On the Interdisciplinarity of Geometry

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Pécsi Tudományegyetem
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University of Pecs
JUBILEE

650

Congratulations and

ad multos annos!

0. Congratulations

Your university was one of the first world-wide!

The Vienna University was founded **1365** (650-years jubilee 2015)

The Charles University in Prague was the first, founded **1348** (670 years ago)

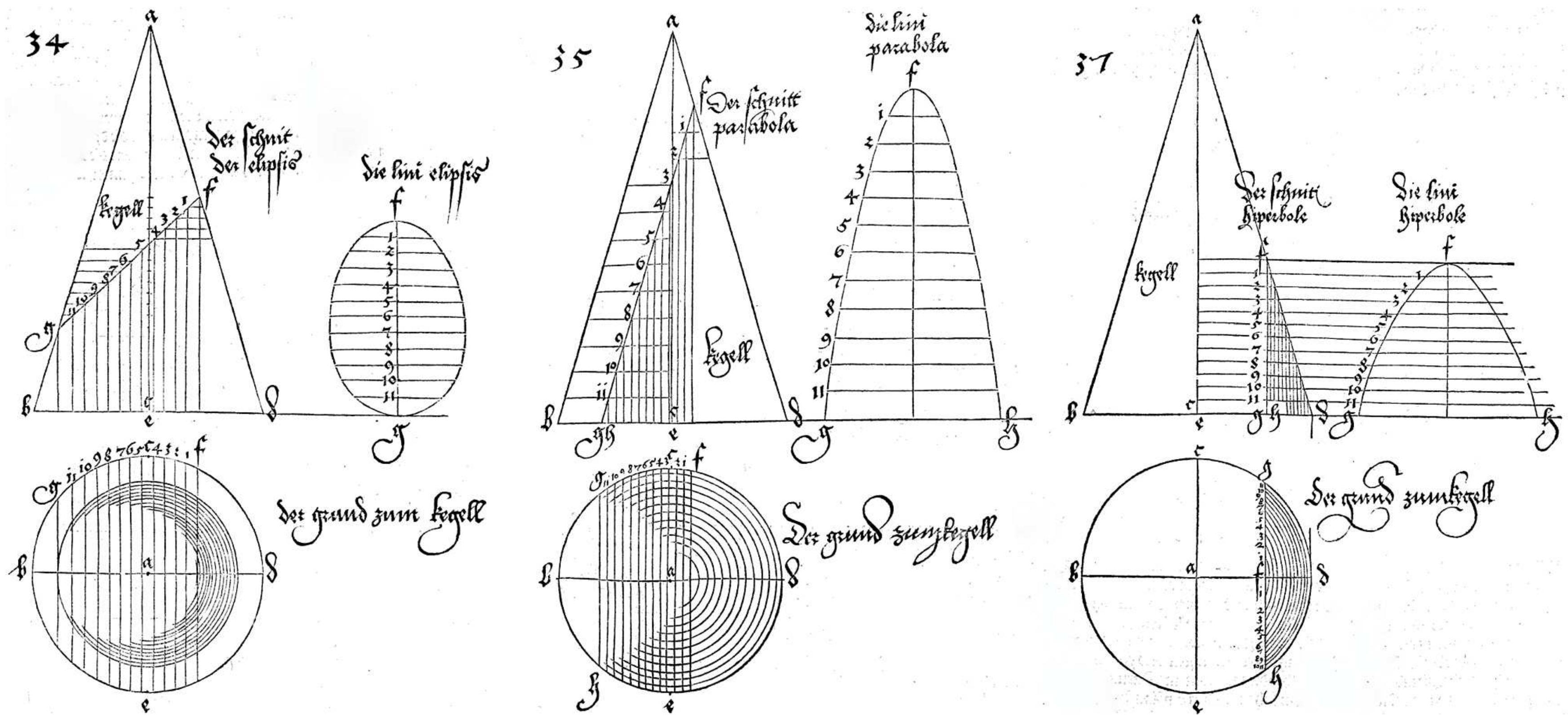
0. Congratulations

The Vienna University of Technology was founded **1815**

(200-years jubilee 2015)

The **first** University of Technology worldwide was founded **1763** in **Selmec**/Hungary (today Banská Štiavnica/Slovakia). The École polytechnique in Paris followed **1794**.

1. Historical development



Menaichmos (380 –320 B.C.) discovered the **conics** as planar sections of cones of revolution (shadow lines). Above: Woodcut of Albrecht **Dürer** (1471–1528)

1. Historical development

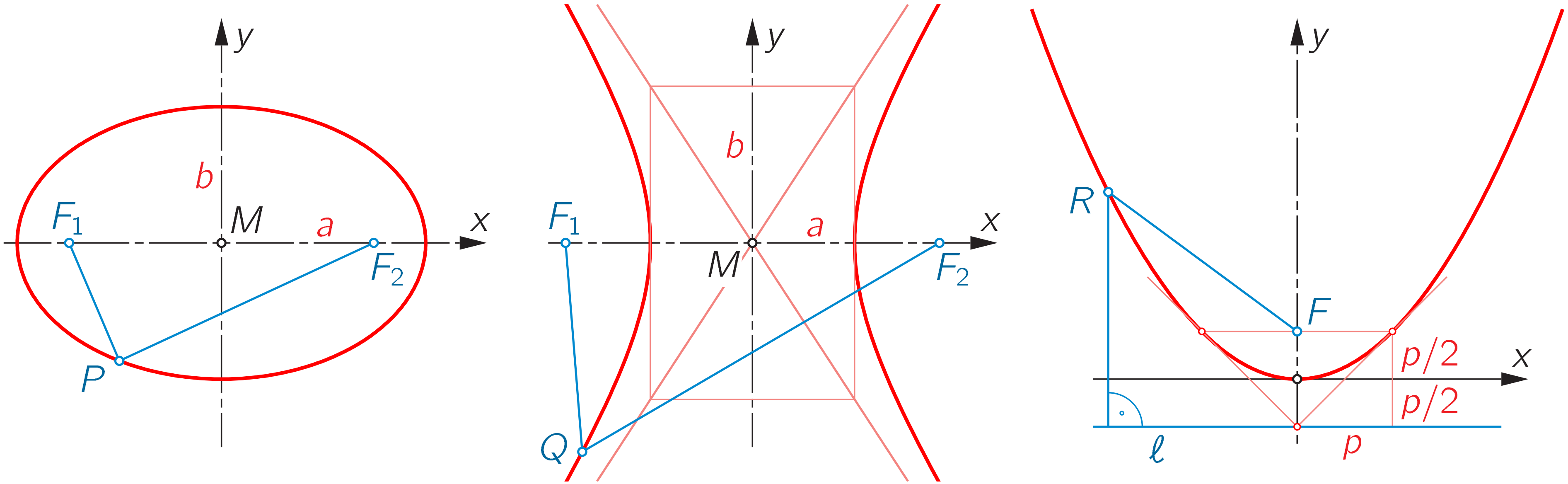
The history of conics started $\sim$350 B.C.; many results are due to **Apollonius of Perge** ($\sim$260–190 B.C.)


Hades abducting Persephone, fresco in the tomb of Philipp III of Macedon (half-brother of Alexander the Great), painted $\sim$ **310 B.C.**, recovered 1980 in Vergina, 80 km West of Thessaloniki. Left: original, right: G. Glaeser's computer simulation

. . . this is a highly impressive and precise perspective view, produced 400 years before the Roman period.

In textbooks, we can often read that the first, but poor perspectives were drawn during the time of the Romans.

1.1 Apollonian definition of conics



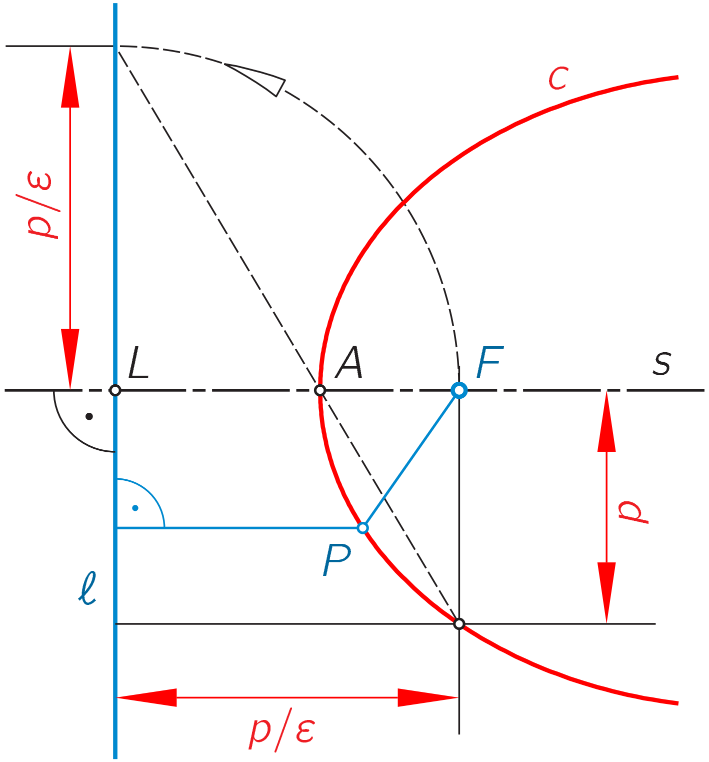
$\overline{PF_1} + \overline{PF_2} = 2a$ $\qquad$ $\left|\overline{PF_1} - \overline{PF_2}\right| = 2a$ $\qquad$ $\overline{PF} = \overline{P\ell}$

In view of the different standard definitions and shapes of conics, it is surprising that there is a **uniform definition**, attributed to Apollonius of Perge (today $\sim$ Antalya).

1.1 Apollonian definition of conics

Apollonian definition of conics $c = \{\, P \mid \overline{PF} = \varepsilon \cdot \overline{P\ell} \,\}$

For $\varepsilon < 1$ the curve c is an *ellipse*, for $\varepsilon = 1$ a *parabola* and for $\varepsilon > 1$ a *hyperbola*.



F **focal point**

ℓ **directrix**

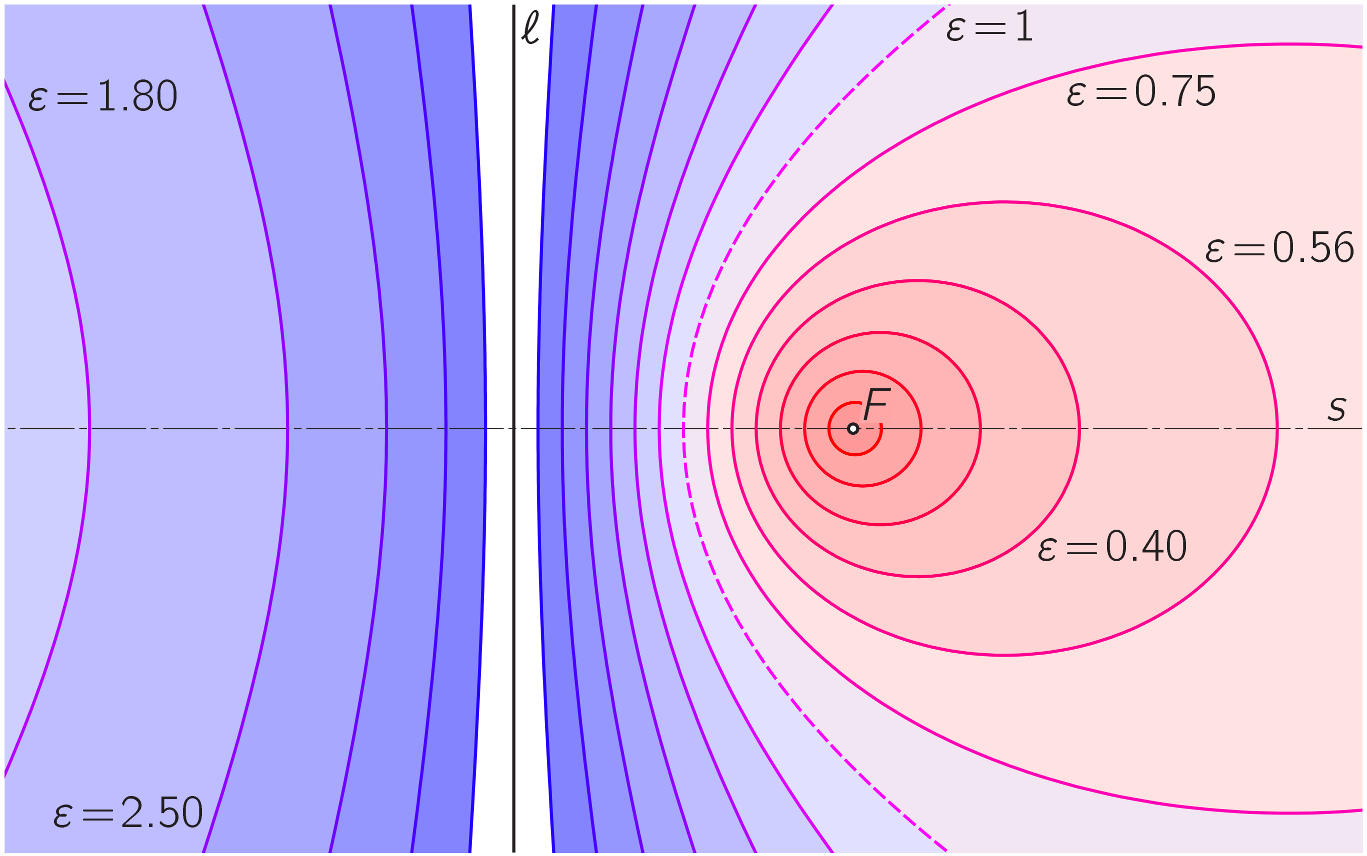
ε numerical eccentricity

A vertex

p parameter (= radius of curvature at A)

$\Longleftrightarrow$ polar equation: $r(\varphi) = \dfrac{p}{1 + \varepsilon \cos\varphi}$.

1.1 Apollonian definition of conics



Conics sharing a focal point F and the corresponding directrix ℓ

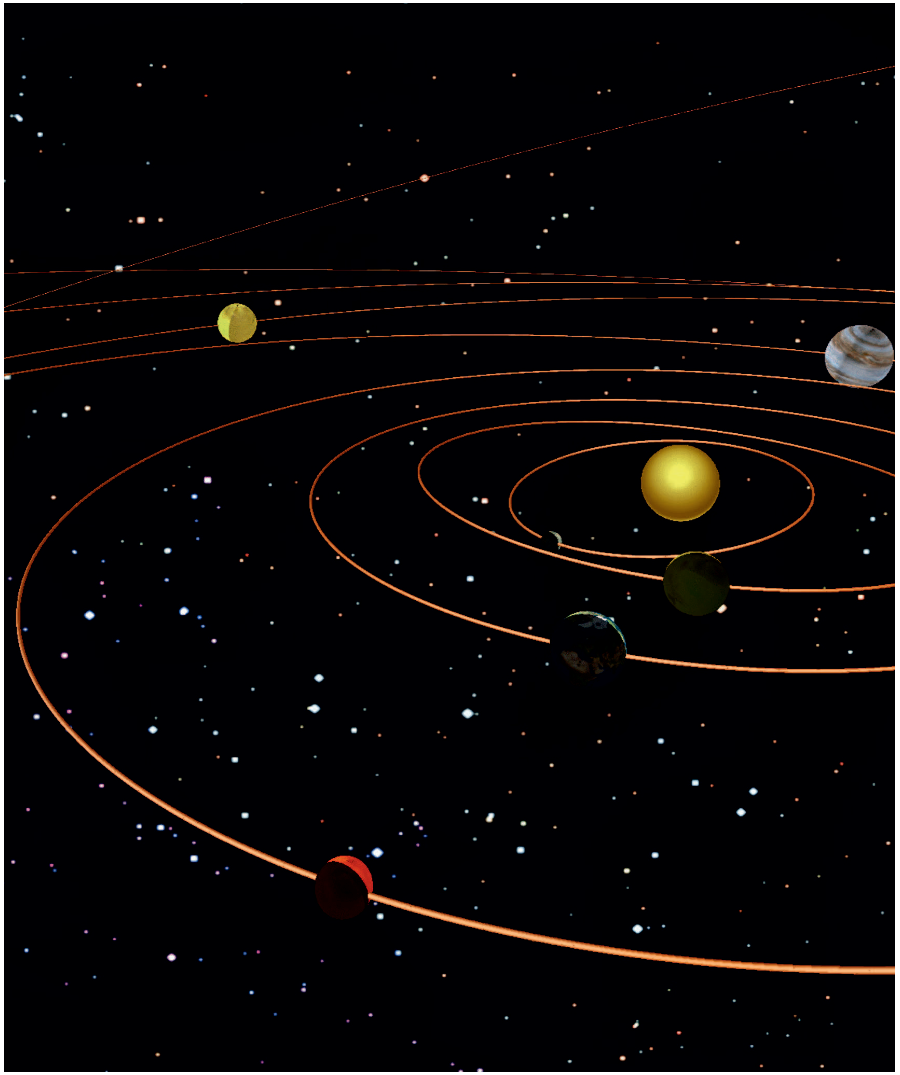
Our world is **full of conics**.

Newton's Law of Gravitation (1687)

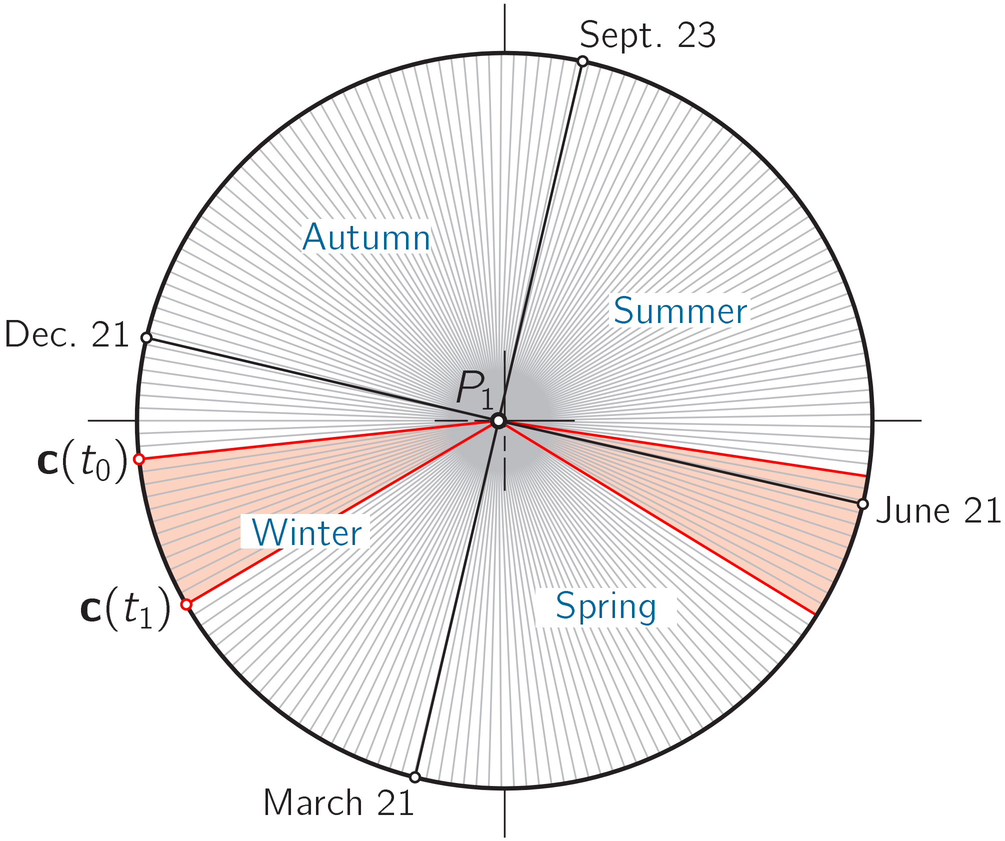
$$\|\overrightarrow{F}\| = G\,\frac{m_1 m_2}{r^2} \quad \Longrightarrow$$

Kepler's First Law (1609)

When the motion of a particle P_2 is determined by the attractive force of a single mass with center P_1, then the orbit of P_2 is either on the line connecting P_1 and P_2, or it is a **conic** having P_1 as a focus.



1.2 Conics as orbits of planets



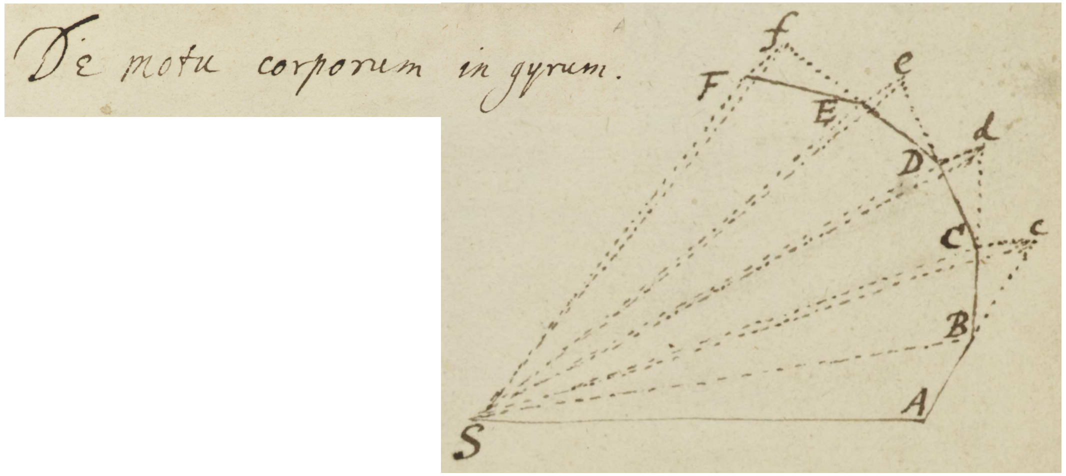
Kepler's Second Law (1609)
When a particle P_2 traverses its orbit around P_1 according to Kepler's First Law, then it moves with **constant areal velocity**. This means that in time intervals of equal duration, the line segment P_1P_2 sweeps sectors of equal areas.

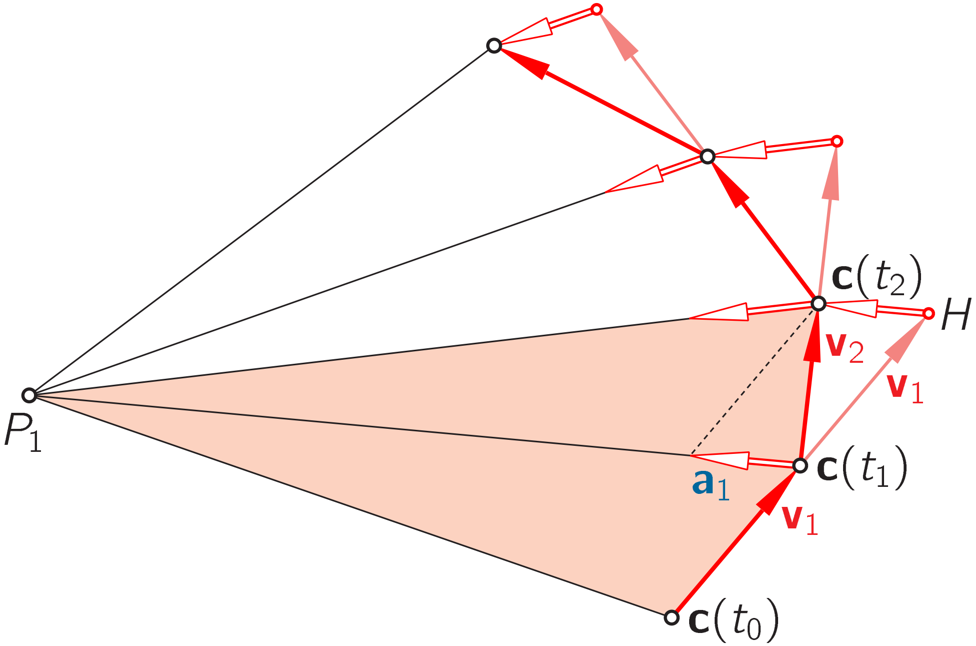
Left: Earth's orbit around the Sun

1.2 Conics as orbits of planets

Newton's proof of the 2nd Law :

Let $t_0, t_1, \ldots, t_n$ be a uniform subdivision of a time intervall, i.e., $t_i - t_{i-1} = h = 1$:





velocity vectors $\mathbf{v}_i \sim \mathbf{c}(t_i) - \mathbf{c}(t_{i-1})$,
acceleration vectors $\mathbf{a}_i \sim \mathbf{v}_{i+1} - \mathbf{v}_i$.

Triangles with equal areas:
$P_1\mathbf{c}(t_0)\mathbf{c}(t_1)$, $P_1\mathbf{c}(t_1)H$, $P_1\mathbf{c}(t_1)\mathbf{c}(t_2)$.

1.2 Conics as orbits of planets

Newton's proof of the 2nd Law :

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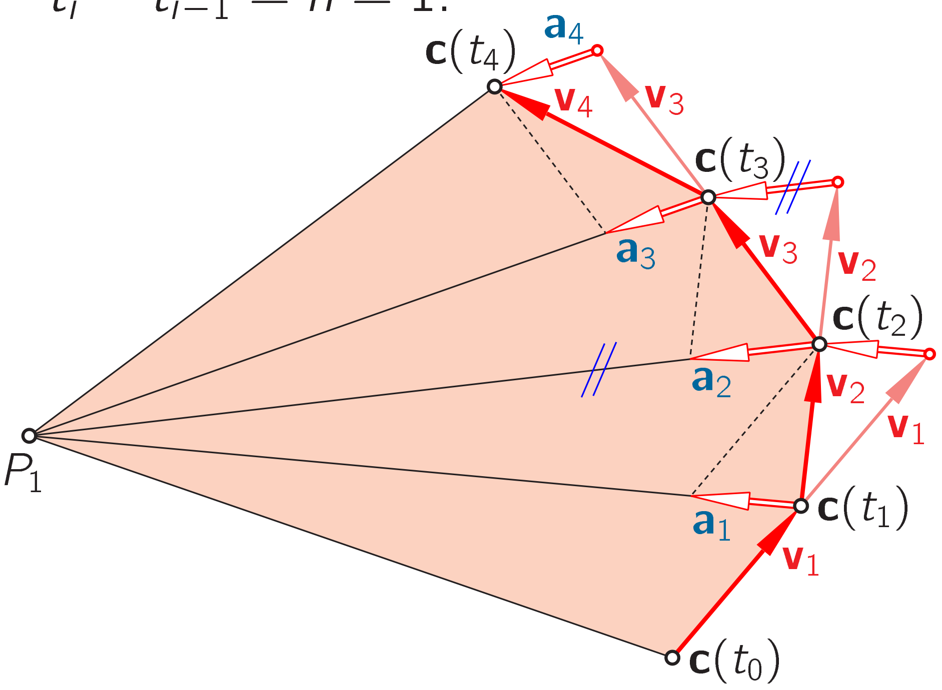

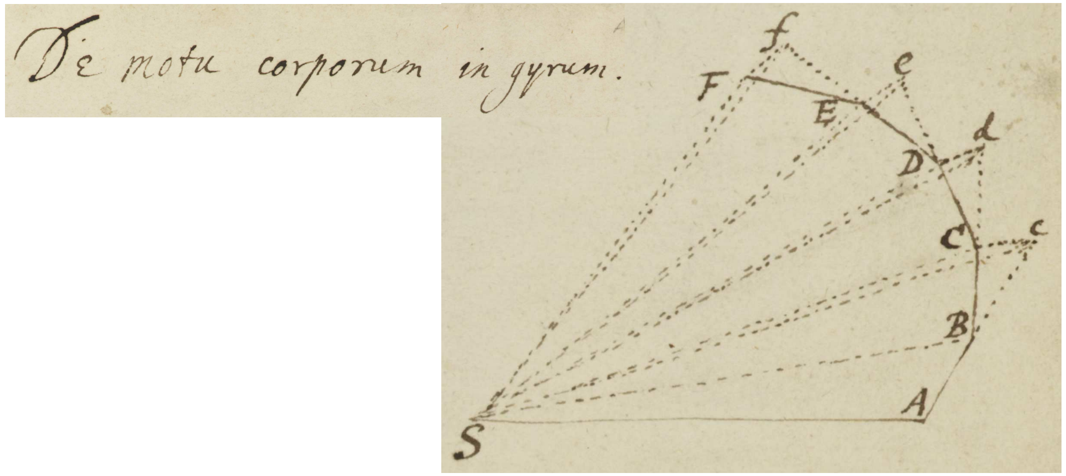

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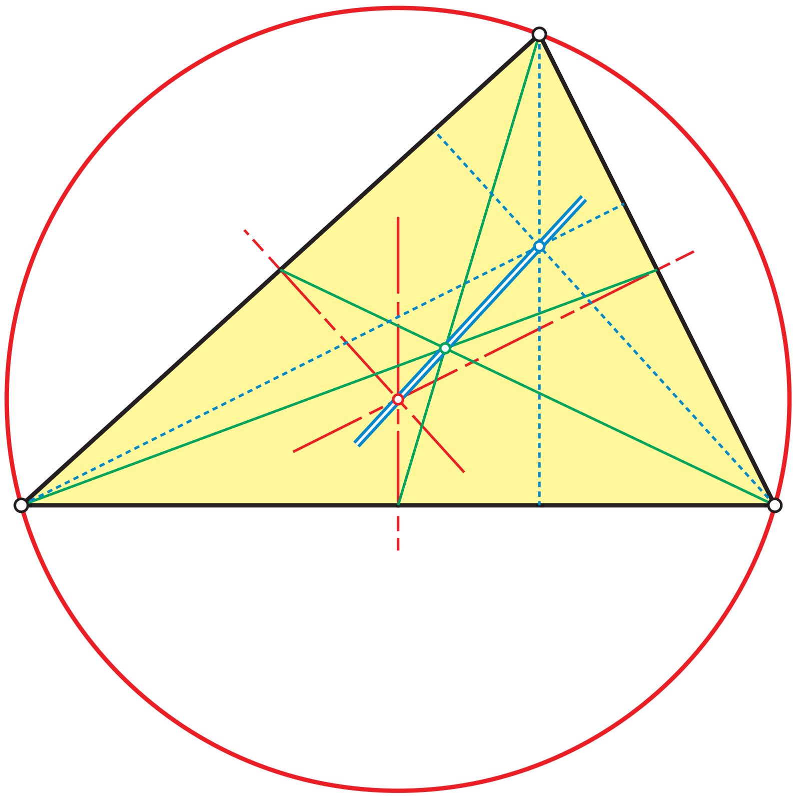
1.3 Geometry of triangles

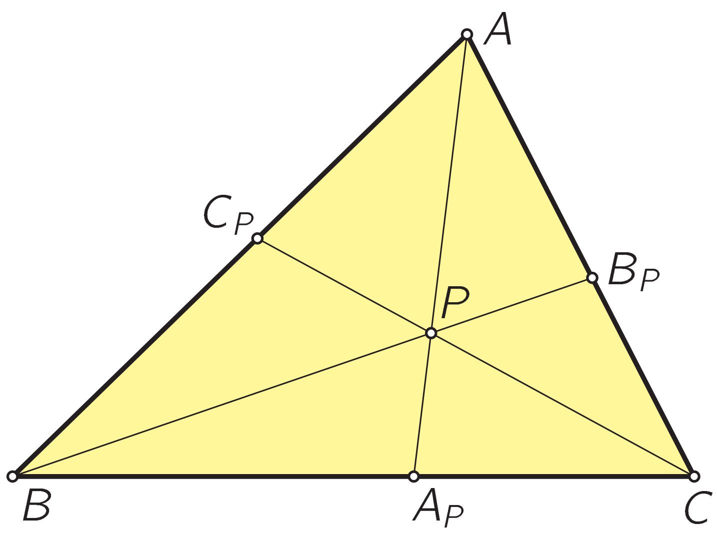


For many people geometry begins and ends with the geometry of **triangles** (orthocenter, circumcenter, centroid, Euler line)

Clark Kimberlings's Encyclopedia of Triangle Centers includes up today **25737** centers!

http://faculty.evansville.edu/ck6/encyclopedia/etc.html

1.3 Geometry of triangles

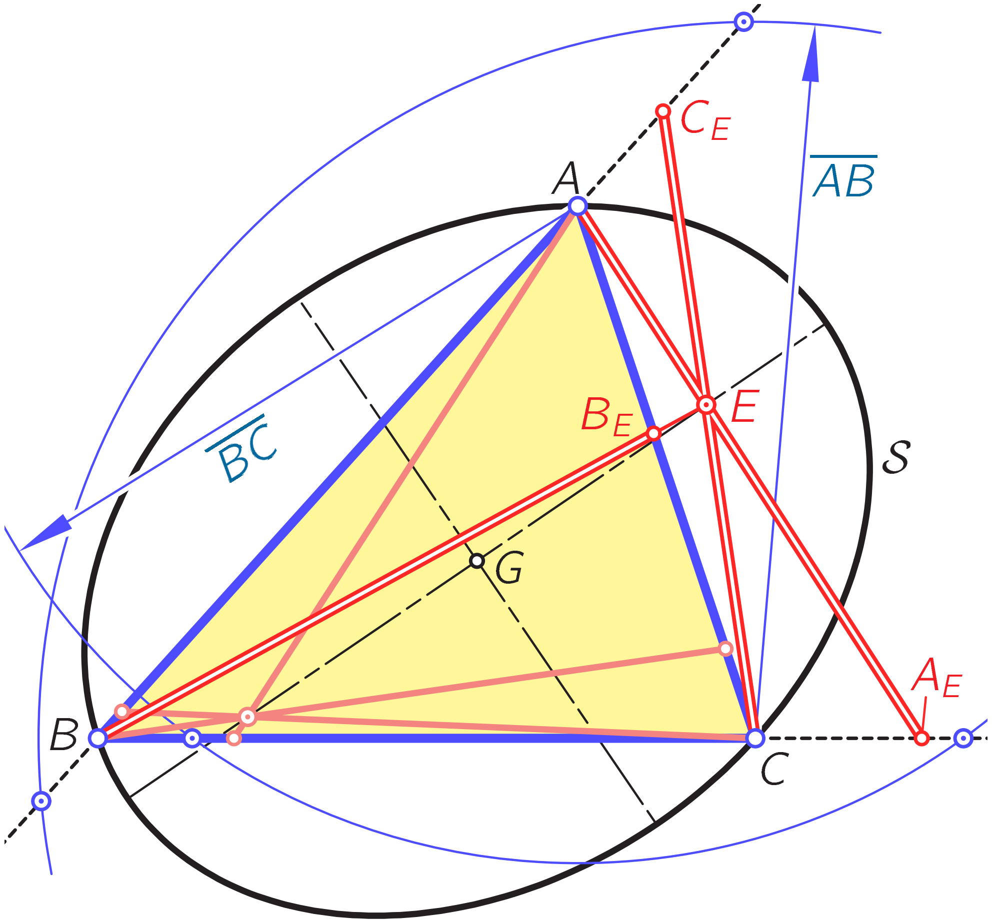


Example: For any point $P \neq A, B, C$ the segments AA_P, BB_P, and CC_P, are called **cevians** of the point P.

Giovanni Ceva, 1647–1734, Milan/Italy.

The point P is called **equicevian**, if its three cevians have the same lengths, i.e., $\overline{AA_P} = \overline{BB_P} = \overline{CC_P}$.

1.3 Geometry of triangles



Theorem: For each triangle ABC, the non-trivial equicevian points are identical with the two real and two complex conjugate focal points of the Steiner circumellipse $\mathcal{S}$.

1.4 Geometry and perspectives

The study of perspectives influenced the development of a new geometry called **Projective Geometry**.

Left: **Jean François Nicéron:**
Ritratto di Luigi XIII
(Portrait of Louis XIII of France)
$\sim$ 1635, Palazzo Barberini, Rome

1.4 Geometry and perspectives

Left: **Johannes Vermeer van Delft**
De Schilderkunst [The Art of Painting]
(1666/1668) Vienna, Kunsthistorisches Museum,
$1.00 \times 1.20\,\mathrm{m}$

My coauthor Gerhard Gutruf, a Viennese artist, opposed against the general opinion that Vermeer used a *camera obscura* for producing the perspective.

Philip Steadman: *Vermeer's Camera*, New York 2001.

1.4 Geometry and perspectives

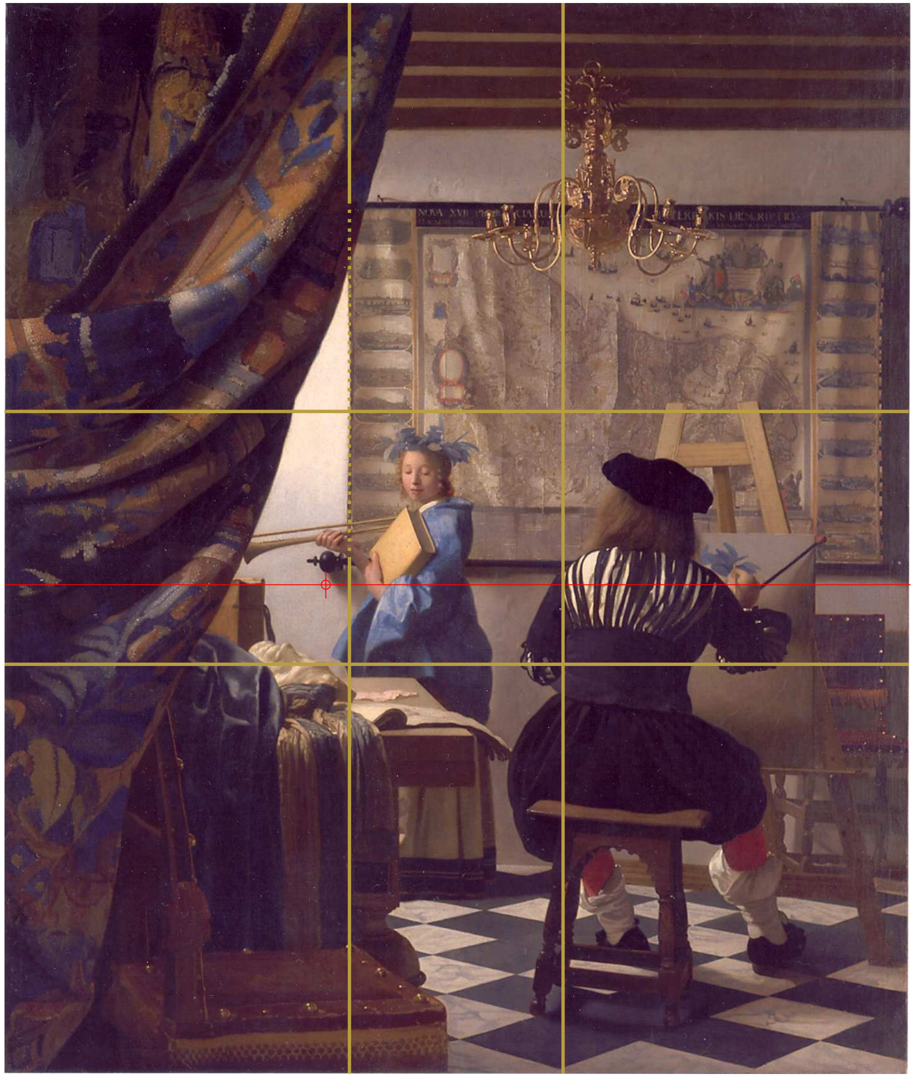
Vermeer himself called it ‘The Art of Painting’.

G. Gutruf:
‘It was a designed masterpiece’.

What is meant with the ‘Art of Painting’ ?

Obviously, it is not a ‘real’ scene like *‘Girl with a pearl earring’*.

The perspective is perfect, apart from a few flaws – due to rules of composition. But there are hidden secrets to be disclosed.



Noting the golden ratio:

- the left line passes through the left border of the wall-map
- The painter seems to paint the central motive on his canvas

Noting the pentagon:

- the curtain follows the left-hand diagonal
- the right-hand diagonal passes exactly through the painter's stick
- the city of Delft on the map is an intersection point of diagonals

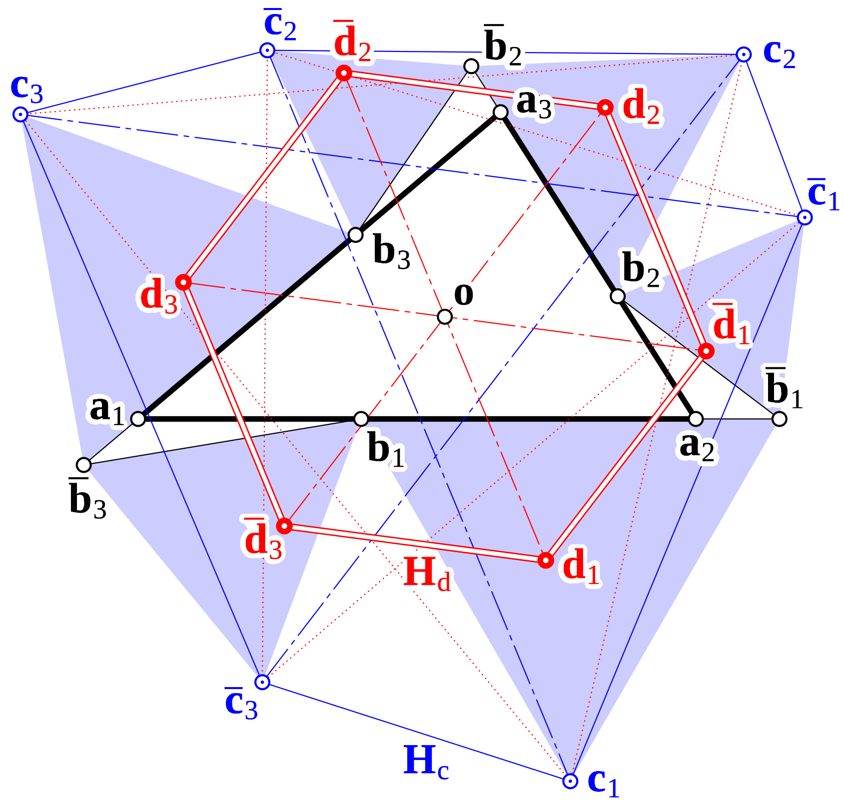
2. What is geometry ?

What is geometry ?

or, more precise:

Where are the borders of geometry within mathematics?
Is geometry a part of mathematics?

“Geometry is what geometers are doing!”



Generalization of a Theorem of Napoleon: Fukuta’s theorem

2. What is geometry ?

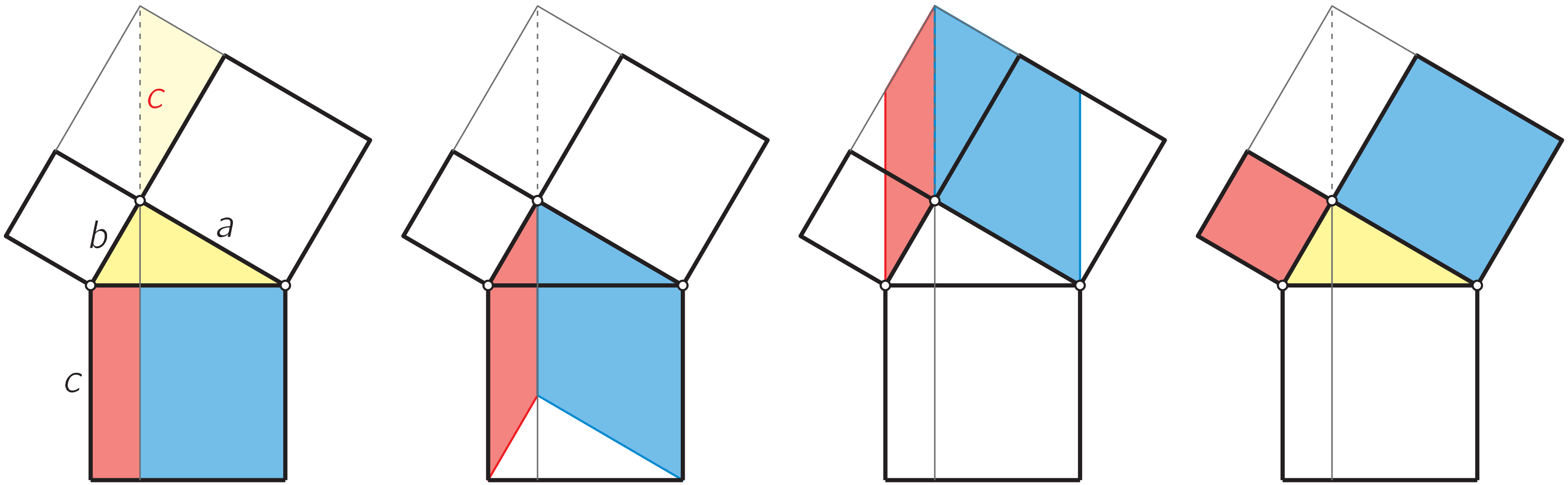
Geometry has to do with images. But images only illustrate;

- they elucidate statements or mathematical ideas,
- they offer a control by inspection,
- they document different steps within any proof.

However, they never replace a proof, because this is based on a sequence of logically rigorous conclusions.

Felix Klein (1849–1925): *'Among all mathematicians, geometers have the advantage to see what they are studying.'*

2. What is geometry ?



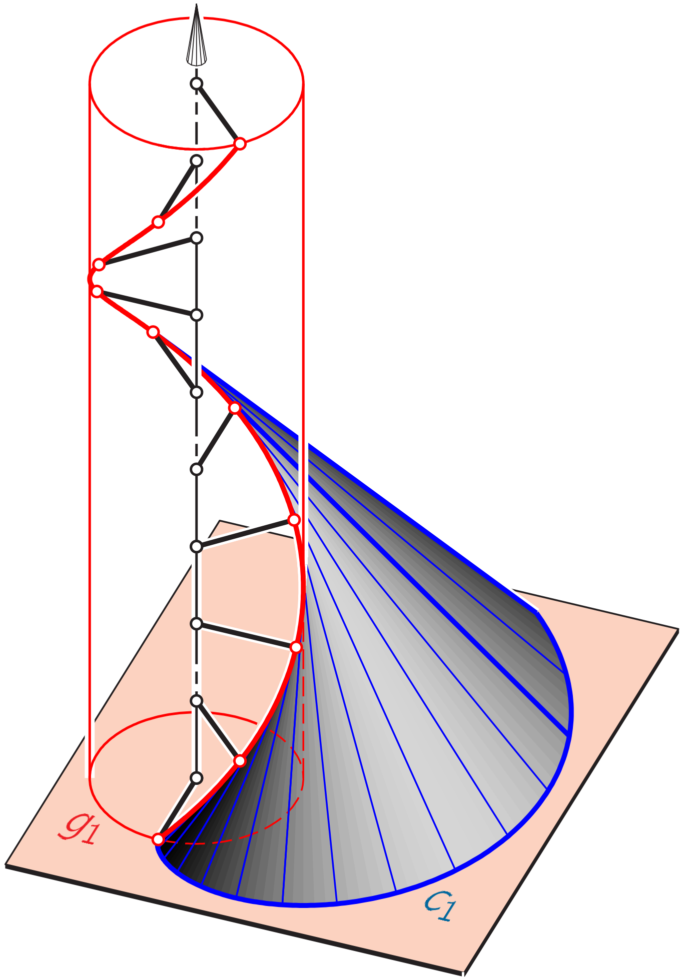
This is an example of a self-explaining proof in mathematics, a visual demonstration of Pythagoras' Theorem: $c^2 = a^2 + b^2$

Such geometric proofs are most elegant, but they often require more attention, skills and creativity than those based on Algebra and Analysis.

2. What is geometry ?

Bill Casselman: *Mathematical Illustrations*
A Manual of Geometry and PostScript
Cambridge University Press 2005

The magic of geometry in mathematics, even at the most sophisticated level, is that geometrical concepts are somehow more visible than others.



2. What is geometry ?

One possible definition is a list of topics within the 2010 Mathematics Subject Classification of the American Mathematical Society

51	GEOMETRY
52	CONVEX AND DISCRETE GEOMETRY
53	DIFFERENTIAL GEOMETRY
14	ALGEBRAIC GEOMETRY
54	GENERAL TOPOLOGY
57	MANIFOLDS AND CELL COMPLEXES

However, this must be completed by geometric topics in technics, natural sciences and computer science.

Right: René Descartes: *La Géométrie*, 1637

LA

GEOMETRIE.

LIVRE PREMIER.

Des problеſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.

TOus les Problеſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin par aprés que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

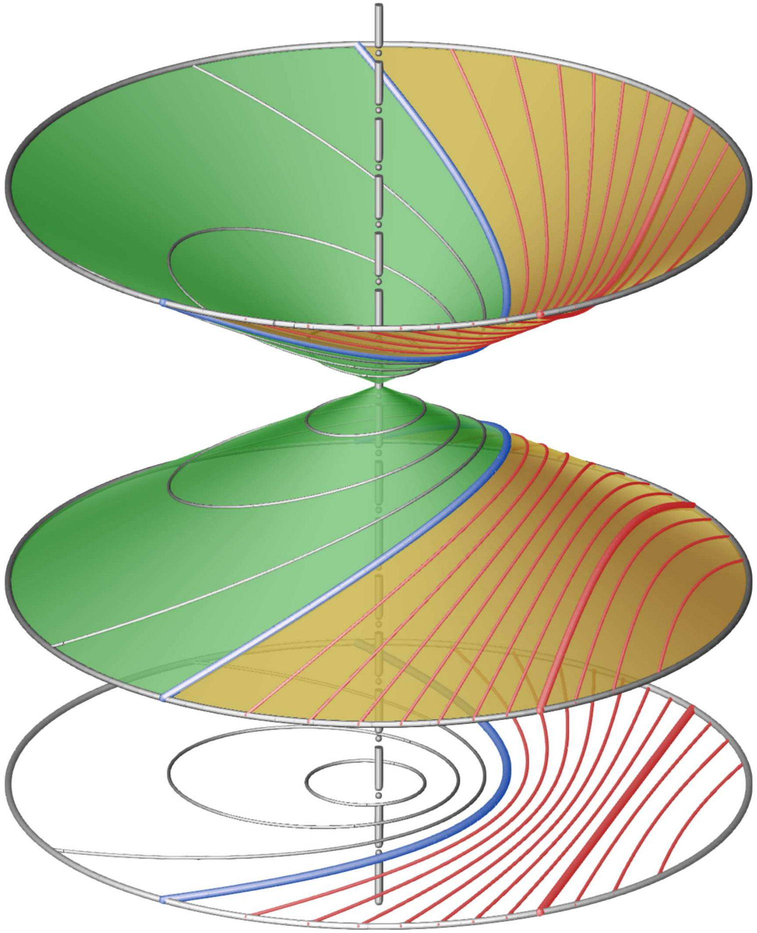
Et comme toute l'Arithmetique n'eſt compoſée, que de quatre ou cinq operations, qui ſont l'Addition, la Souſtraction, la Multiplication, la Diuiſion, & l'Extraction des racines, qu'on peut prendre pour vne eſpece de Diuiſion : Ainſi n'a-t'on autre choſe a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a eſtre connuës, que leur en adiouſter d'autres, ou en oſter, Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eſtre priſe a diſcretion, puis en ayant encore deux autres, en trouuer vne quatrieſme, qui ſoit à l'vne de ces deux, comme l'autre eſt a l'vnité, ce qui eſt le meſme que la Multiplication, oubien en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux, comme l'vnité

Commēt le calcul d'Arithmetique ſe rapporte aux operations de Geometrie.

2.1 Geometry, a cultural asset over millenia

G. Aumann: *Euklids Erbe*, Darmstadt 2006

'Geometry is much more than a collection of more or less interesting theorems. It is an essential part of our culture.'



Geometry is a basic science. This follows also from its historical development.

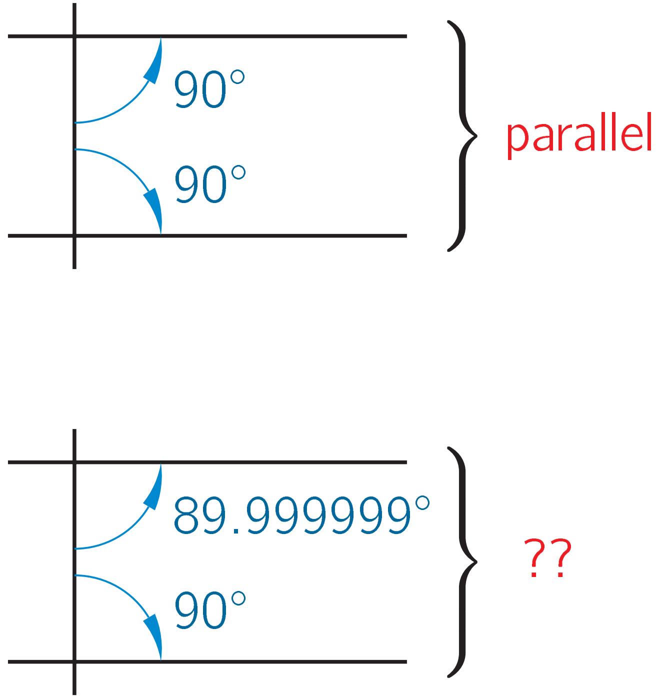
Geometry was the first science with a rigorously deductive structure, based on axioms.

2.1 Geometry, a cultural asset over millenia

Geometry takes place in “ideal spaces”, which need not have any relation to our physical space.

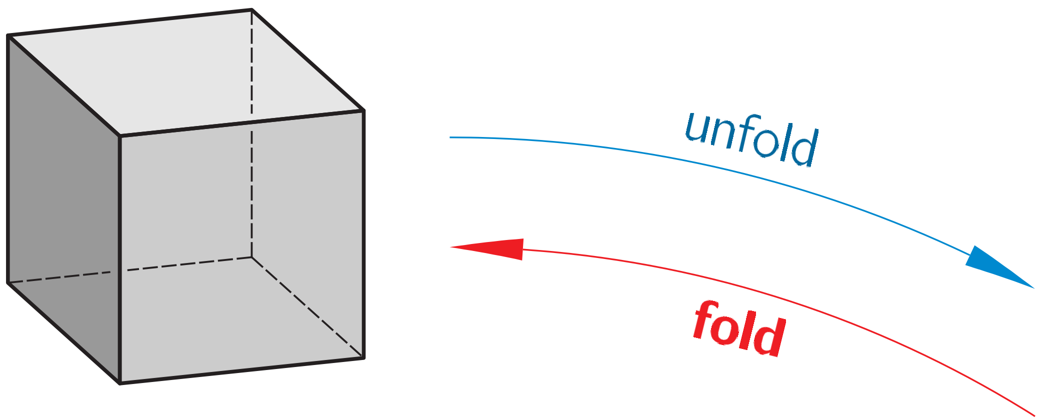
Already Euklid ($\sim$365–300 B.C.) was aware of the fact that the **parallel postulate** can never be proved experimentally.

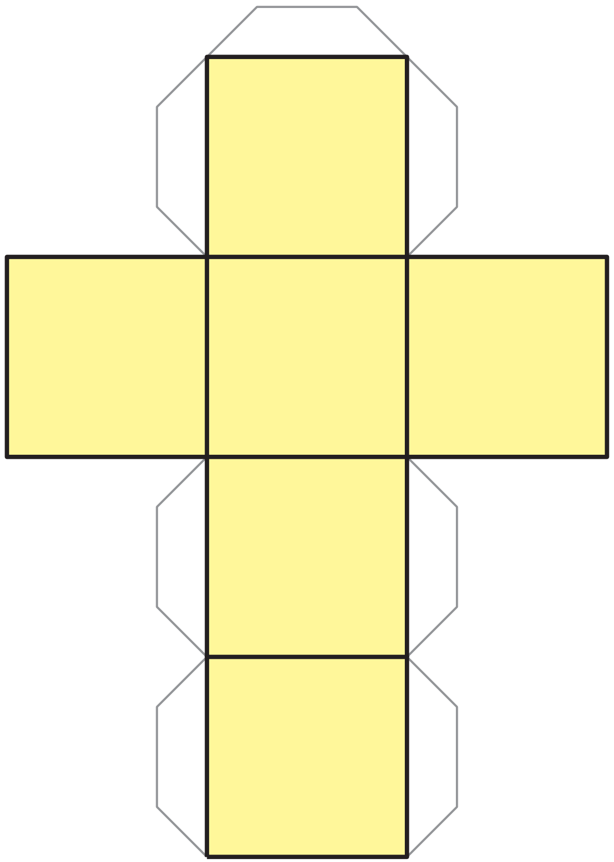
2000 years later **János Bolyai** (1802–1860) proved that there exists a *non-Euclidean Geometry*, where the negation of the parallel postulate is valid.



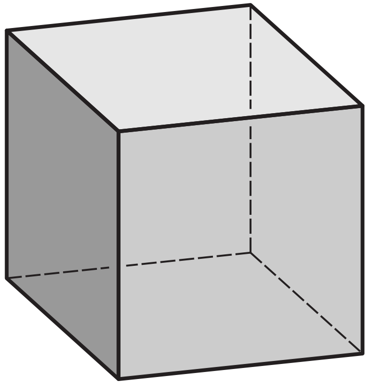
2.1 Geometry, a cultural asset over millenia

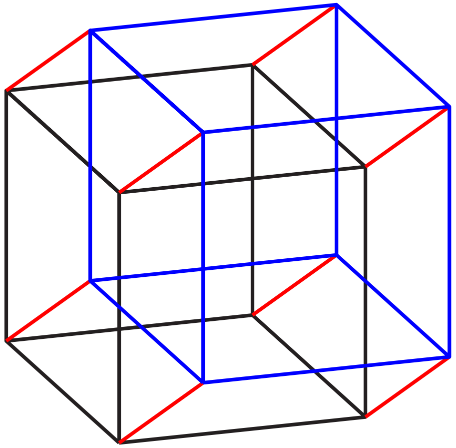
Geometry acts in spaces of all kinds and dimensions.
Here an Euclidean example:

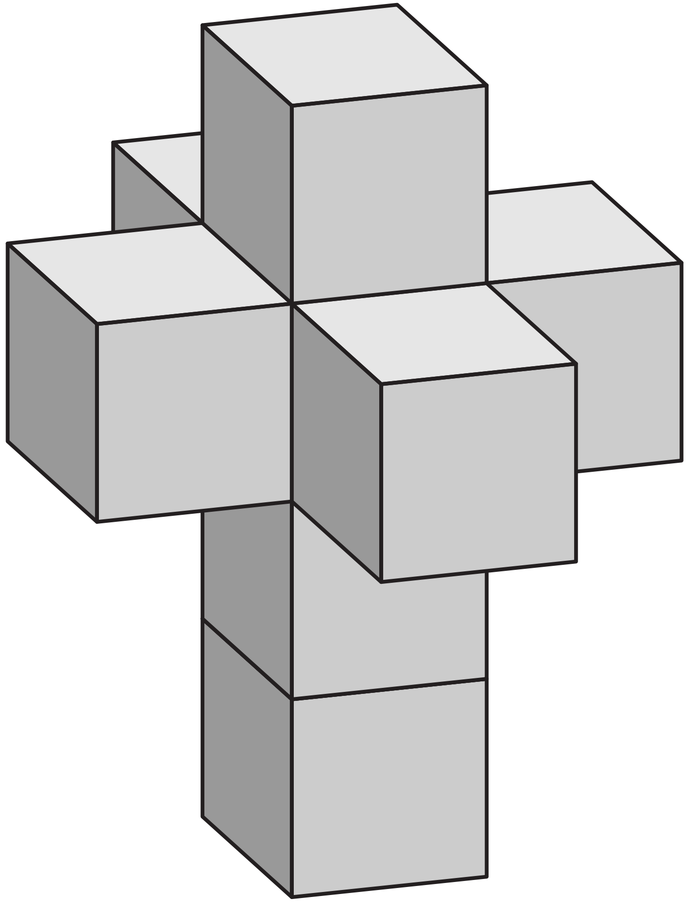




While the result of folding a polyhedron is unique, the inverse problem, i.e., the determination of a folded structure from a given unfolding leads to a system of algebraic equations. The corresponding spatial object needs not be unique.







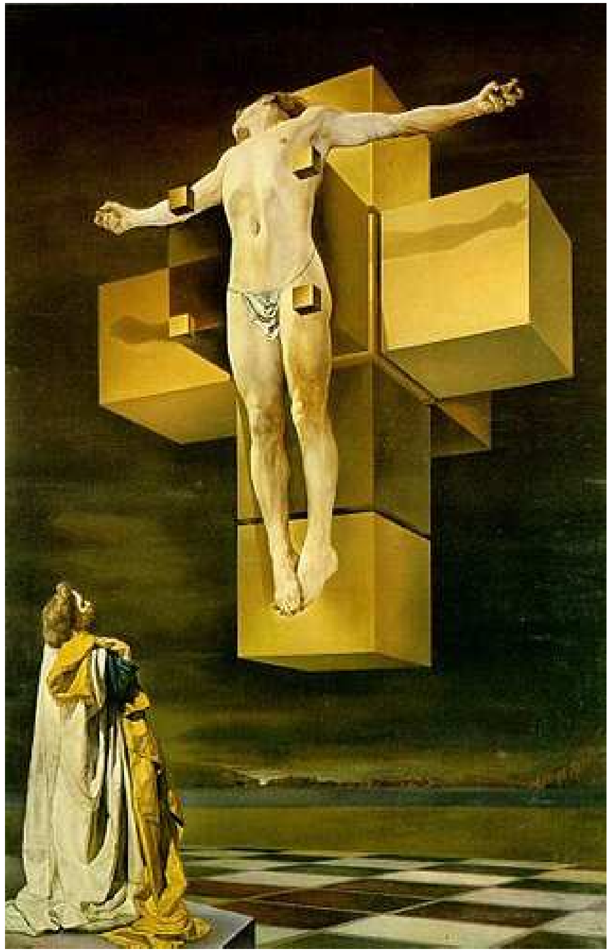
A cube together with its translated copy (in blue) in the 4-space and the trajectories of the vertices (in red) form a **hypercube**.

It has 8 cells (= 3-cubes). Each of the 24 faces is the meet of two cells.

Iterated rotations of cells about a face into the hyperplane of the neighboring cell results in a three-dimensional unfolding.

2.1 Geometry, a cultural asset over millenia

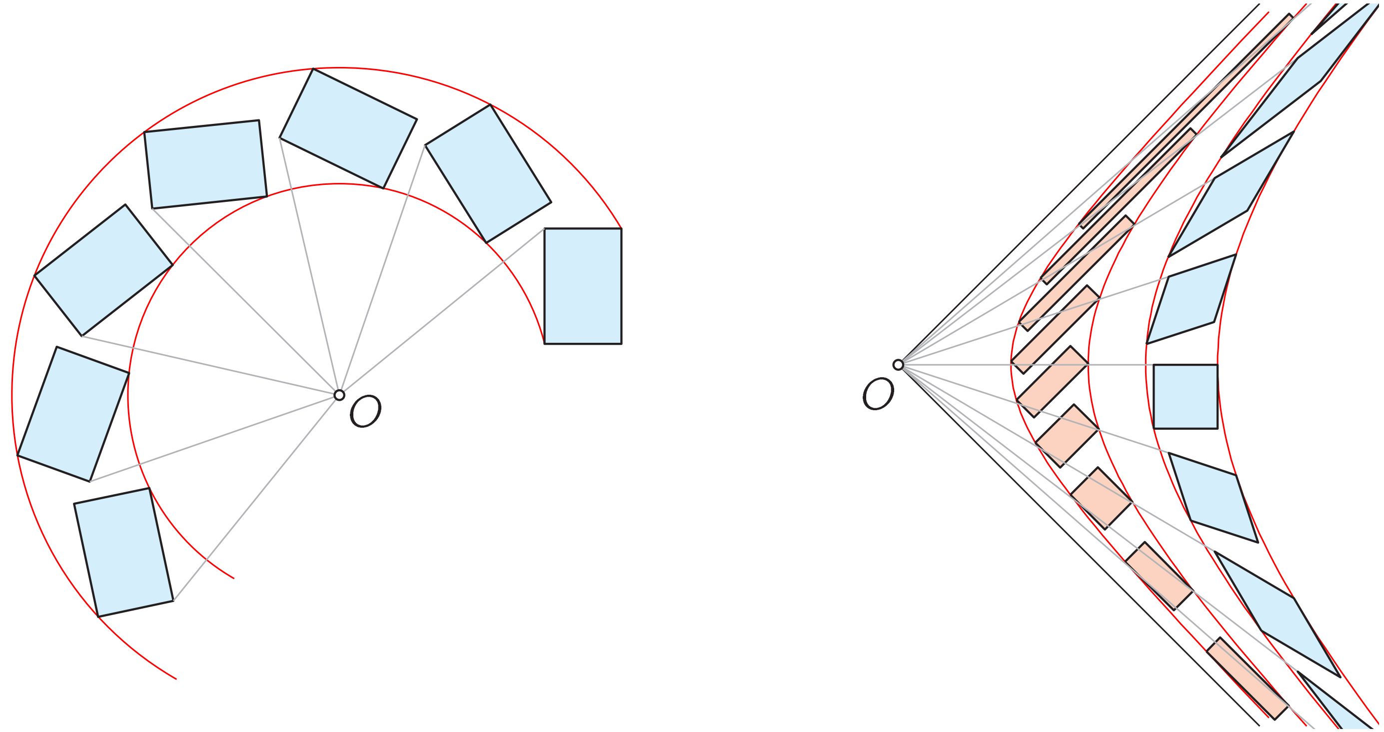
Salvador Dalí: *Corpus Hypercubus*, 1954
$194 \times 124\,\mathrm{cm}$, Metropolitan Museum of Art, New York



Salvador Dalí
(1904–1989)

2.1 Geometry, a cultural asset over millenia

Already before **A. Einstein** detected relativity, a geometric model of Einstein's space time was already available, known as *pseudo-Euclidean* or *Minkowski Geometry*.



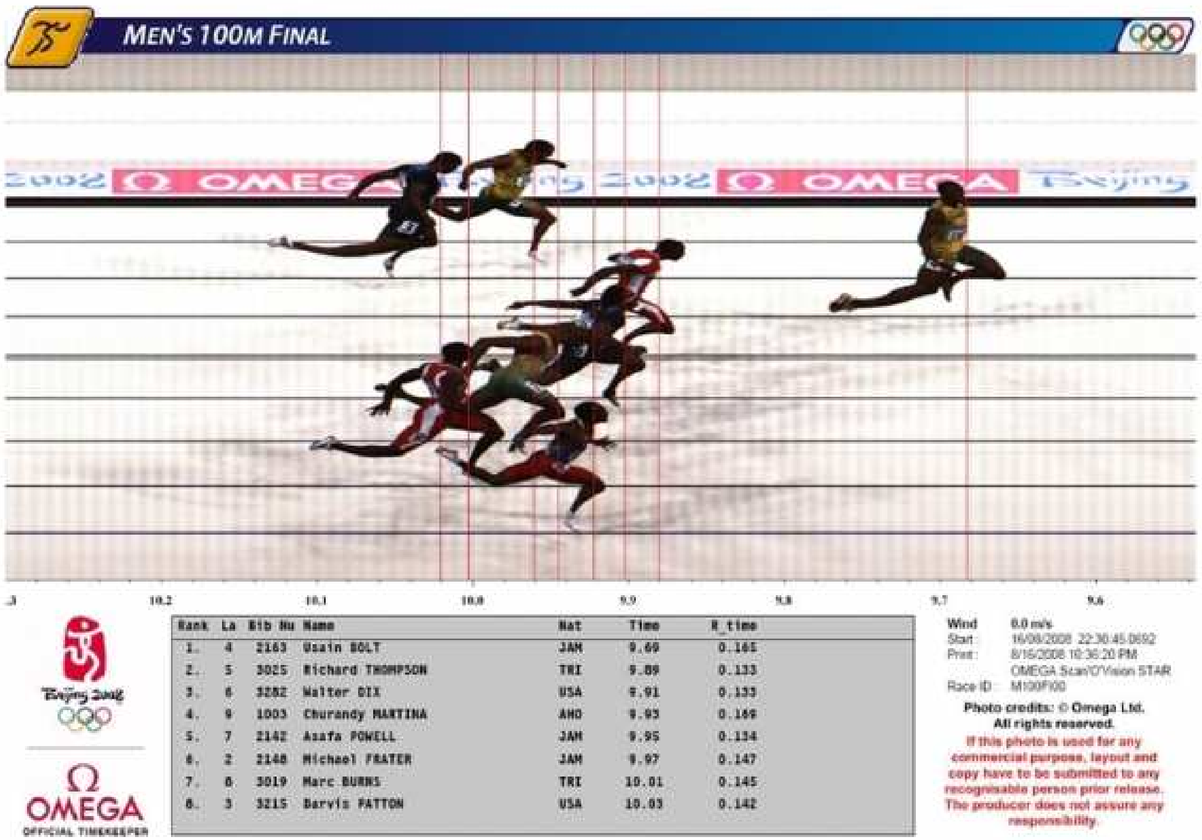
Euclidean (left) and pseudo-Euclidean rotations (right) about the center O.

2.1 Geometry, a cultural asset over millenia

We are facing images of four-dimensional space-time at photo finish record.

In strip photography the camera captures only the sequence of events in the vertical plane through the finish line.

The horizontal axis shows the time scale.



2.2 Descriptive Geometry

With the foundation of Technical Universities around 1800 started the development of Descriptive Geometry.

Gaspard Monge (1746–1818), the founder of the science of Descriptive Geometry, was one of the most prominent mathematicians, but also an effective manager as principal manager of the **École polytechnique** in Paris.

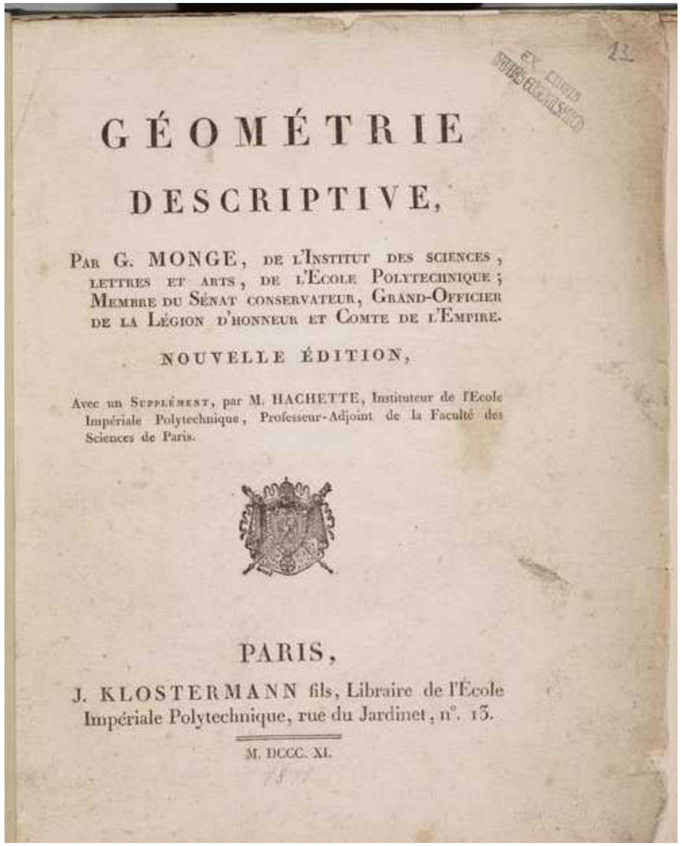
He enjoyed the confidence of Napoleon and occupied leading political positions. After the fall of Napoleon, Monge died in state of mental derangement.

2.2 Descriptive Geometry

La Géométrie descriptive a deux objets:

– **le premier**, de donner les méthodes pour représenter sur une feuille de dessin qui n'a que deux dimensions, savoir, longueur et largeur, tous les corps de la nature qui en ont trois, longueur, largeur et profondeur, pourvu néanmoins que ces corps puissent être définis rigoureusement.

– **Le second objet** est de donner la manière de reconnaître, d'aprés une description exacte, les formes des corps, et d'en déduire toutes les vérités qui résultent et de leur forme et de leurs positions respectives.



GÉOMÉTRIE

DESCRIPTIVE,

Par G. MONGE, de l'Institut des sciences, lettres et arts, de l'École Polytechnique; Membre du Sénat conservateur, Grand-Officier de la Légion d'honneur et Comte de l'Empire.

NOUVELLE ÉDITION,

Avec un Supplément, par M. HACHETTE, Instituteur de l'Ecole Impériale Polytechnique, Professeur-Adjoint de la Faculté des Sciences de Paris.

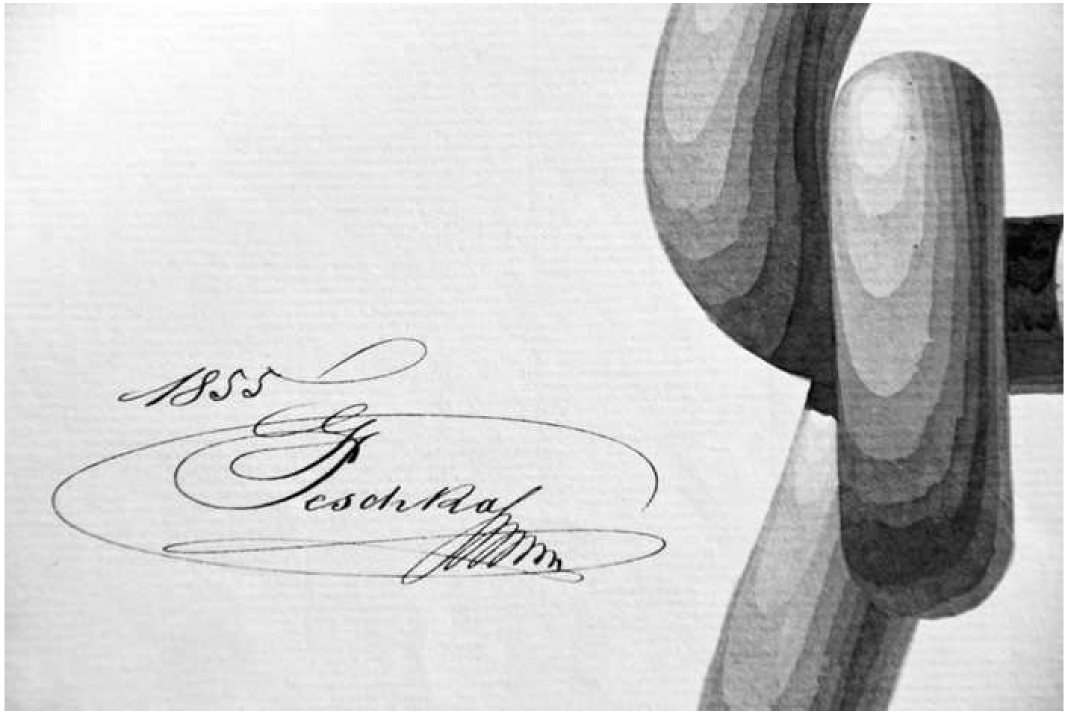
PARIS,

J. KLOSTERMANN fils, Libraire de l'École Impériale Polytechnique, rue du Jardinet, n°. 13.

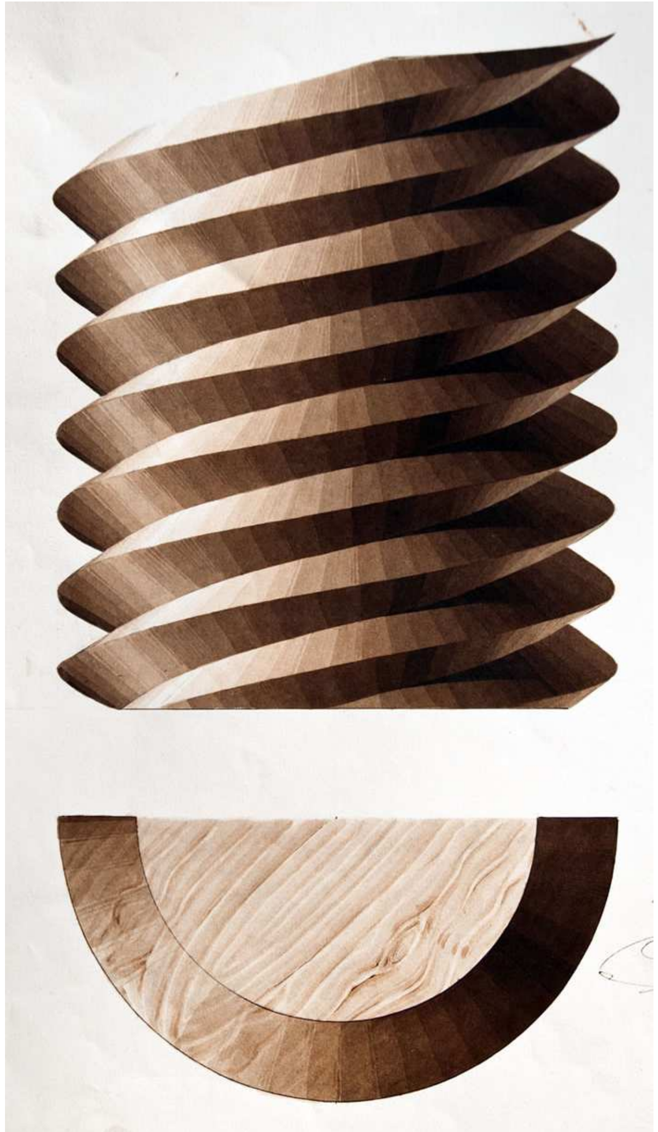
M. DCCC. XI.

2.2 Descriptive Geometry

At the beginning, the goal was to obtain realistic images of 3D-objects.



Students work under Gustav A.V. Peschka (1830–1903) in Brno

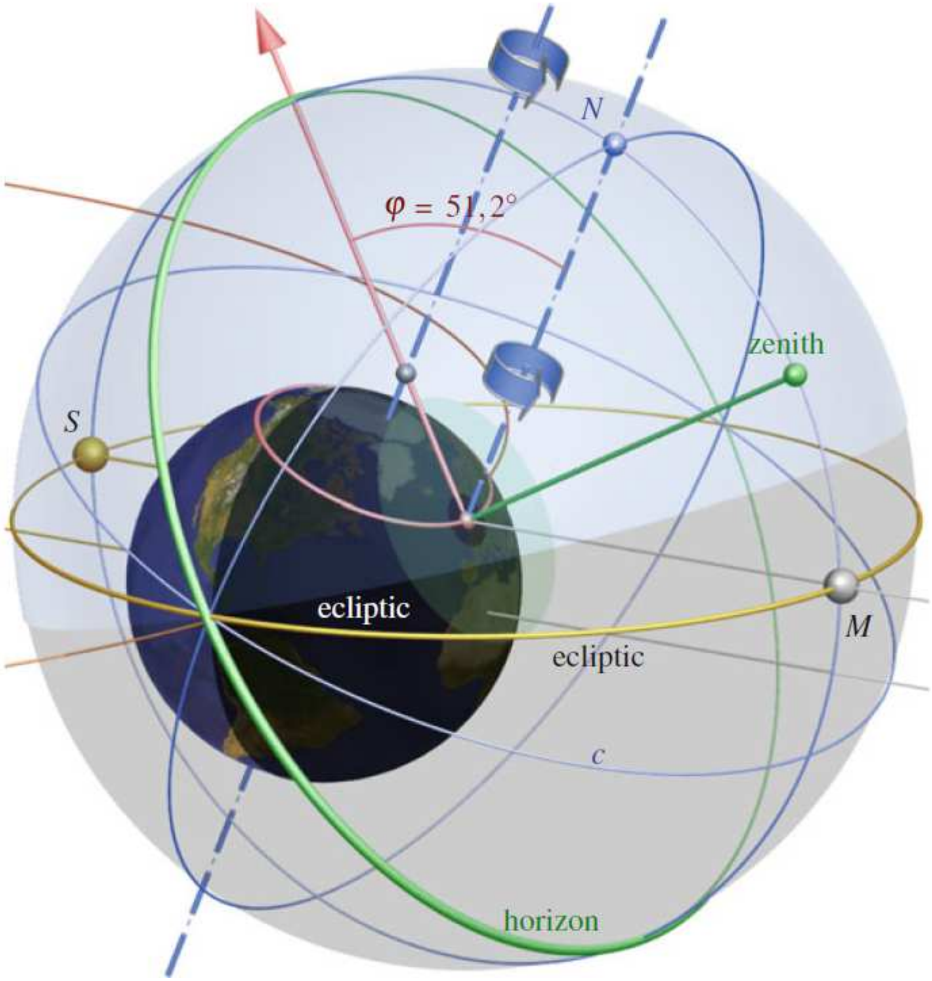

MASC


3. Geometry today

Today, Descriptive Geometry is a method to study 3D geometry through 2D images.

It provides insight into structure and metrical properties of spatial objects, processes and principles.

Are we always able to grasp all the animated and realistically rendered objects ?



3. Geometry today

Spatial ability is the capacity to understand, reason and remember the spatial relations among objects in space. There are four common types of spatial abilities:

- spatial or visuo-spatial perception, i.e., the ability to perceive spatial relationships in respect to the orientation of one's body
- spatial visualization, i.e., complicated multi-step manipulations of spatially presented information.
- mental rotation, i.e., the mental ability to manipulate and rotate 3D objects,
- spatial working memory, i.e., the ability to temporarily store a certain amount of visual-spatial memories under attentional control in order to complete a task.

These skills are not inborn. It needs a lot of **training** until students master signs and symbols of engineering and technical graphics and get the spatial objects behind.

3. Geometry today

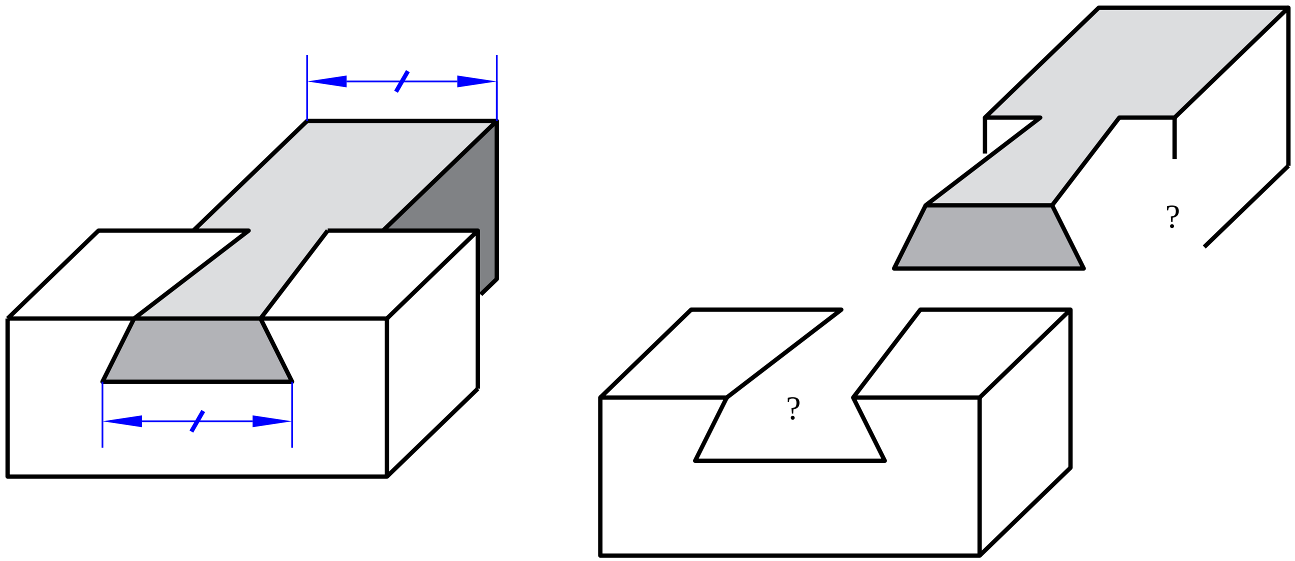
The training of spatial visualization should bring about the ability

- to comprehend spatial objects from given principal views,
- to specify and grasp particular views (auxiliary views),
- to get an idea of geometric idealization (abstraction), of the variety of geometric shapes, and of geometric reasoning.

The first two items look elementary. However, these intellectual abilities are so fundamental that many people forget how hard they were to achieve.

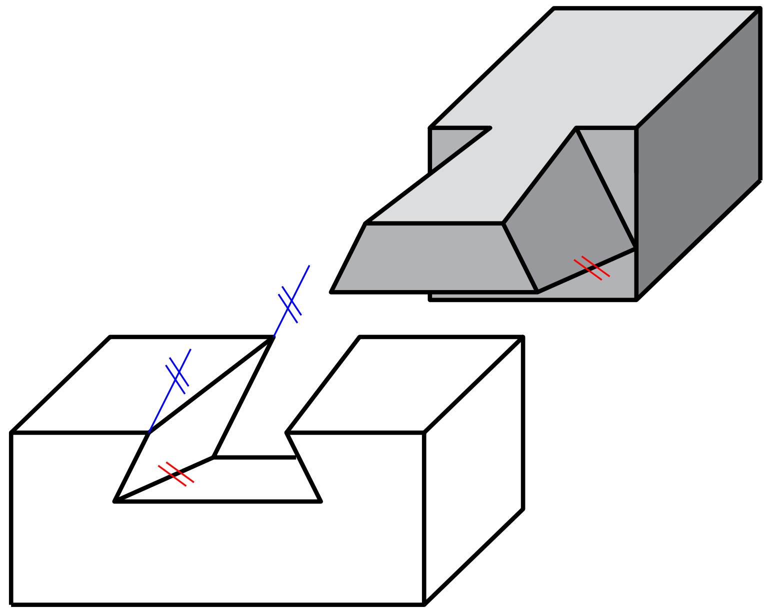
3.1 Improving spatial ability

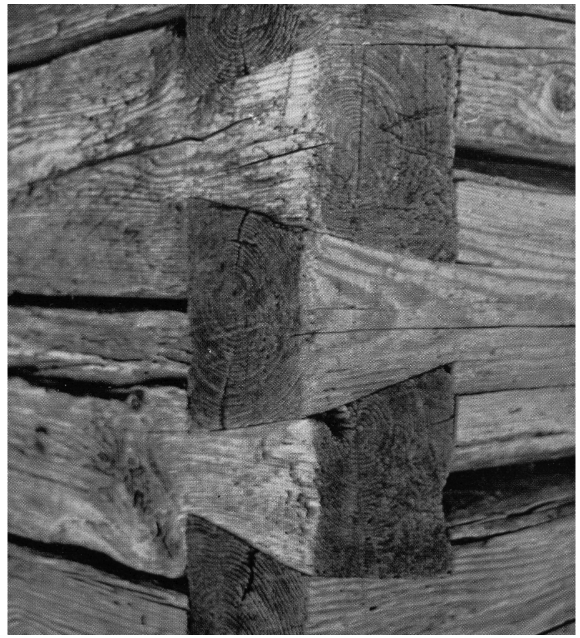
Example 1: *Show the two components of the wooden joint displayed below.*



While the two components are pulled apart, they remain in contact along three sliding planes. Two of these planes can be figured out from the view on the left hand side; each is spanned be two visible lines.

3.1 Improving spatial ability





The connection of the wooden beams displayed on the photo looks similar. However, this wooden structure is assembled layer by layer, and one cannot finally pull out a single beam from the blockhouse.

3.1 Improving spatial ability

Spatial intelligence is the ability to think in three dimensions. Core capacities include mental imagery, spatial reasoning, image manipulation, graphic and artistic skills, and an active imagination.

Who is able to manipulate 3D objects mentally in space without using any drawing ?

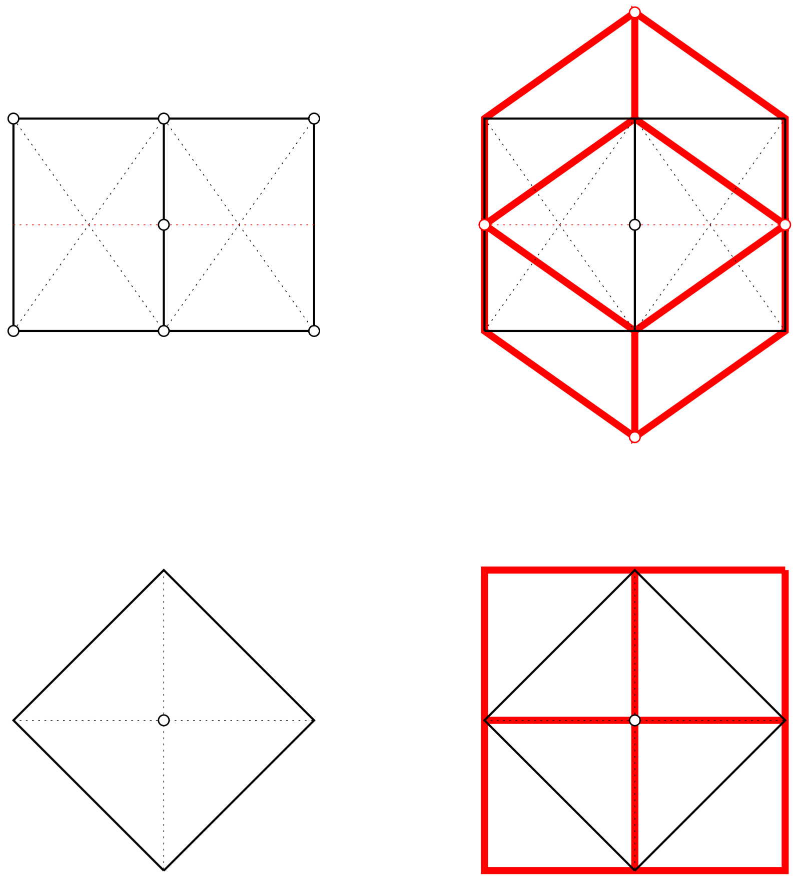
Maybe, pilots, sculptors, surgeons, painters, and architects all exhibit spatial intelligence.

3.1 Improving spatial ability

Example 2: The *rhombic dodecahedron* can be built by erecting quadratic pyramides with 45° inclined planes over each face of a cube.

Any two coplanar triangles can be glued together forming a rhomb.

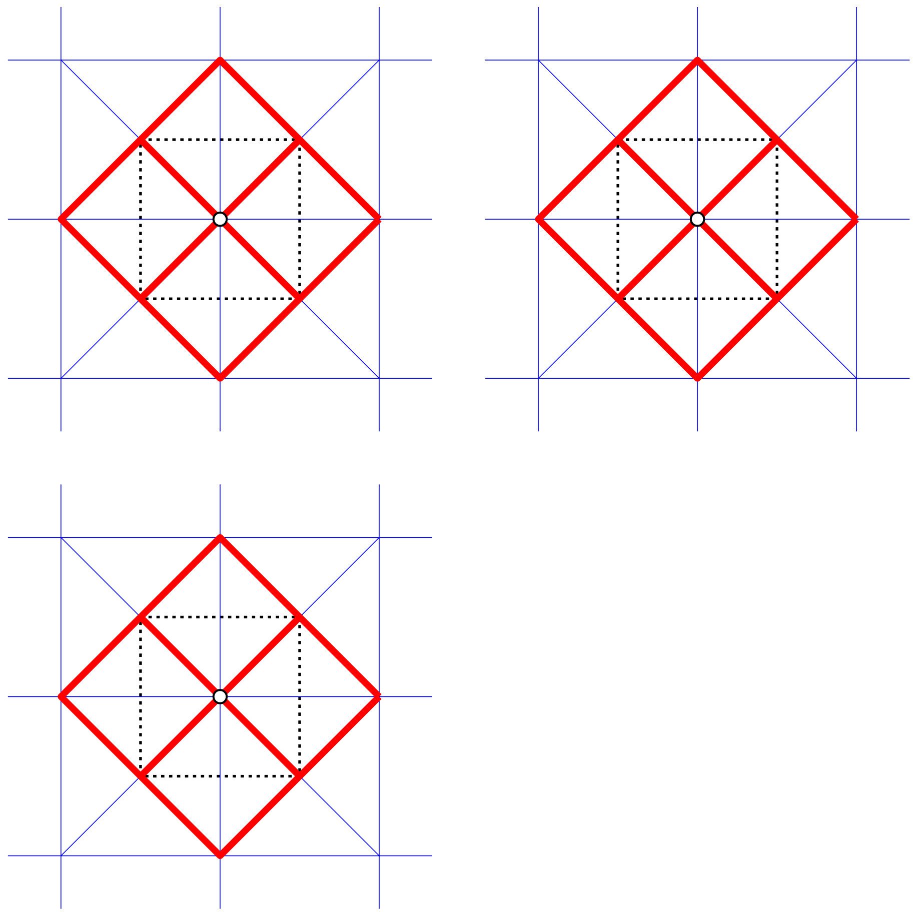
Question:
How does this polyhedron look like from above when it is resting with one face on a table?



Cube and rhombic dodecahedron

3.1 Improving spatial ability

The rhombic dodecahedron is the *intersection of three quadratic prisms* with pairwise orthogonal axes.

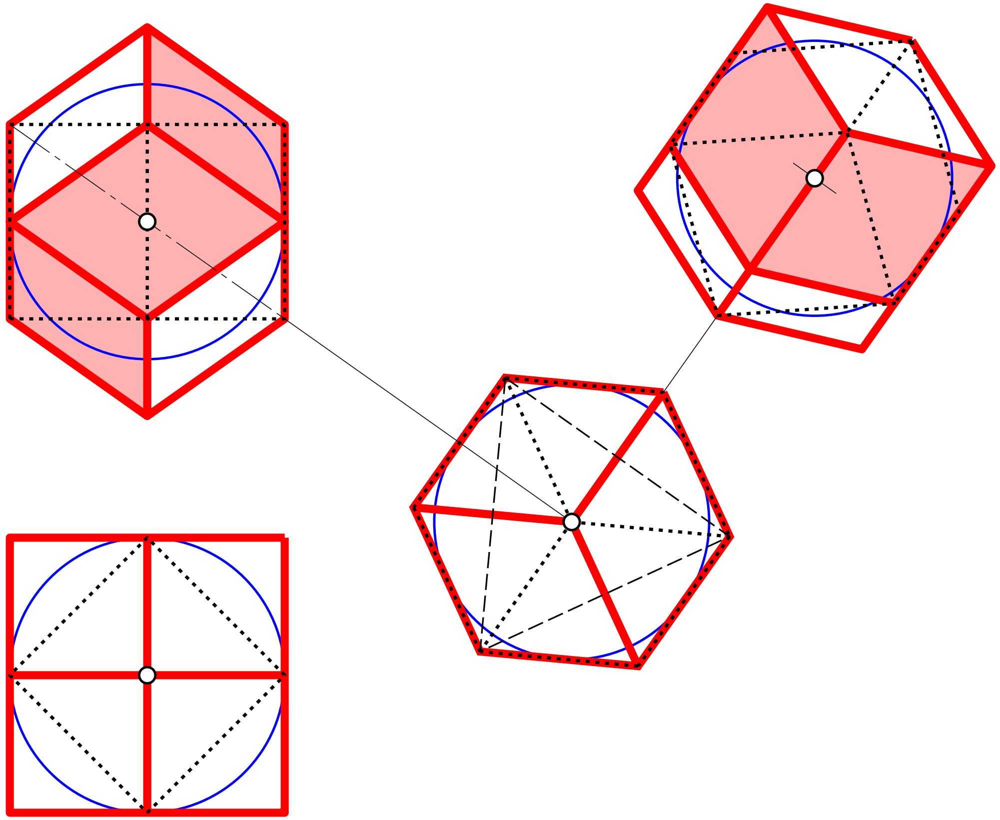


3.1 Improving spatial ability

The rhombic dodecahedron is the *intersection of three hexagonal prisms* with axes in direction of the cube-diagonals.

The side and back walls of a *honey comb* belong to a rhombic dodecahedron.

Each *dihedral angle* has 120°, and there is an in-sphere (contacting all faces of the initial cube).



3.1 Improving spatial ability

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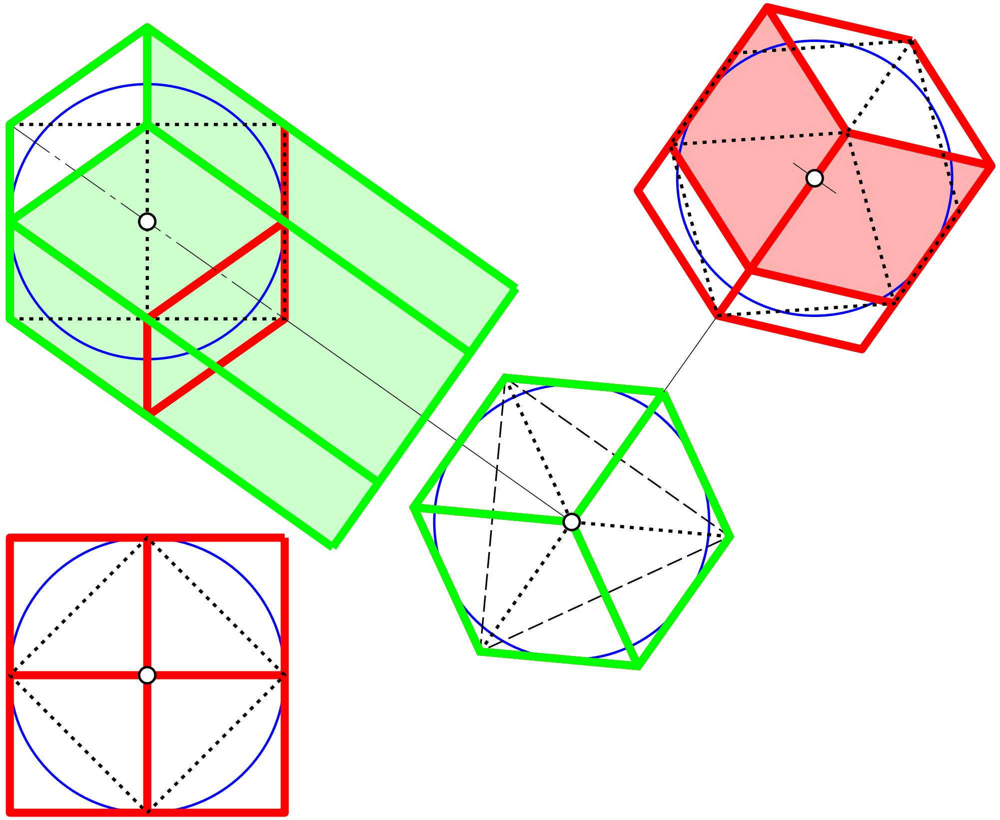


3.1 Improving spatial ability

The rhombic dodecahedron is a space-filling polyhedron.

Proof:

- Start with a *'3D-chessboard'* built from black and white cubes.
- Then the 'white' cubes can be *partitioned* into 6 quadratic pyramides with the vertex at the cube's center.
- Glue each pyramide to the *adjacent* 'black' cube thus enlarging it to a rhombic dodecahedron. □

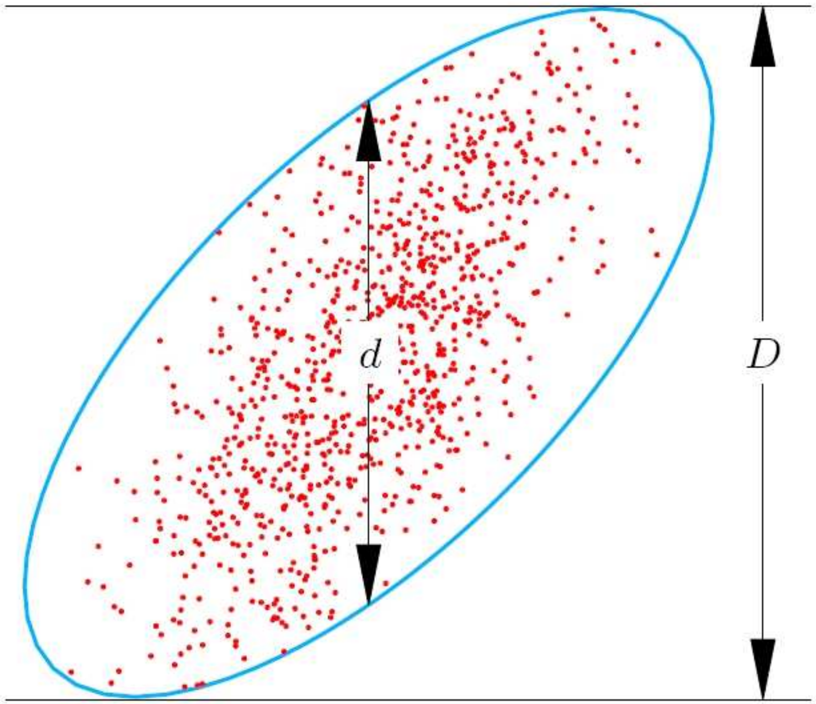
3.2 Applications of Geometry

The most important role of Geometry within mathematics, technical sciences, medicine and computer science is to offer modelle and thus provide visualizations.

This works in different *spaces*, the space of functions, of colors, of data.

The word 'geometry' usually stands for metrical properties or dimensions of objects.

Right: an example from Statistics

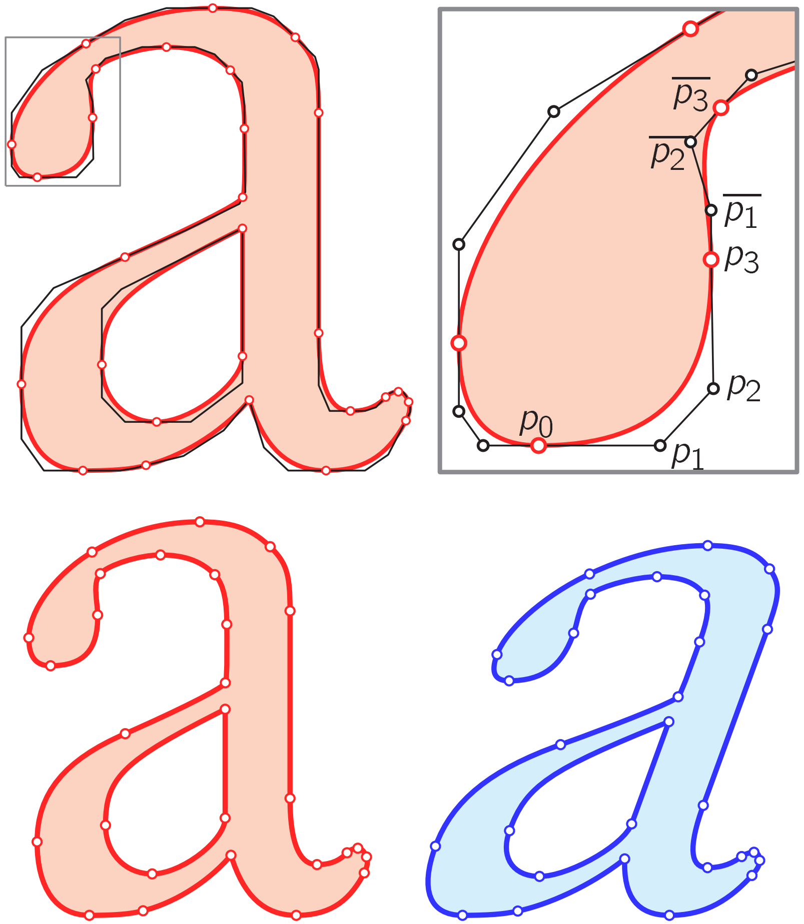


correlation coefficient $r = \sqrt{1 - (d/D)^2}$,
ellipse of concentration

3.2 Applications of Geometry

Computer-aided geometric design:

Béziercurves and -surfaces as well as B-spline-curves and surfaces offer new methods of design



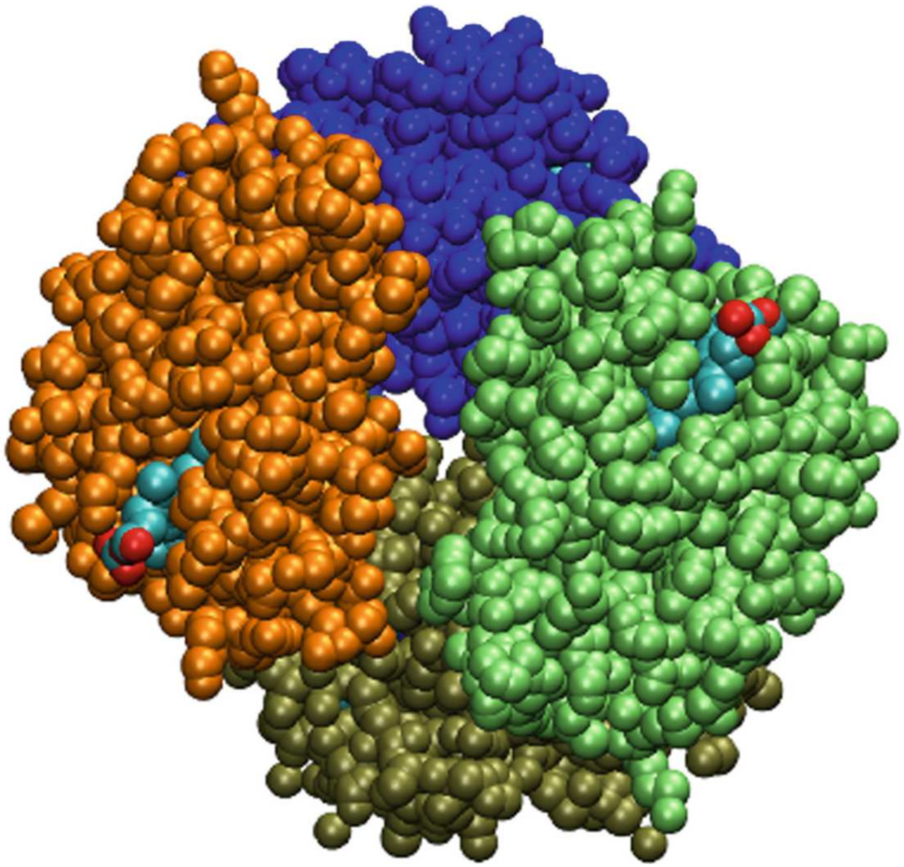
3.2 Applications of Geometry

In **Freeform Architecture** most of the surfaces are designed as *quad meshes*

— like the Capital Gate in Dubai (height = 160 m, inclination 18°)
Steel-glass construction: Wagner Biro, Austria

3.2 Applications of Geometry

Molecular Geometry:



Guido Raos, Polytecnico di Milano:

'Molecular Geometry is of pervasive importance for chemistry, from the 19th century up to the present day. Advances in molecular graphics, alongside those in experimental and computational methods, allow chemists, materials scientists and biologists to reveal structure and properties of ever more complex materials.'

Left: The four polypetide chains making up Hemoglobin

3.2 Applications of Geometry

Kinematics and Robotics
are emerging fields of applied geometry.

Right: Design of a film-pull mechanism in a professional (analog) motion picture camera of ARRI

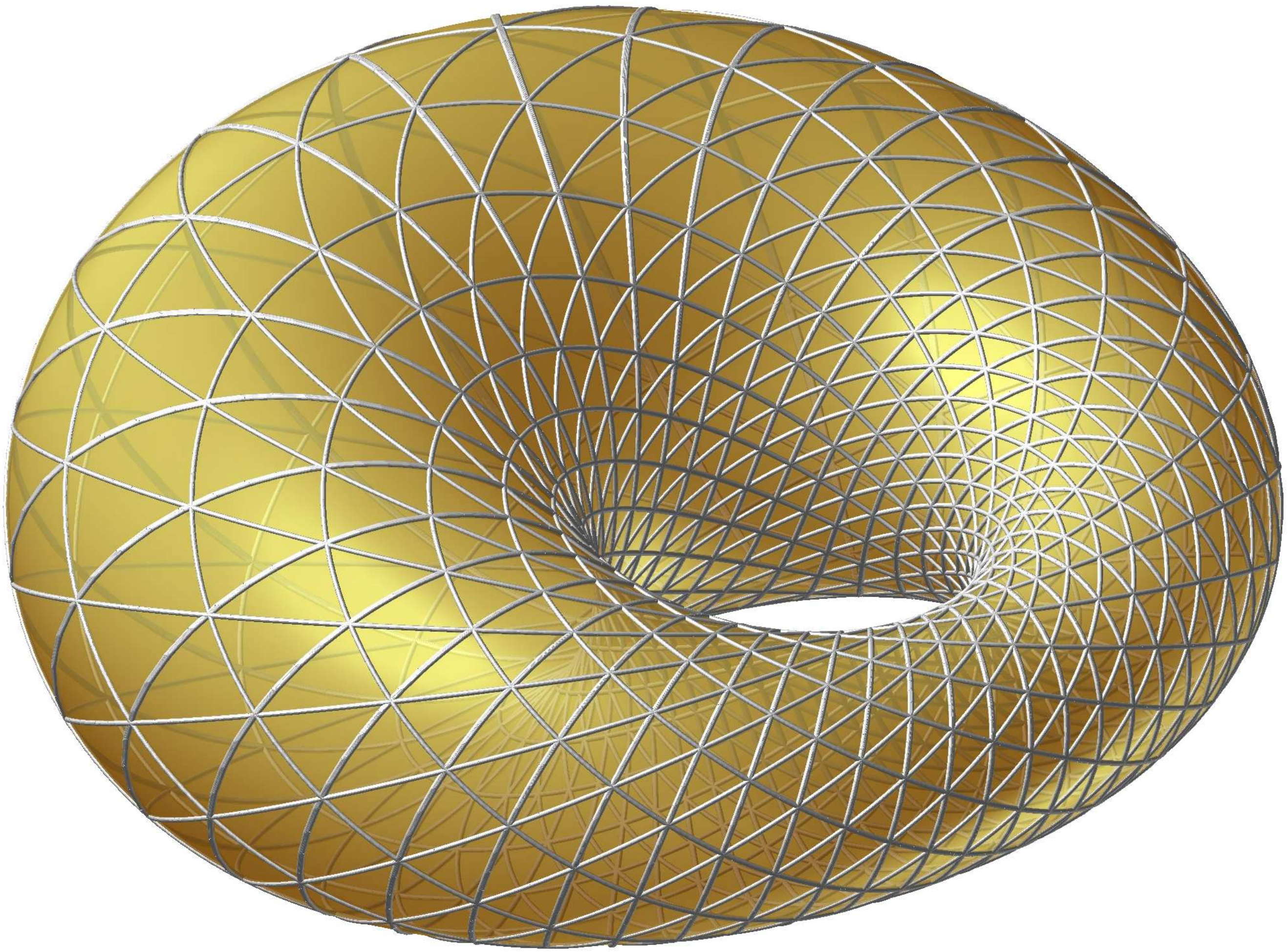
3.2 Applications of Geometry

Biology:

Spiral grids play a role in *Phyllotaxis*, a topic of plant morphogenesis.

The grid approximates the position of leaves.

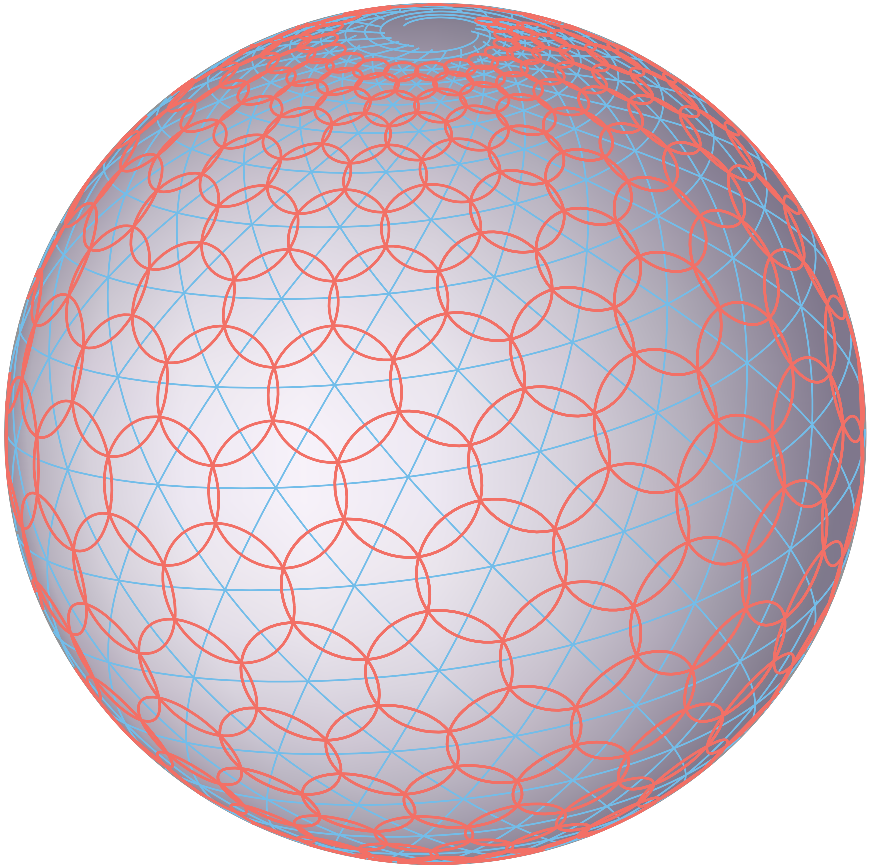
3.2 Applications of Geometry



Sometimes, the research in geometry is drawn from an appreciation for aesthetics:

Left: A **3-web** of isogonal trajectories on a Dupin cyclide (by courtesy of Georg Glaeser).

3.2 Applications of Geometry



spiral arrangement of circles

Thank you

for your attention !