Movement of conics on quadrics

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Table of contents

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1. Introduction

Heinrich Brauner (1928–1990)

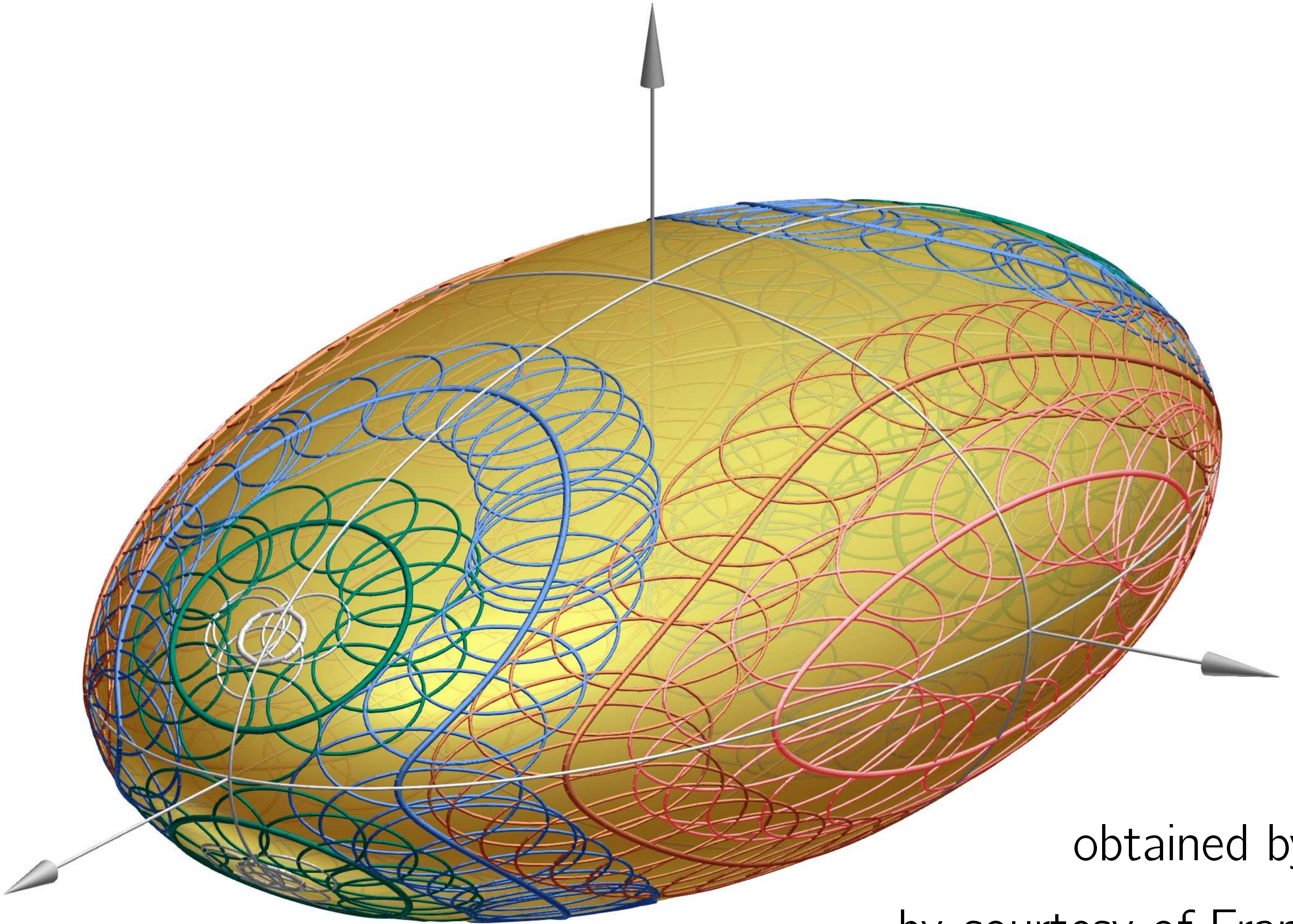
Given an ellipsoid, there is a three-parametric set of cutting planes, but the size of an ellipse depends only on its two semiaxes.

1980 Heinrich Brauner (1928–1990) posed the question for how to move ellipses on ellipsoids ?

This was a period where the kinematic generation of surfaces or recovering kinematic nets on given surfaces where fields of research in differential geometry.

H. Brauner: *Quadriken als Bewegflächen*, Monatsh. Math. 59 (1955), 45–63.

1. Introduction



obtained by a numerical algorithm

by courtesy of Franz Gruber (1982–2019)

1. Introduction

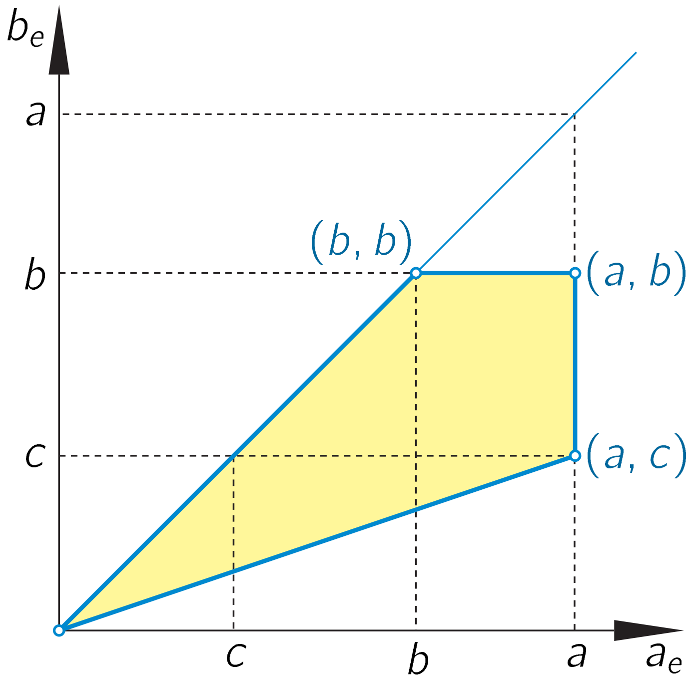
Trivial examples are surfaces of revolution.

Not only the meridians, but each conic with one plane of symmetry passing through the axis of rotations, generates a portion of a quadric.

Other examples are paraboloids, which are *surfaces of translation*, even on infinitely many different ways.

The following method for analyzing the case of a triaxial ellipsoid is straightforward. Main results were already found by Brauner. New is the dominant role of elliptic coordinates.

1. Introduction

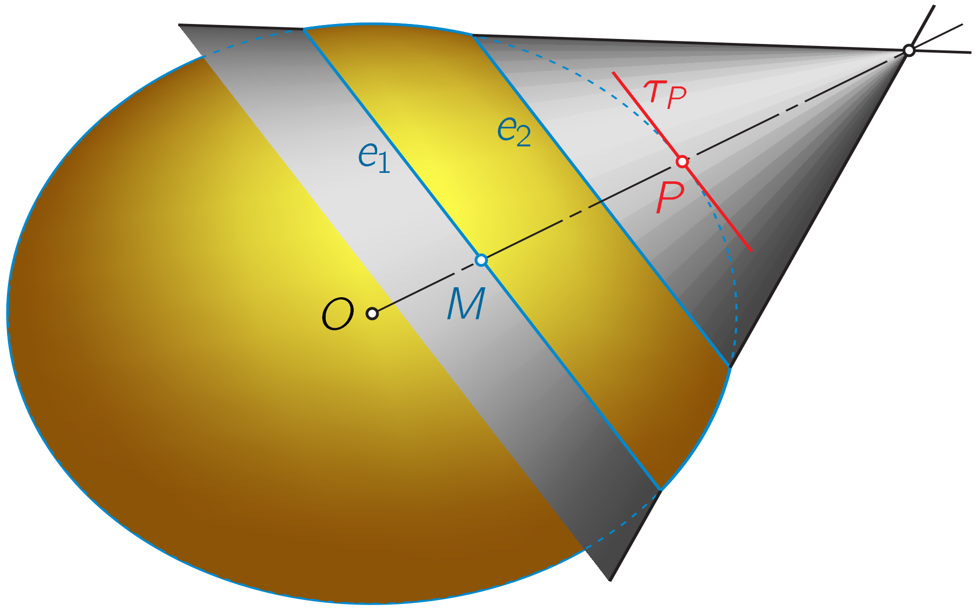


This diagram shows the semiaxes (a_e, b_e) of ellipses on an ellipsoid $\mathcal{E}$ with semiaxes a, b, and c, where $a > b > c$.

It turns out that each ellipse corresponding to an interior point of the yellow area is movable on $\mathcal{E}$.

$$1 < \frac{a_e}{b_e} < \frac{a}{c}.$$

1. Introduction



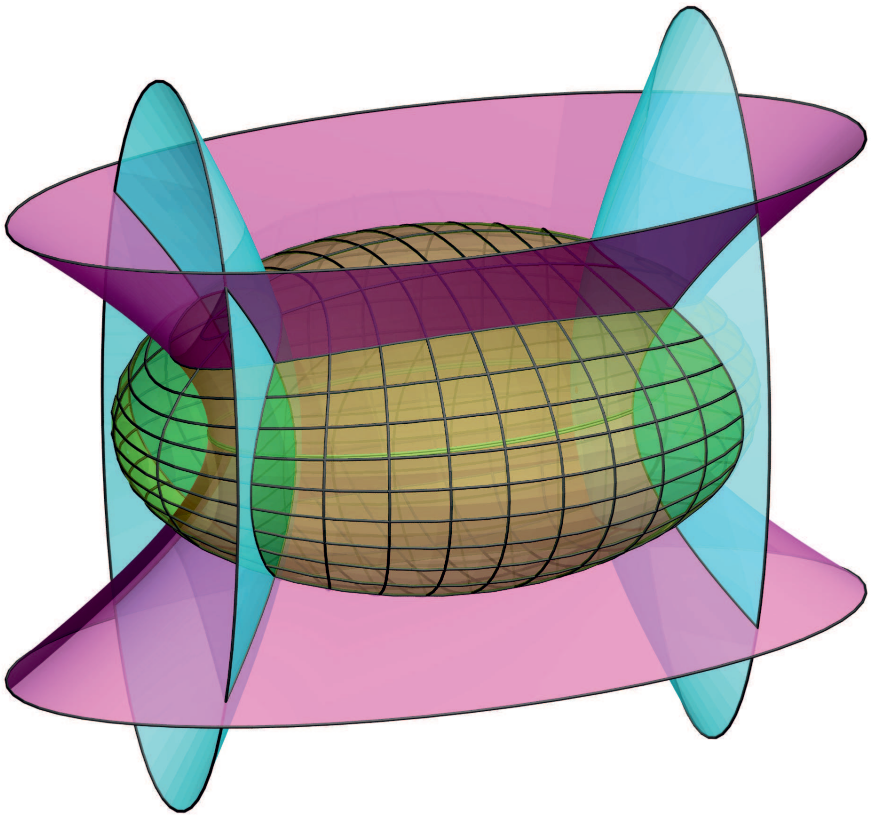
For each quadric holds:

Conics e in parallel planes ε are homothetic and centered on a diameter OP. They are also homothetic to the **Dupin indicatrix** at point P with tangent plane τ_P parallel to ε.

- The axes of e are parallel to the principal curvature directions at P, and

For the following, we recall some properties of confocal quadrics and elliptic coordinates.

1. Introduction



Two quadrics are **confocal** $\iff$ they have common axes and intersect each plane of symmetry along confocal conics.

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1\,, \quad a > b > c\,.$$

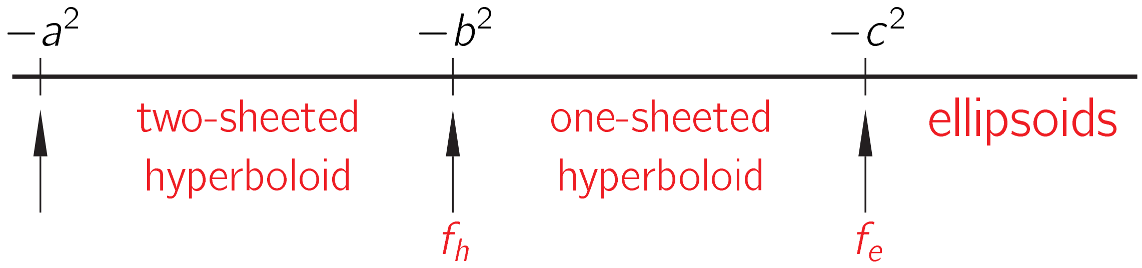
$\infty > k > -c^2$	tri-axial ellipsoid
$-c^2 > k > -b^2$	one-sheeted hyperboloid
$-b^2 > k > -a^2$	two-sheeted hyperboloids

As limits for $k \to -c^2$ and $k \to -b^2$

focal ellipse $\quad \dfrac{x^2}{a^2-c^2} + \dfrac{y^2}{b^2-c^2} = 1,\ z = 0\,,$

focal hyperbola $\quad \dfrac{x^2}{a^2-b^2} - \dfrac{z^2}{b^2-c^2} = 1,\ y = 0\,.$

1. Introduction

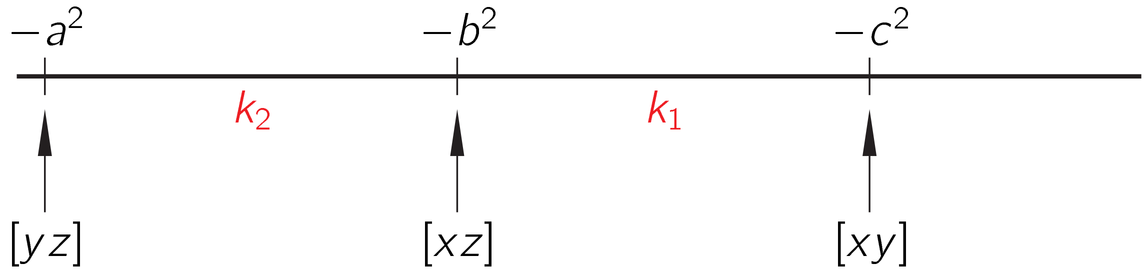


Confocal central quadrics: $\dfrac{x^2}{a^2+k}+\dfrac{y^2}{b^2+k}+\dfrac{z^2}{c^2+k}=1\,,\quad a>b>c\,.$

Through each point $P=(\xi,\eta,\zeta)$ outside the coordinate planes pass three surfaces of this family, one out of each type.

We focus on the ellipsoid $\mathcal{E}:\ k=0$ and use the parameters (k_1,k_2) of the two hyperboloids as **elliptic coordinates**.

1. Introduction



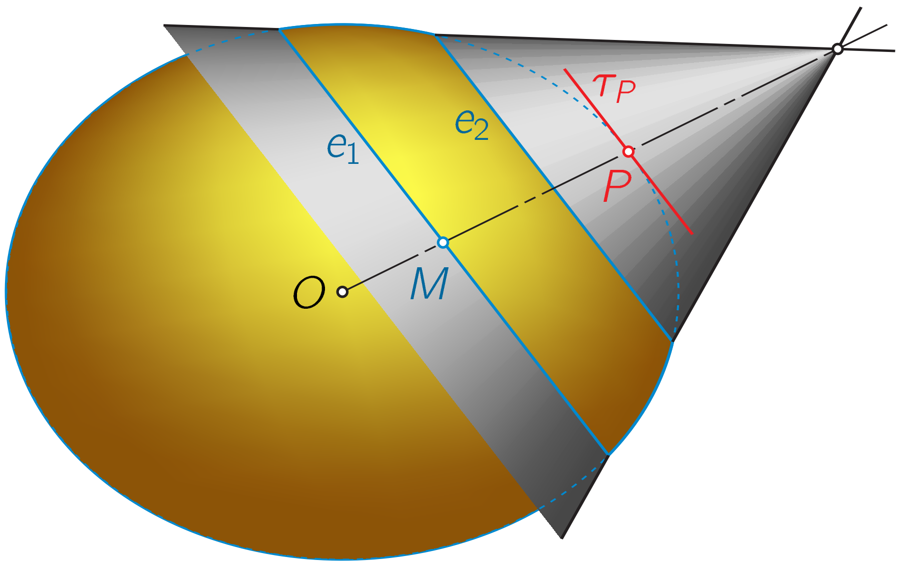
$$\mathcal{E}: \ \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1, \qquad \mathcal{H}_i: \ \frac{\xi^2}{a^2 + k_i} + \frac{\eta^2}{b^2 + k_i} + \frac{\zeta^2}{c^2 + k_i} = 1,$$

where $-c^2 > k_1 > -b^2 > k_2 > -a^2$.

$$\xi^2 = \frac{a^2(a^2 + k_1)(a^2 + k_2)}{(a^2 - b^2)(a^2 - c^2)}, \quad \eta^2 = \frac{b^2(b^2 + k_1)(b^2 + k_2)}{(b^2 - c^2)(b^2 - a^2)}, \quad \zeta^2 = \frac{c^2(c^2 + k_1)(c^2 + k_2)}{(c^2 - a^2)(c^2 - b^2)}.$$

The elliptic coordinates (k_1, k_2) of $P = (\xi, \eta, \zeta)$ satisfy $k^2 + L\,k + M = 0$ with coefficients depending on the cartesian coordinates.

2. Moving ellipses on ellipsoids

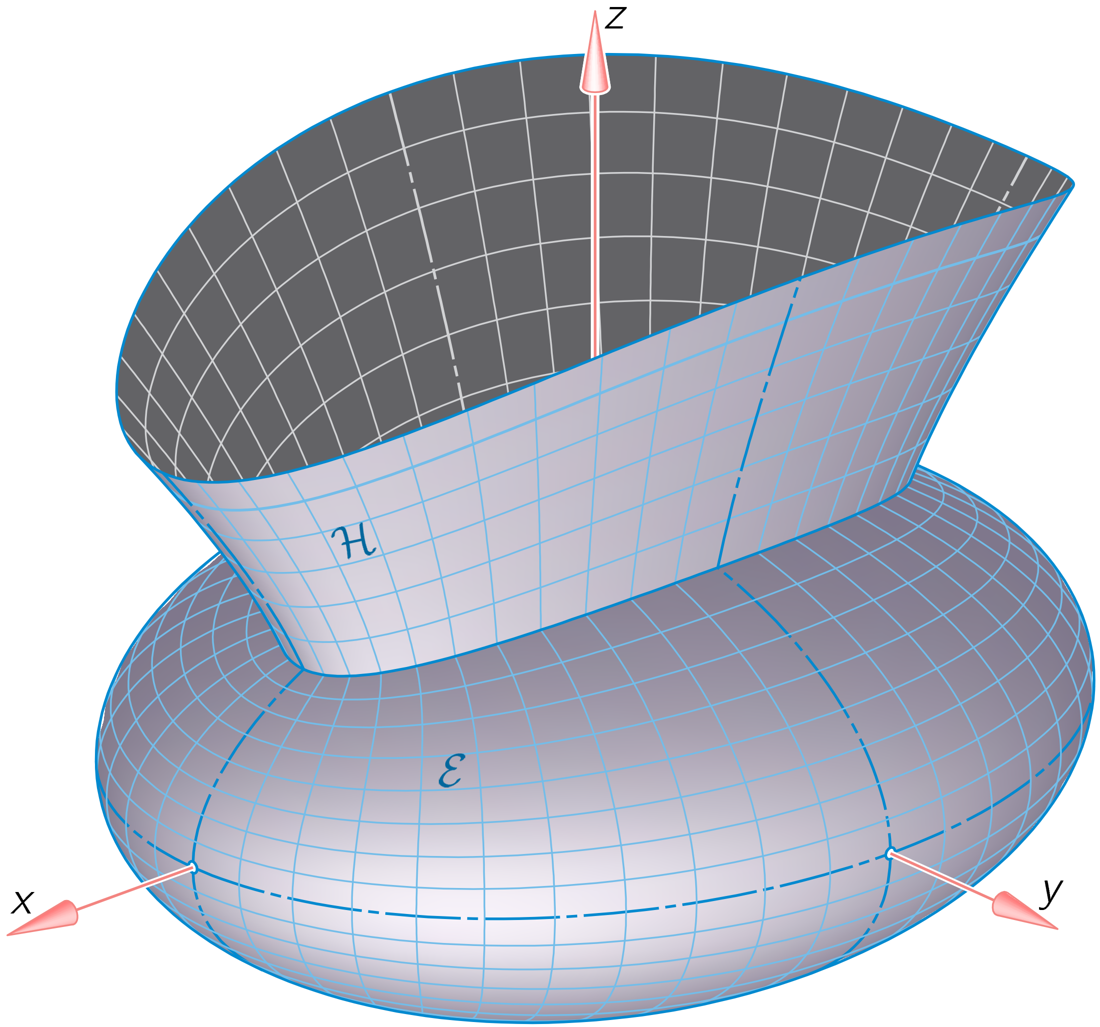


We are going to determine

- the principal curvature directions at P,
- the principal curvatures (κ_1, κ_2) at P,
- the semiaxes (a_P, b_P) of the ellipse in the parallel plane through O.

All are related to the family of confocal quadrics.

2. Moving ellipses on ellipsoids



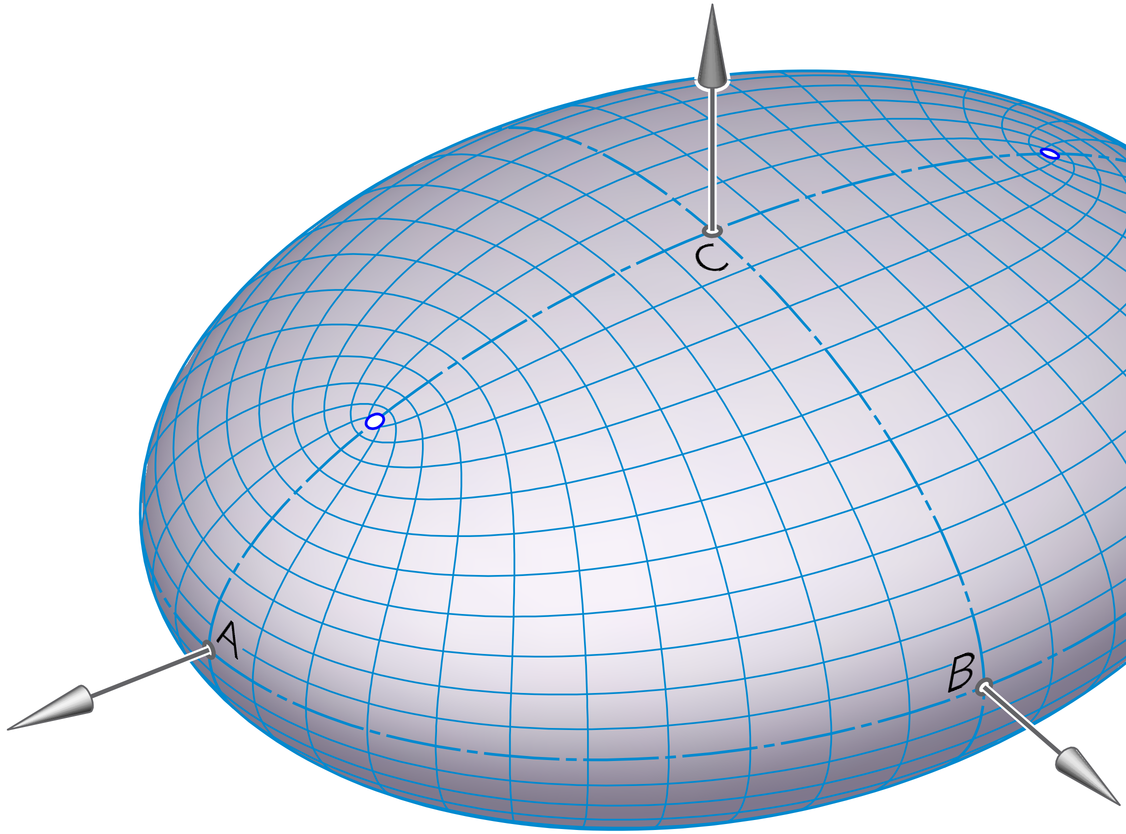
Confocal surfaces form a triply orthogonal system.

Charles Dupin (1784–1873): Confocal surfaces intersect each other along their lines of curvature.

The principal curvature directions at $P \in \mathcal{E}$ are normal vectors of the two hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$ through P:

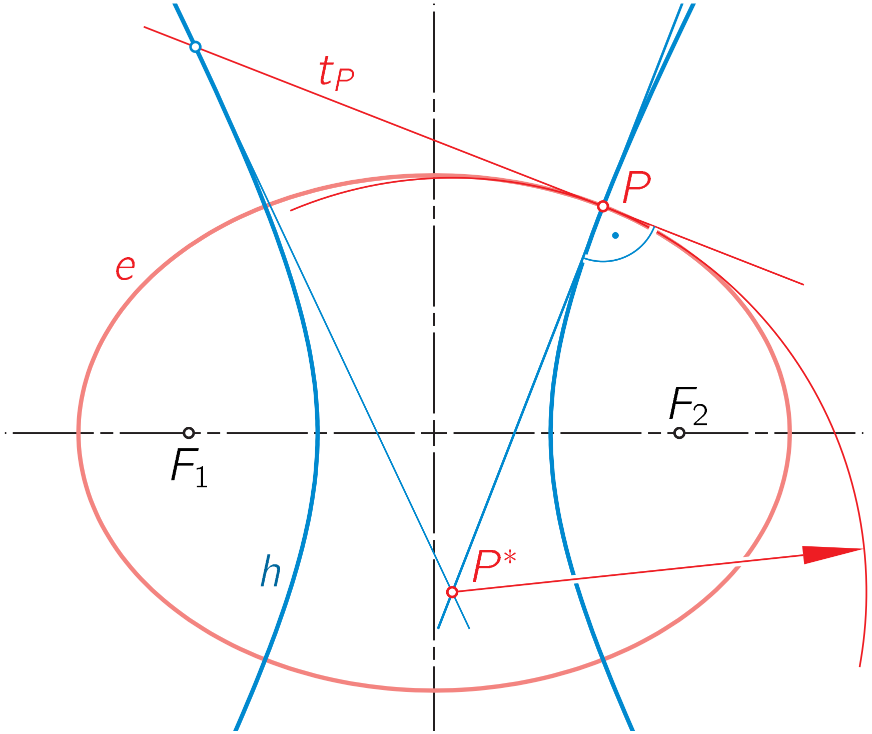
$$\left(\frac{\xi}{a^2+k_i}, \frac{\eta}{b^2+k_i}, \frac{\zeta}{c^2+k_i}\right).$$

2. Moving ellipses on ellipsoids



The curvature lines are parameter lines of elliptic coordinates with the umbilic points as singular points.

2. Moving ellipses on ellipsoids



The center of curvature P^* to P is the pole of the tangent t_P w.r.t. the confocal hyperbola.

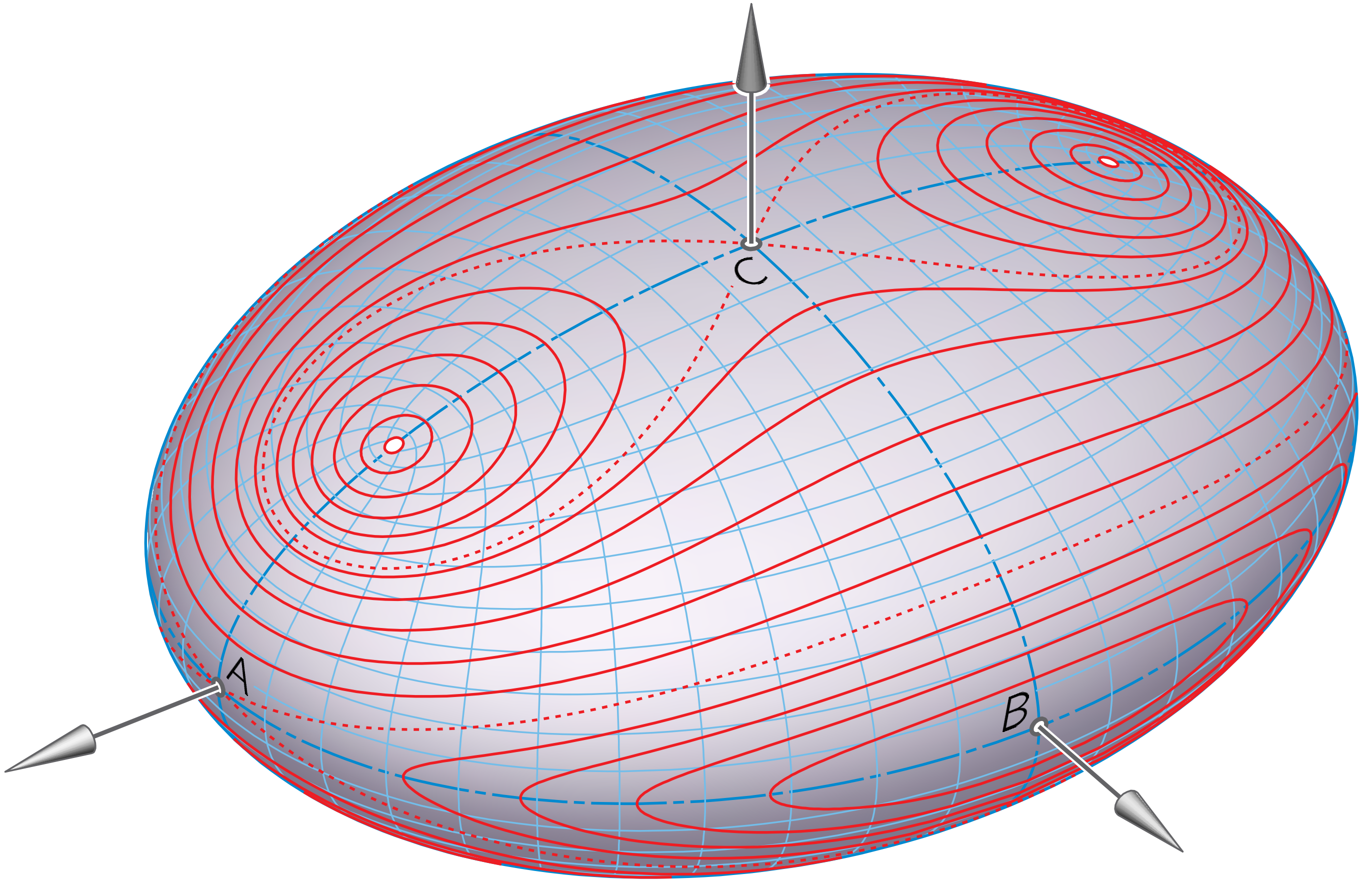
A similar result holds for **quadrics**:

The pole T_i of τ_P w.r.t. $\mathcal{H}_i$ is the center of curvature of the principal sections through $P \Longrightarrow$.

The **principal curvatures** at P are related to the elliptic coordinates of P

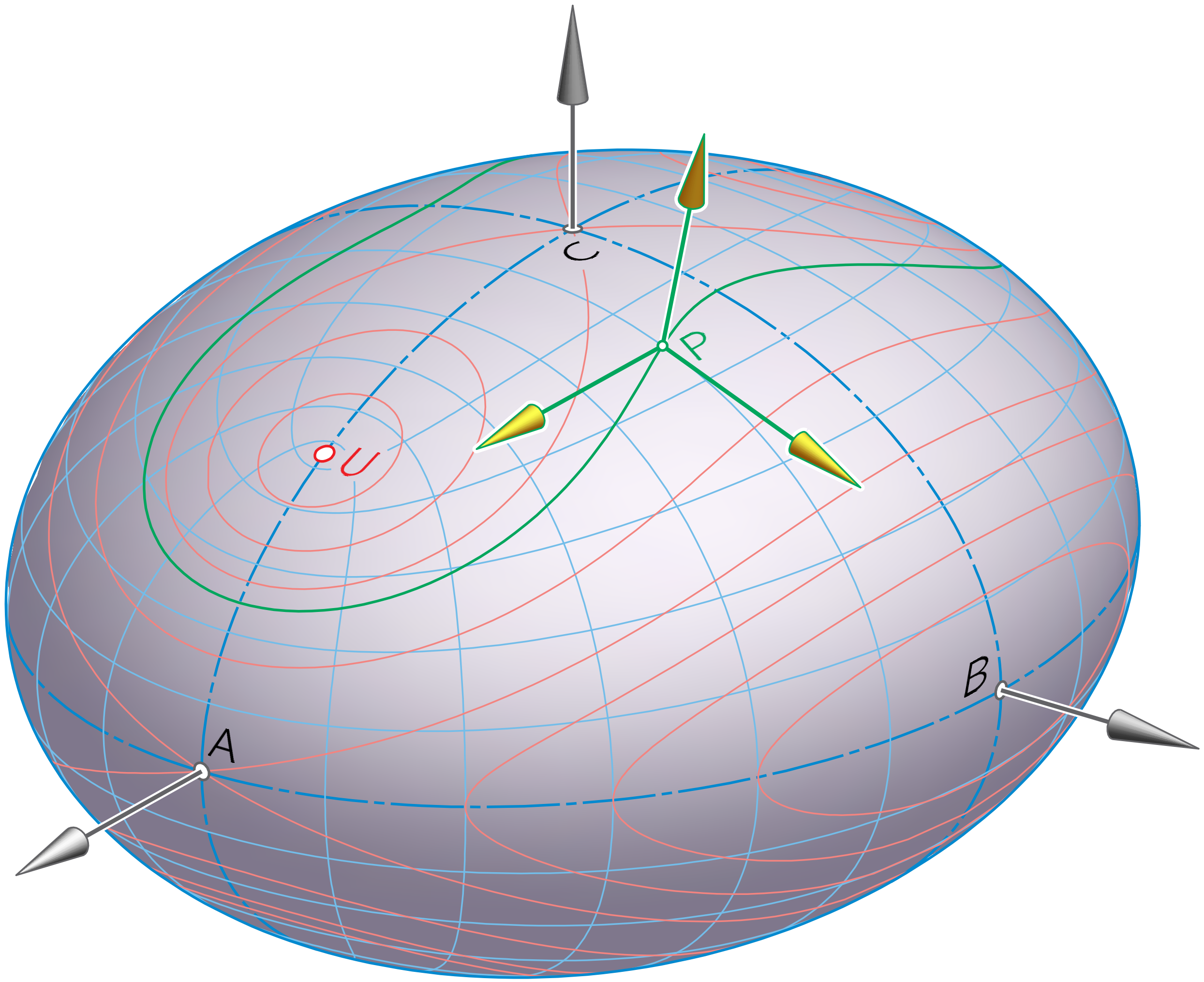
$\kappa_i = 1/\overline{PT_i} = -\frac{h}{k_i}$, $i = 1, 2$, $h = \overline{O\tau_P}$.

2. Moving ellipses on ellipsoids



The trajectories of P are curves of **constant ratio** $k_1 : k_2$ of elliptic coordinates.

2. Moving ellipses on ellipsoids

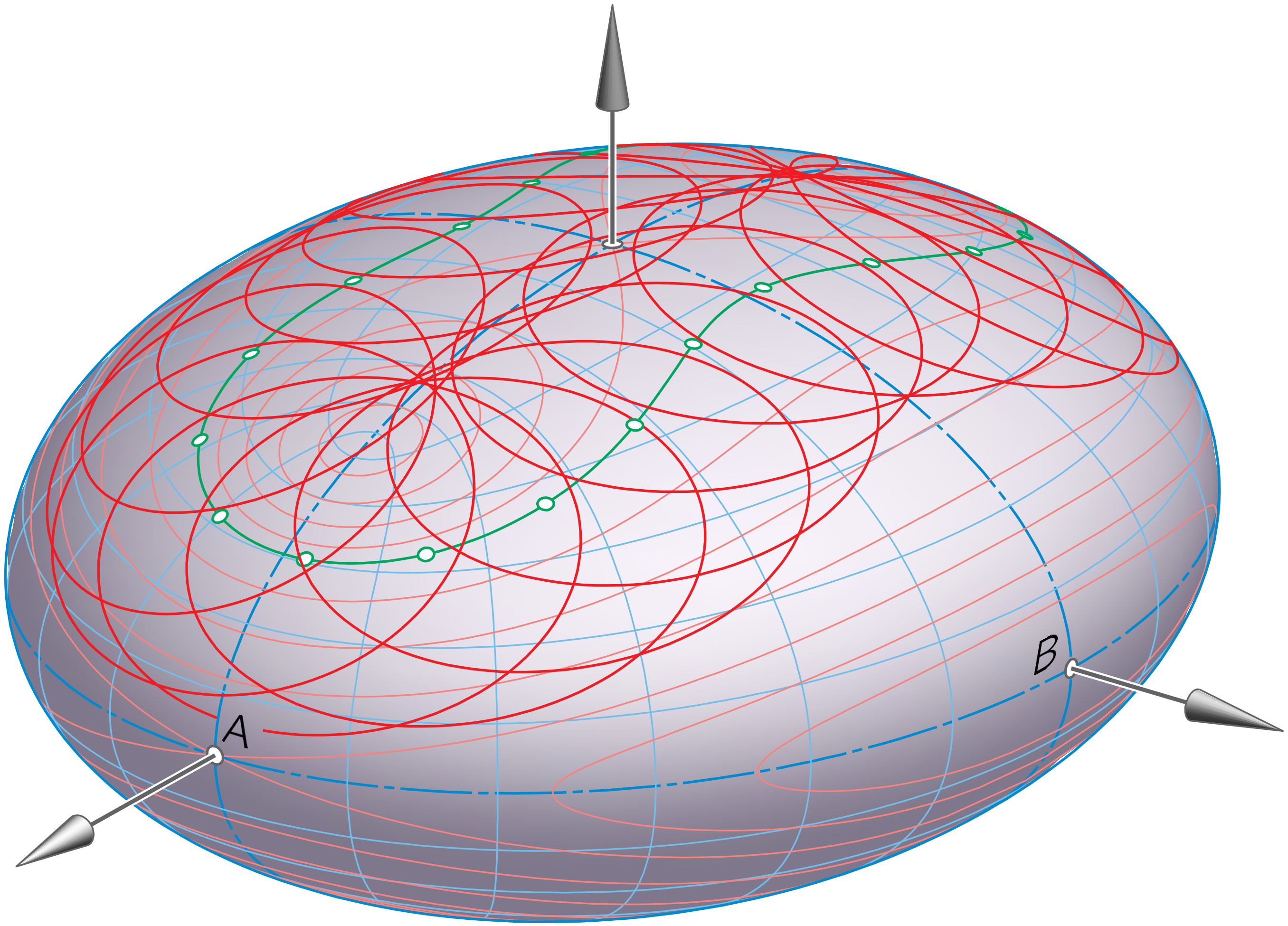


The principal curvature directions at P are parallel to the axes of the moving ellipse.

The ellipse in the diameter plane parallel τ_P has the semiaxes

$$a_P = \sqrt{-k_2},$$
$$b_P = \sqrt{-k_1}.$$

2. Moving ellipses on ellipsoids

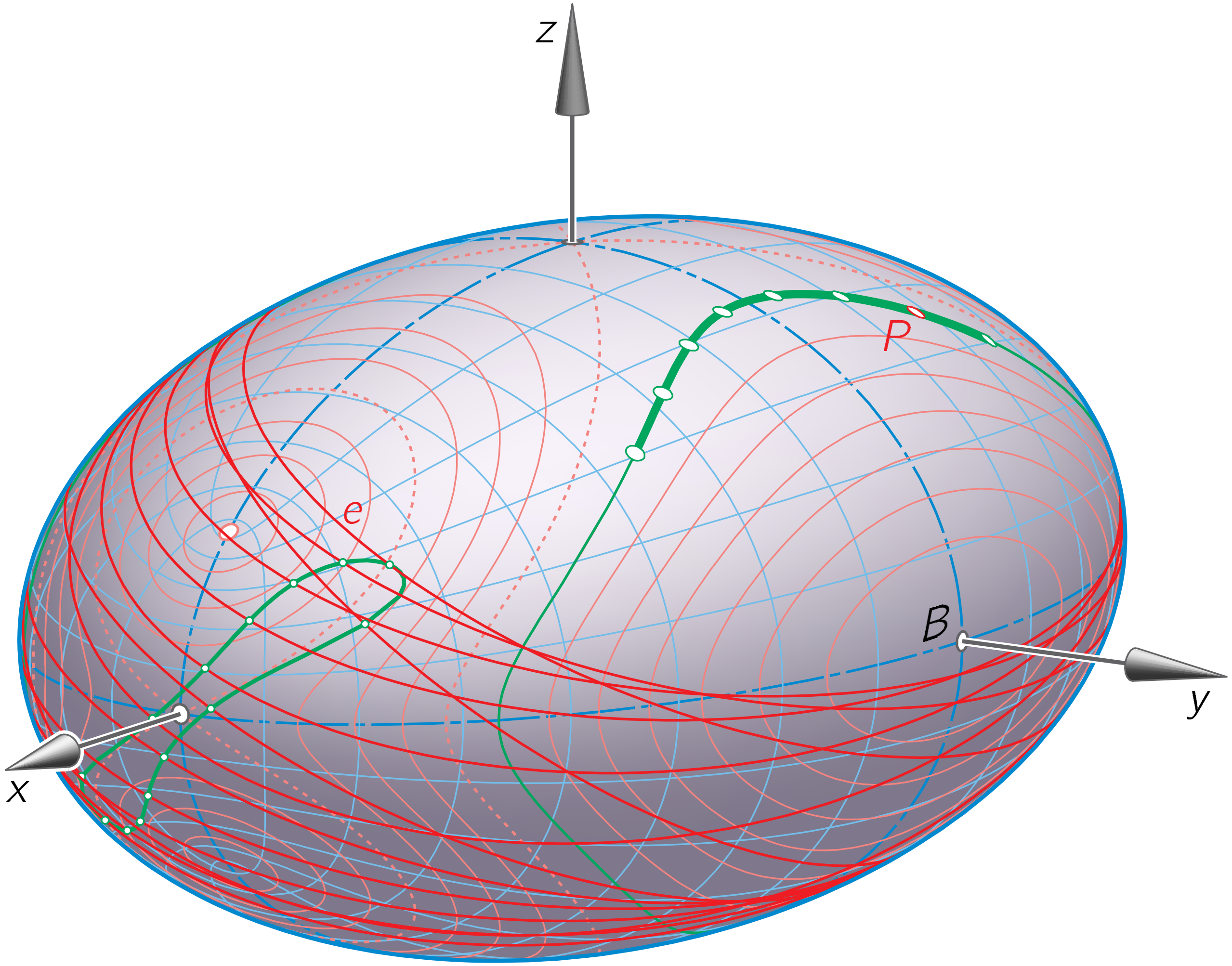


The position of the center M of e on the diameter OP is defined as

$$\mathbf{m}_e = \sqrt{1 - \frac{a_e^2}{k_2}}\,\mathbf{p}$$

At poses with $M = O$, i.e., $a_P = a_e$, for continuing the motion of e, the point P must jump to its antipode.

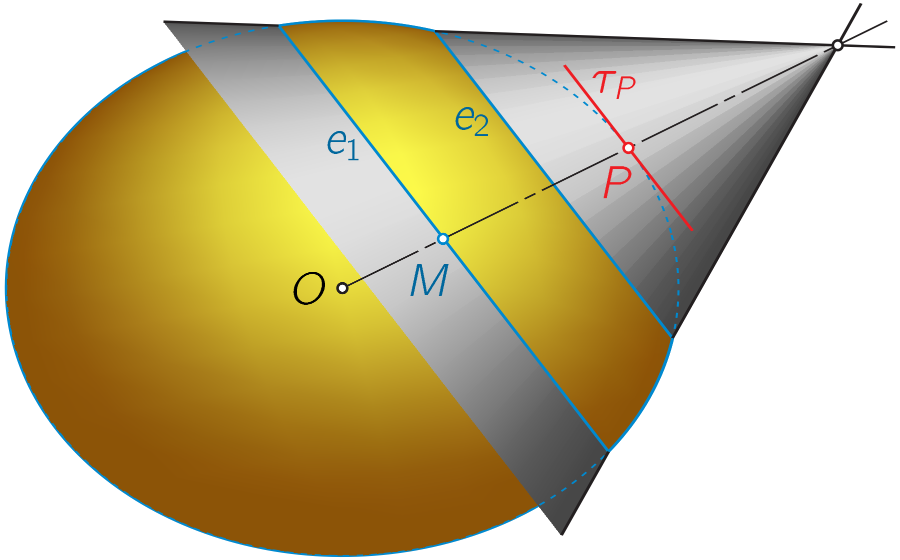
2. Moving ellipses on ellipsoids



Here, the midpoint of the moving ellipse e reaches the center of the ellipsoid.
Point P does not run the complete curve of constant ratio $\kappa_1 : \kappa_2$.

The green curve on the left-hand side is the trajectory of a principal vertex.

3. Moving conics on other quadrics



For parametrizing the movement of ellipses on ellipsoid we used the homothety of the ellipse in ε to

- the principal curvature directions at P ($\kappa_i = -h/k_i$, $h = \overline{O\tau_P}$),

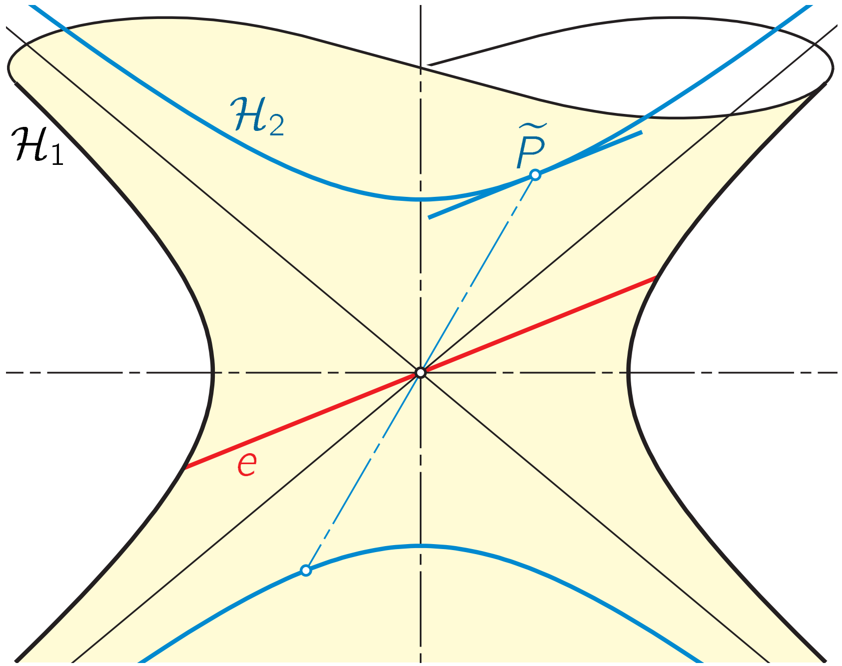
and

- the ellipse in the plane through O with semiaxes

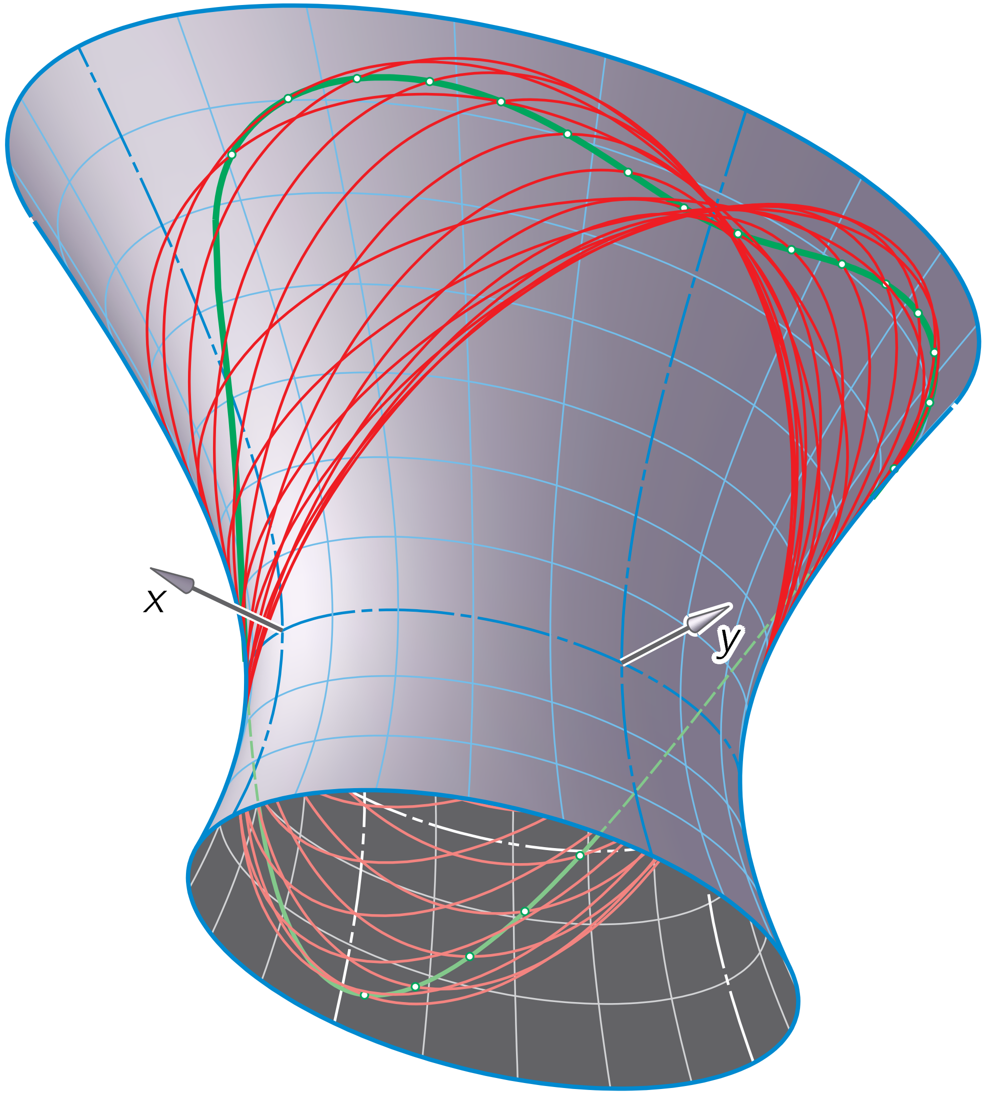
$$a_P = \sqrt{-k_2},\ b_P = \sqrt{-k_1}.$$

At other quadrics P can be either missing or at infinity. Paraboloids have no center.

3. Moving conics on other quadrics



For ellipses e on a one-sheeted hyperboloid $\mathcal{H}_1$ there does not exist a point P with the tangent plane τ_P parallel to e. But we find a useful point $\widetilde{P}$ on the 'conjugate' two-sheeted hyperboloid $\mathcal{H}_2$, sharing the asymptotic cone with $\mathcal{H}_1$.



If k_1 and k_2 are (shifted) elliptic coordinates of $\widetilde{P}$ w.r.t. the two-sheeted hyperboloid $\mathcal{H}_2$, then again

$$a_{\widetilde{P}} = \sqrt{k_2}, \;\; b_{\widetilde{P}} = \sqrt{k_1}.$$

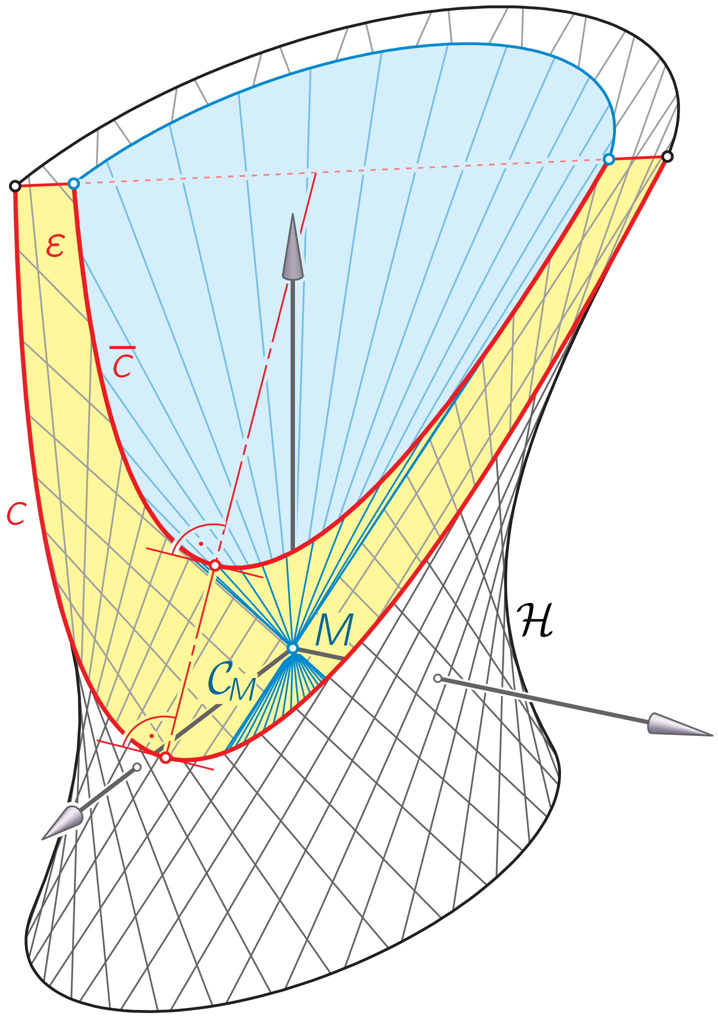
are the axes of the parallel section of $\mathcal{H}_1$ through the center.

The center M of e on the diameter $O\widetilde{P}$ is defined as

$$\mathbf{m}_e = \sqrt{\frac{a_e^2}{k_2} - 1}\;\widetilde{\mathbf{p}}$$

Reflections in $y = 0$ or $z = 0$ result in other movements of the same ellipse e. The green curve is the trajectory of the vertex. Both principal vertices of e trace the same curve.

3. Moving conics on other quadrics

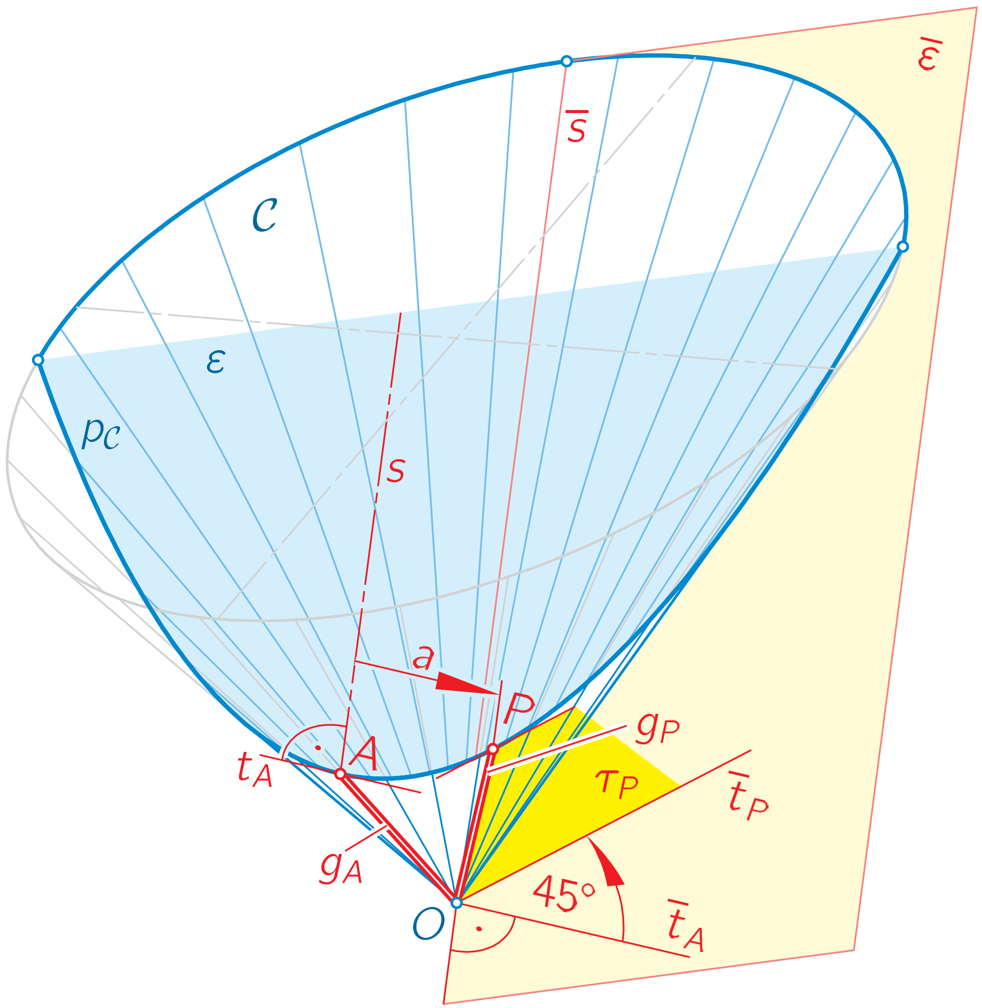


For parabolas on any quadric the point P is at infinity.

The curves of intersection c and $\overline{c}$ of the hyperboloid $\mathcal{H}$ and its asymptotic cone $\mathcal{C}_M$ with the plane ε are congruent parabolas with a common axis.

In the projective setting, c and $\overline{c}$ hyper-osculate at their point at infinity.

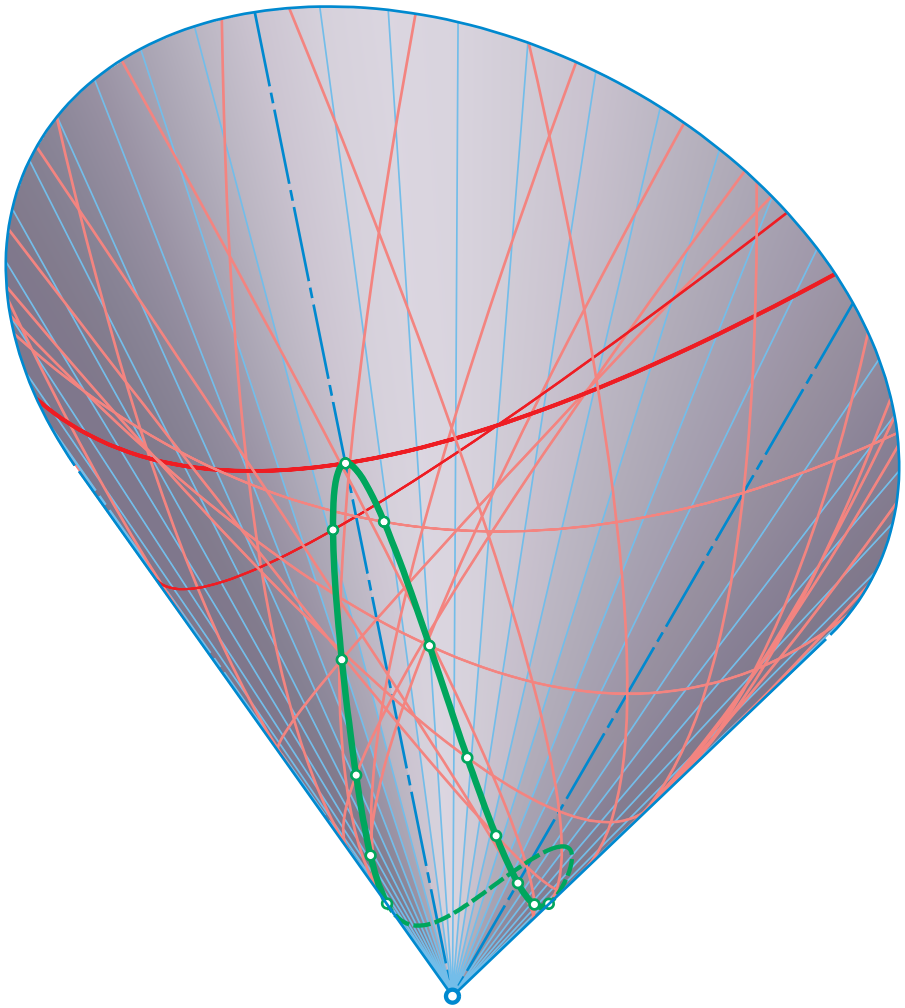
3. Moving conics on other quadrics



In order to find congruent parabolas on a hyperboloid, it is necessary to search at first on its asymptotic cone.

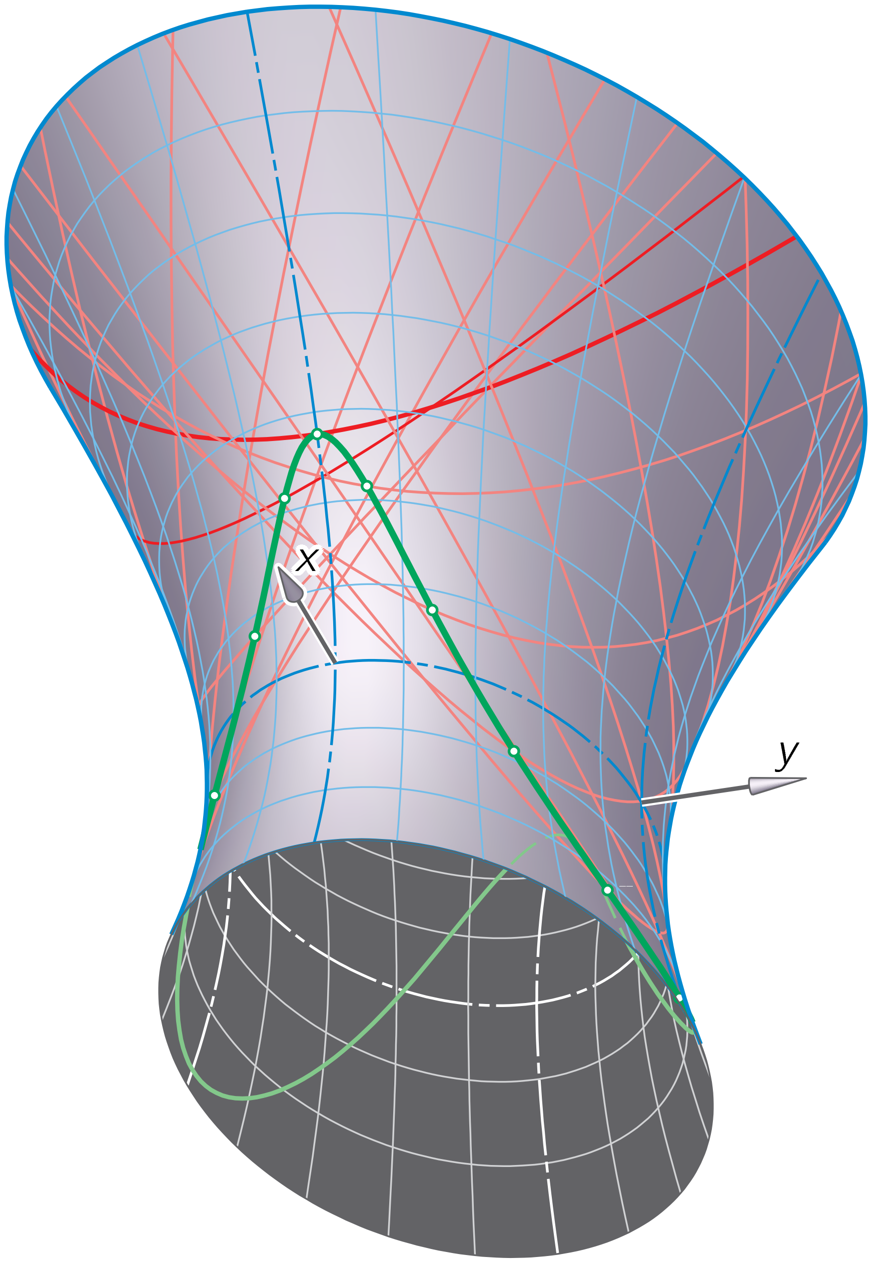
Given a generator $\overline{s}$, the parabola $p_{\mathcal{C}}$ with axis s parallel $\overline{s}$ and prescribed parameter a can be found by a linear construction.

3. Moving conics on other quadrics



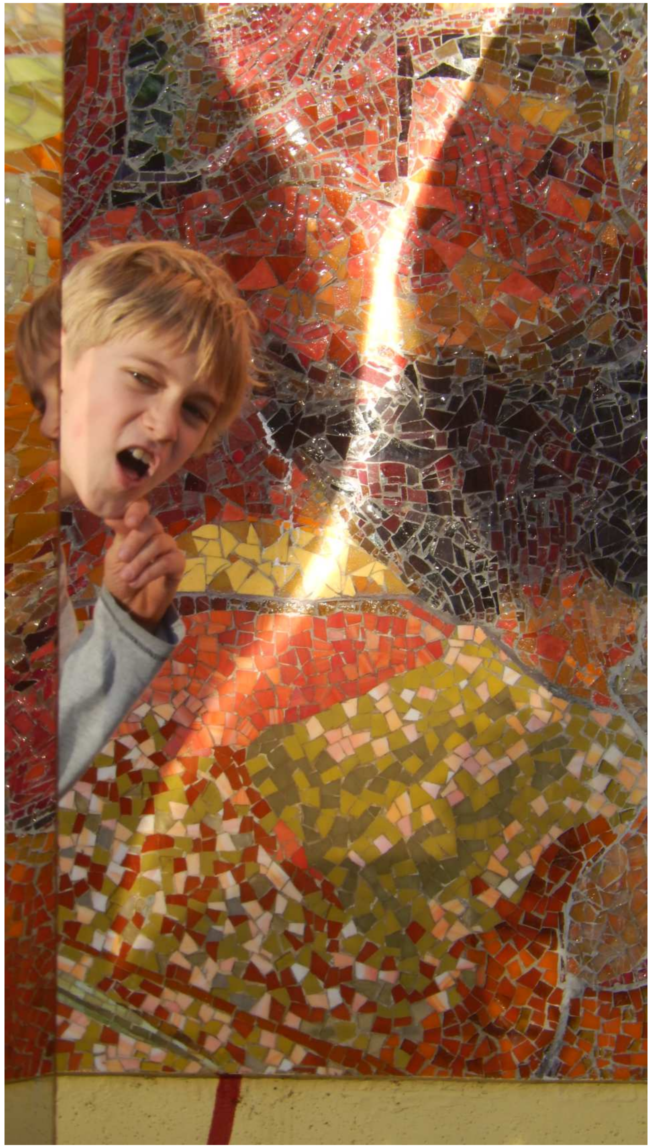
The movement of a parabola on a quadratic cone is rational, apart from a second one-parameter movement obtained by reflection in the vertex.

The green curve is the trajectory of the vertex.



The movement of a parabola on a one-sheeted hyperboloid is also rational, apart from a second one-parameter movement obtained by reflection in the center.

The green curve is the trajectory of the vertex.



Thank you for your attention!

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