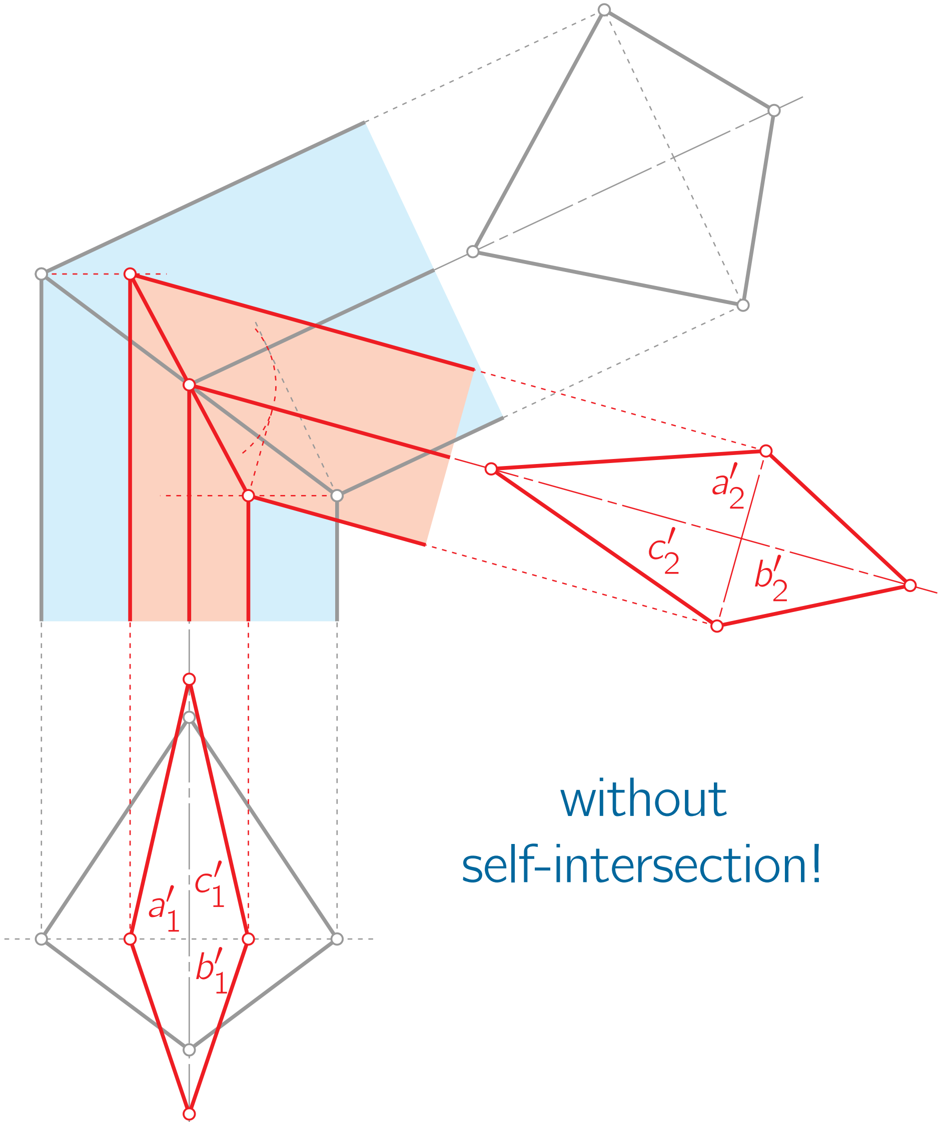
Front, top and side view of a flexible octahedron with two vertices at infinity.

This is the only nontrivial example of a flexible octahedron in $\mathbb{E}^3$ with infinite vertices (Nawratil 2010).

without
self-intersection!

$$\frac{a_1}{\cos\alpha_1} = \frac{a_2}{\cos\alpha_2}, \qquad b_2 = b_1, \qquad c_2 = c_1$$

4.1 Continuously flexible octahedra



without
self-intersection!

The dimensions $a_1', \ldots, c_2'$ and α_1', α_2' of the flexion are for $t \in (1-\varepsilon,\, 1+\varepsilon)$

$$a_1' = \sqrt{a_1^2 - t},$$

$$b_1' = \sqrt{b_1^2 + t}, \quad c_1' = \sqrt{c_1^2 + t}$$

The planar section remains planar.

$$\tan \alpha_i' = \frac{a_i}{a_i'} \tan \alpha_i, \quad i = 1, 2$$

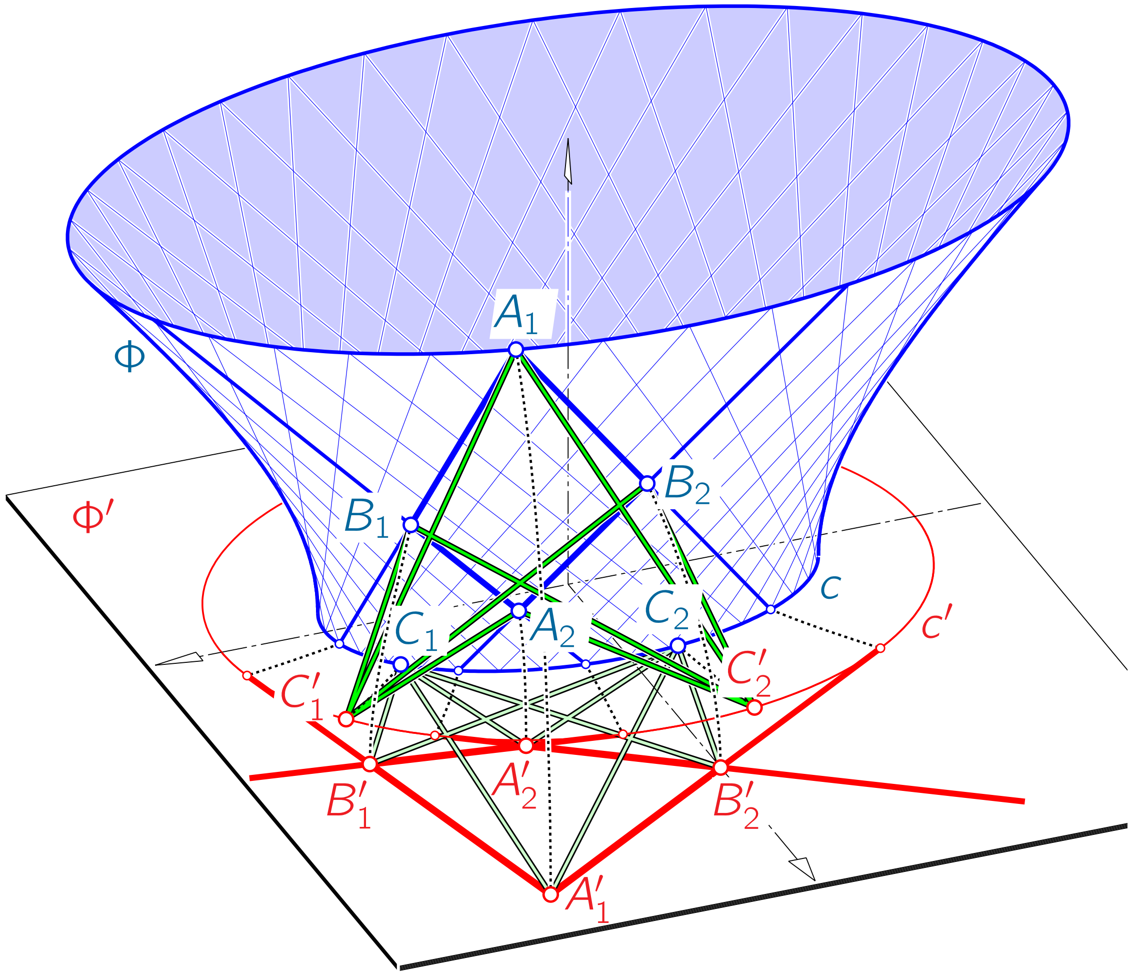
and still

$$\frac{a_1'}{\cos \alpha_1'} = \frac{a_2'}{\cos \alpha_2'}, \quad b_2' = b_1', \quad c_2' = c_1'$$

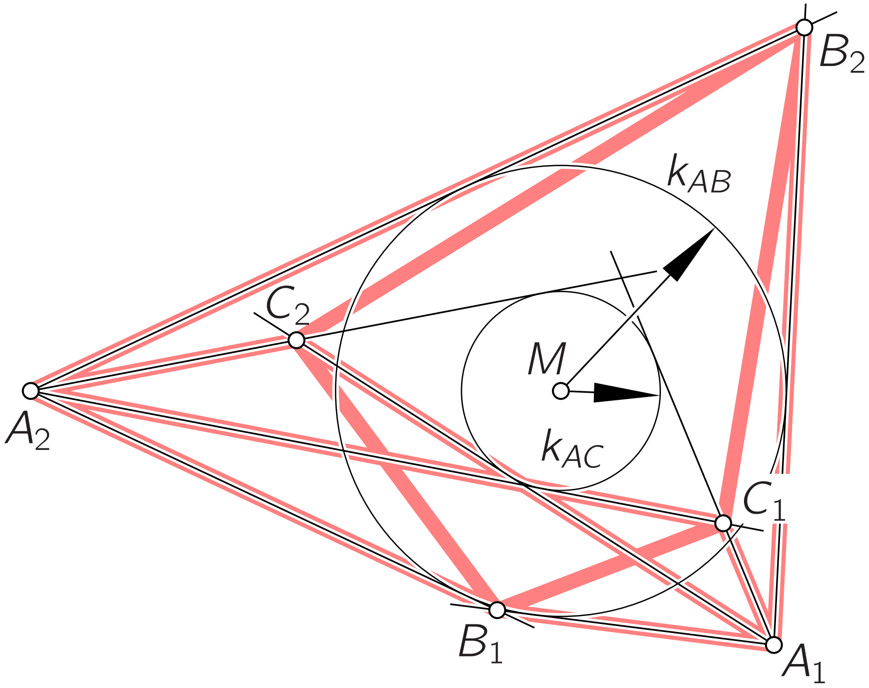
4.1 Continuously flexible octahedra

Ivory's Theorem even shows the flexibility of the unsymmetric Type 3 of R. Bricard's flexible octahedron (1897).

The two confocal 'surfaces' for applying Ivory's Theorem are a one-sheet hyperboloid and its focal ellipse.



4.1 Continuously flexible octahedra



Bricard octahedron of type 3
in a flat pose.

The analogues of Bricard octahedra are also flexible in the hyperbolic 3-space $\mathbb{H}^3$ — also in the case where some vertices are on the absolute or outside.

A long-standing open problem, whether there exist flexible cross-polytopes in higher dimensions, has recently be solved for Euclidean, hyperbolic and elliptic spaces; the answer is 'yes'.

Alexander A. **Gaĭfullin**: *Flexible cross-polytopes in spaces of constant curvature.* arXiv:1312.7608, 38 p.

4.1 Continuously flexible octahedra

Also the bellows conjecture for Euclidean spaces of dimension ≥ 4 found a positive answer. There exists a 'Sabitov-Polynomial' in all dimensions:

Alexander A. **Gaĭfullin**: *Sabitov polynomials for volumes of polyhedra in four dimensions*. Adv. Math. 252 (2014), 586–611
(arXiv:1108.6014, Oct.2011)

Alexander A. **Gaĭfullin**: *Generalization of Sabitov's Theorem to Polyhedra of Arbitrary Dimensions*. arXiv:1210.5408, May2014

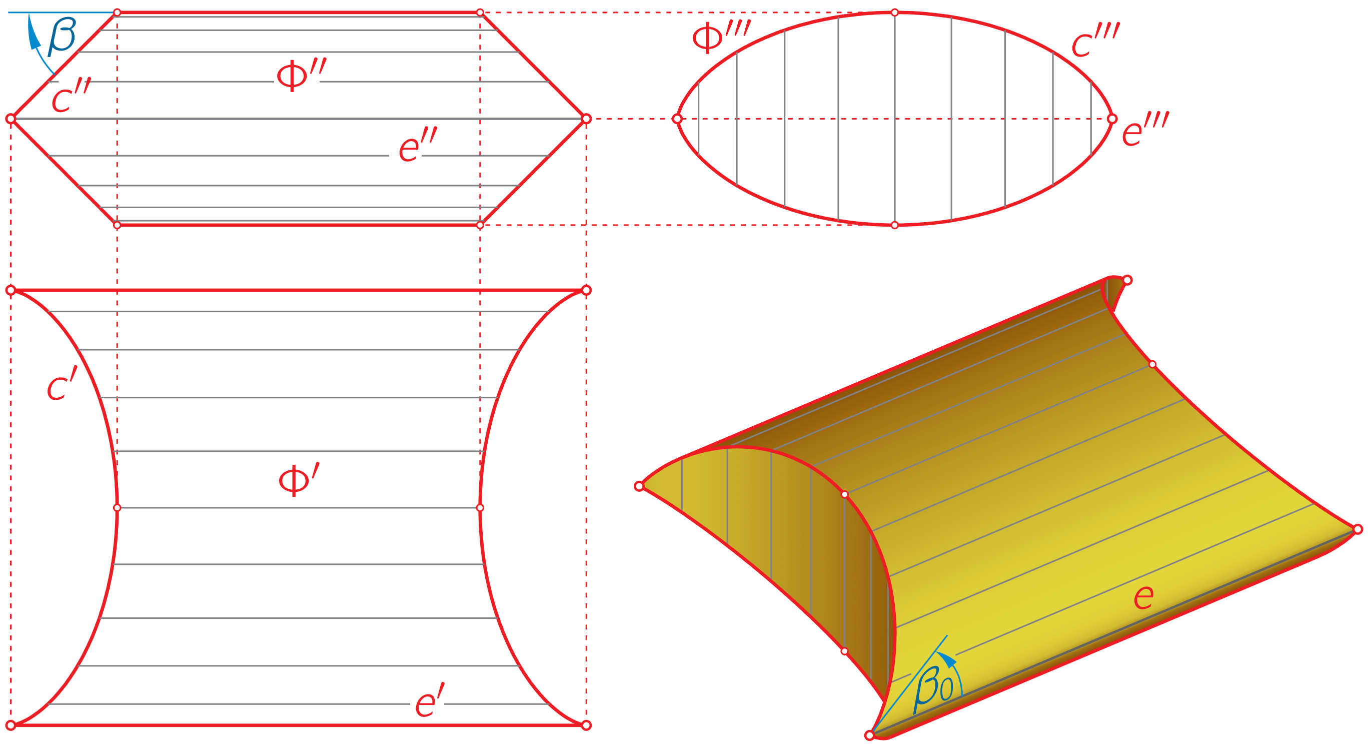
4.2 Curved folding, Example 1

A common way of producing small boxes is to push up appropriate planar cardbord forms Φ_0 with prepared creases. Below the case of creases along circular arcs c_0.

planar version with circular creases

corresponding box with planar creases

4.2 Curved folding, Example 1



The creases at the spatial form are planar and meridians of surfaces of revolution with constant Gaussian curvature

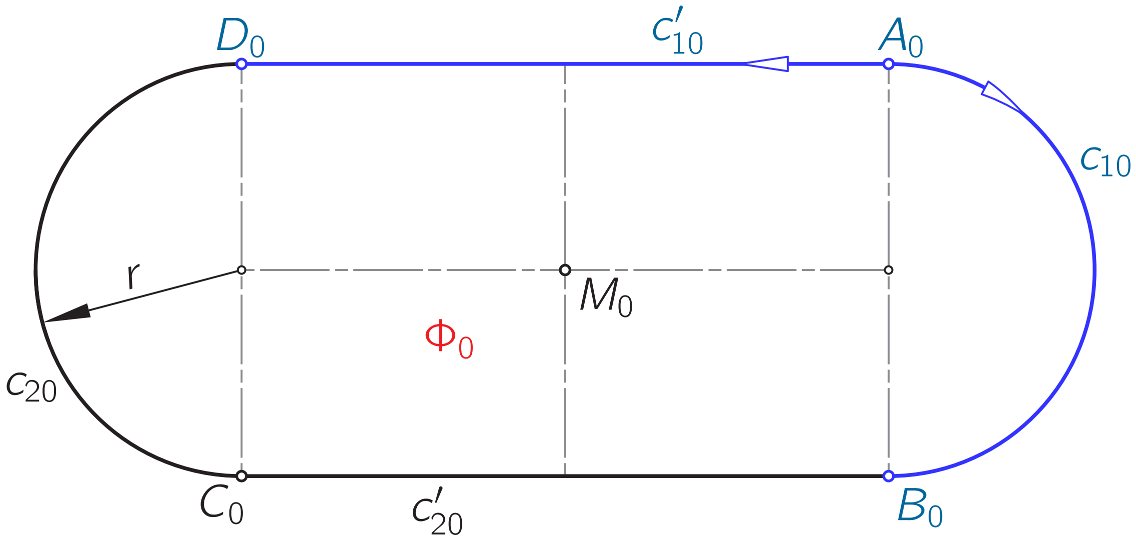
4.3 Curved folding, Example 2

Unfolding and corresponding spatial form (photos: **G. Glaeser**)

The crucial point is here that the ruling is **unknown.**

M. Kilian, S. Flöry, Z. Chen, N.J. Mitra, A. Sheffer, H. Pottmann: *Curved Folding.* ACM Trans. Graphics **27**/3 (2008), Proc. SIGGRAPH 2008.

4.3 Curved folding, Example 2



A **physical model** shows:

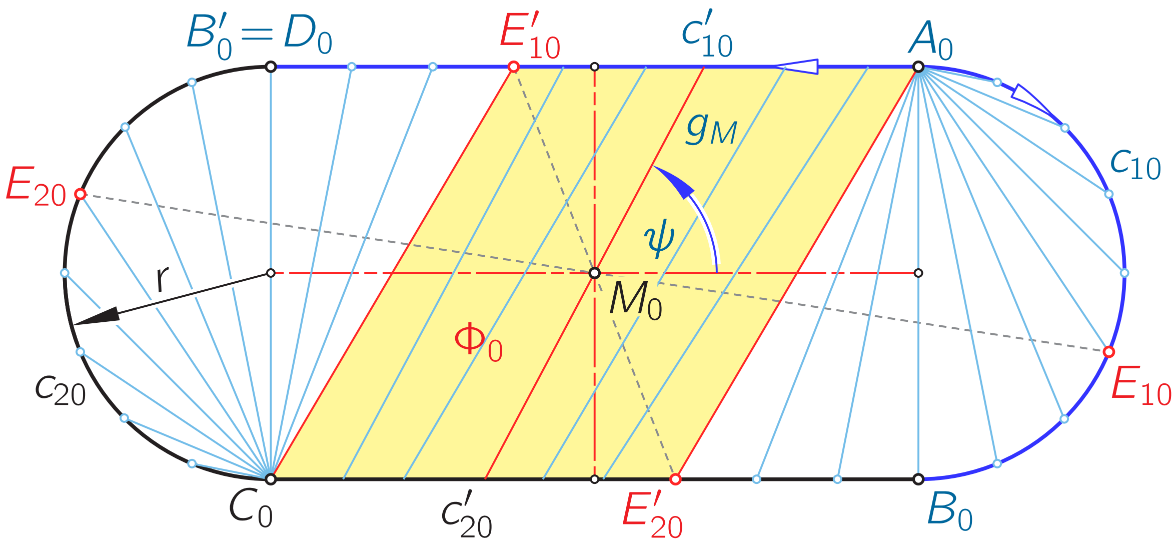
- The spatial body with its developable boundary Φ is convex and uniquely defined.

- The helix-like curve $c = c_1 \cup c_2$ is a proper edge of Φ; the resulting solid is the convex hull of c.

- The semicircular disks are bent to cones with apices A and C. Hence, Φ is a C^1-compound of two cones and a torse between.

- The body has an axis a of symmetry which connects the midpoint M with the remaining transition point $B = D$ on c.

4.3 Curved folding, Example 2

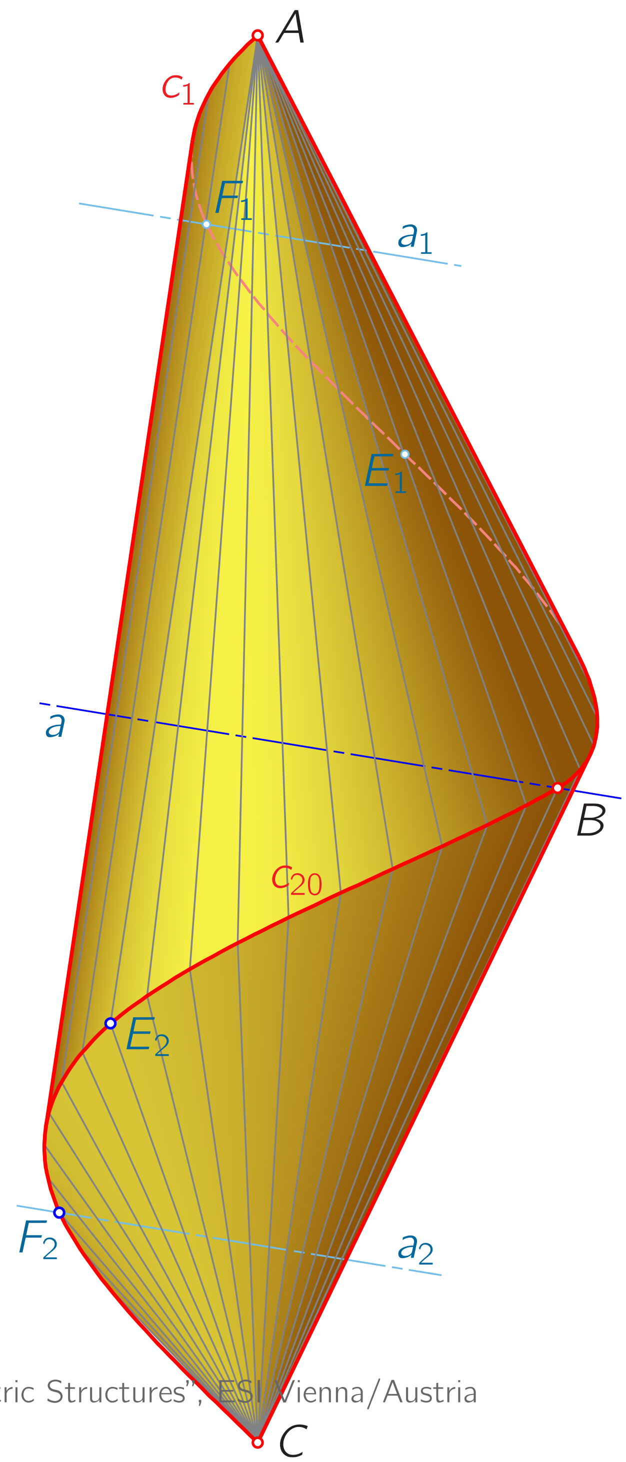


- The tangent at the point $E_2 \in c_2$ of transition between the cone with apex A and the torse must be parallel to t_A.

- The tangent at the analogue point $E_1 \in c_1$ is parallel to the final tangent t_C of c_2.
- The subcurves $AE_1 \subset c_1$ and $E_2C \subset c_2$ have conciding tangent indicatrices.

Approximation 2 shows an excellent accordance with the physical model.

... but there remains a contradiction.



Thank you for your attention!