Plücker's conoid, hyperboloids of revolution and orthogonal hyperbolic paraboloids

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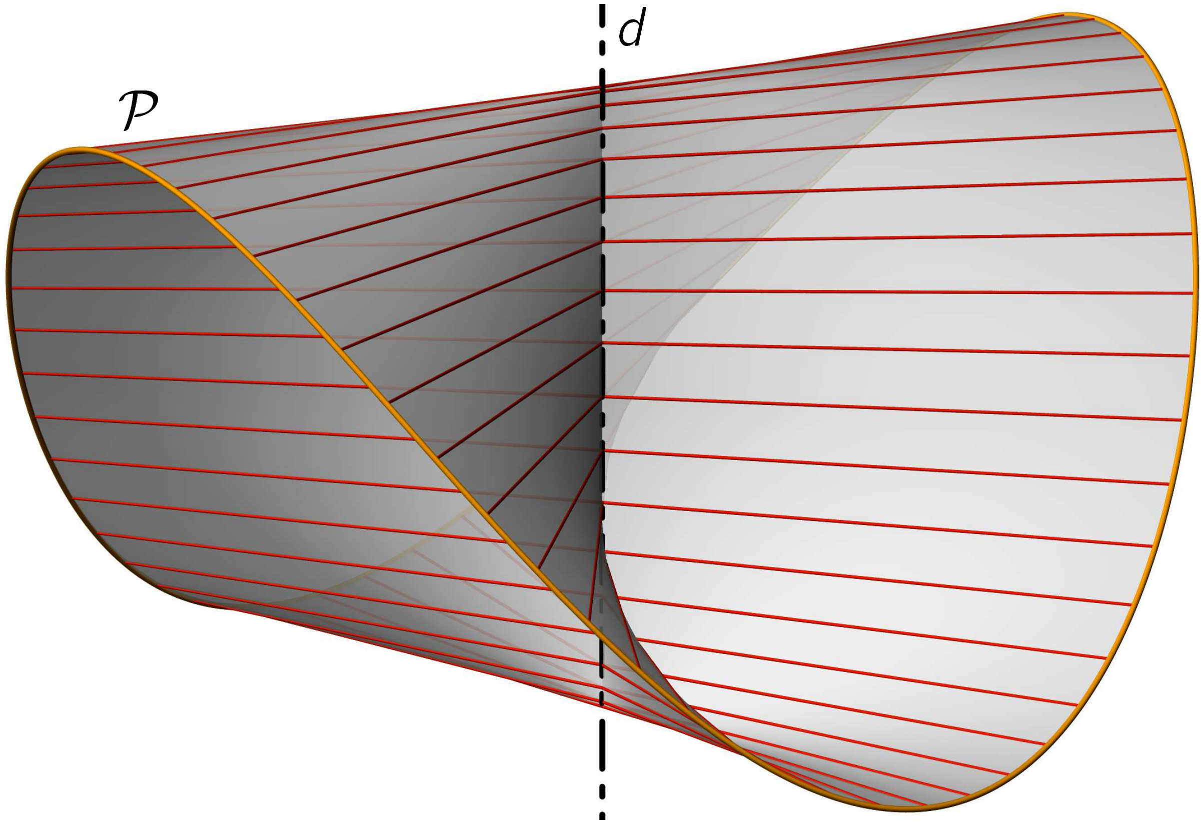
1. Plücker's conoid

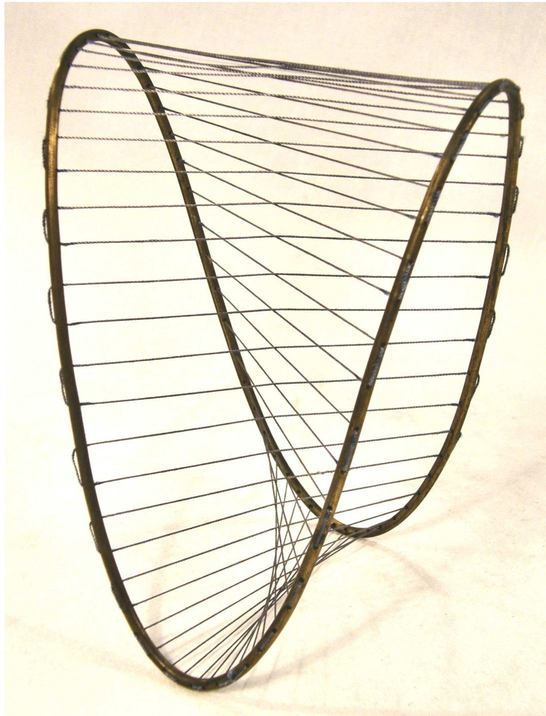


Plücker's conoid (or **Cylindroid**) $\mathcal{C}$ is a ruled surface of degree three with a finite double line.

In cylinder coordinates (r, φ, z) the conoid satisfies $z = c \sin 2\varphi$ with $c =$ const.

Cartesian equation: $(x^2 + y^2)\, z - 2c\, xy = 0.$

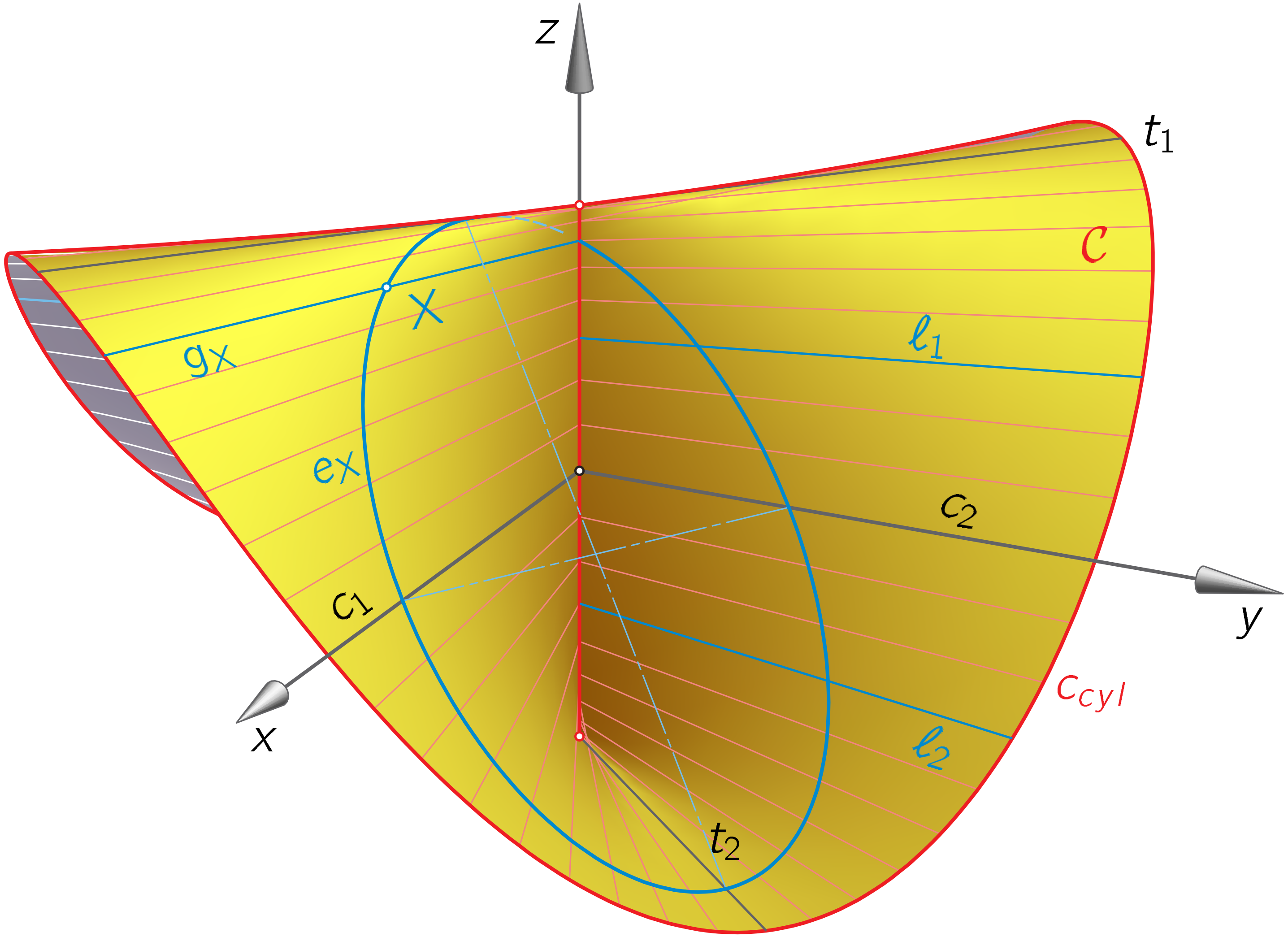
1. Plücker's conoid



two string models of the **Plücker conoid**

The generators connect opposite points of two periods of a Sine-curve wrapped around a right cylinder.

1. Plücker's conoid



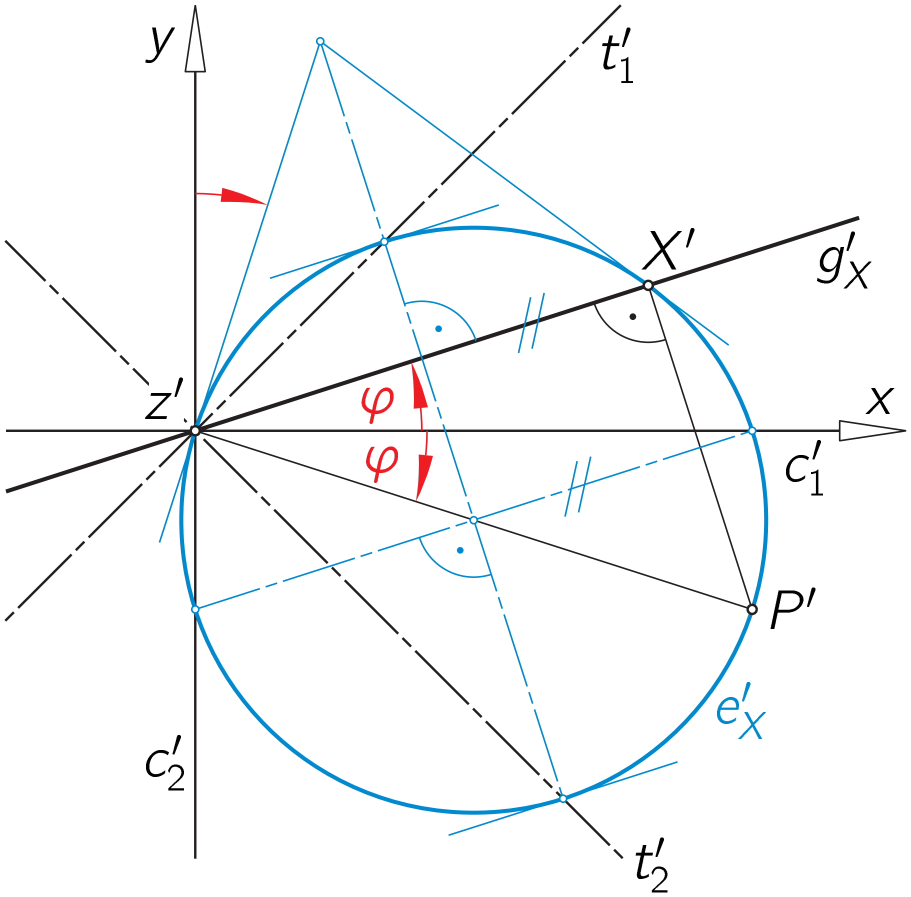
$\mathcal{C}$ has

two torsal generators t_1, t_2
and

two central generators c_1, c_2 in the $[xy]$-plane.

$$\mathcal{C} \cap \tau_X = g_X \cup e_X$$

1. Plücker's conoid

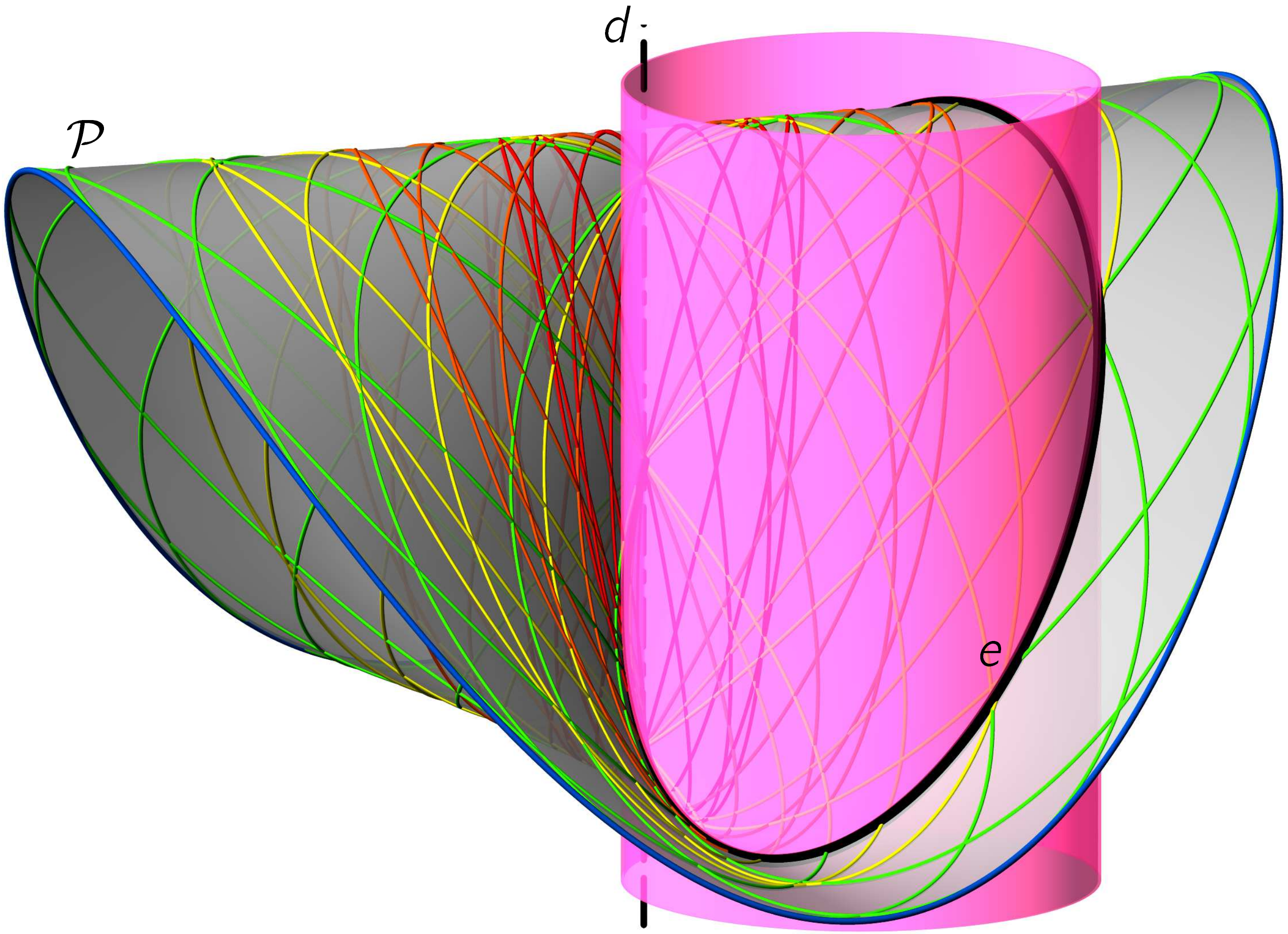


Top view along the z-axis:

The tangent plane τ_X at $X \in \mathcal{C}$ intersects the conoid along the generator g_X through X and an ellipse e_X with a circle as top view.

This ellipse has the principal vertices on the torsal lines and the minor axis in the $[xy]$-plane.

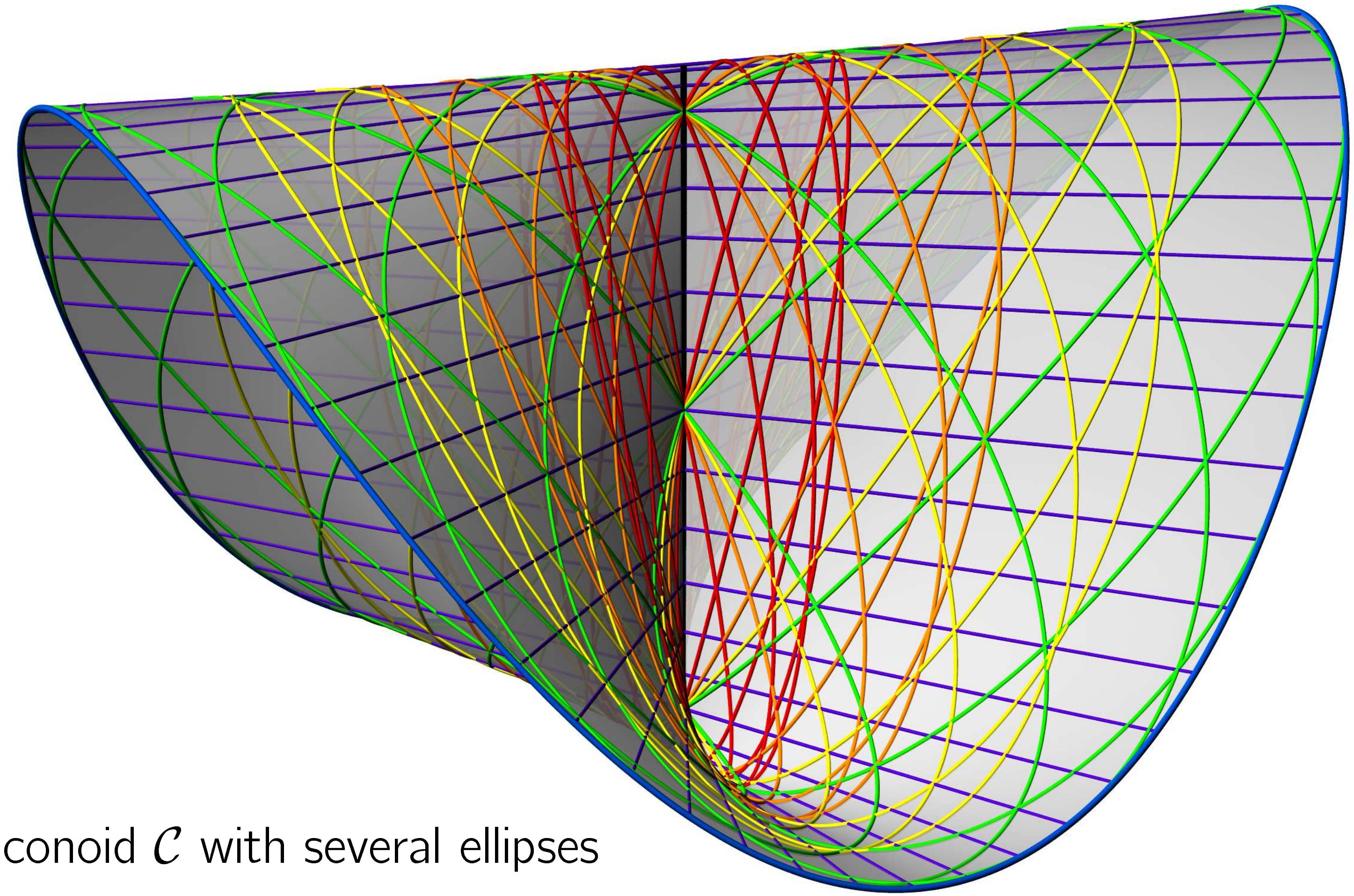
1. Plücker's conoid



Each ellipse $e \subset \mathcal{C}$ is the intersection of $\mathcal{C}$ with a cylinder of revolution through the directrix d.

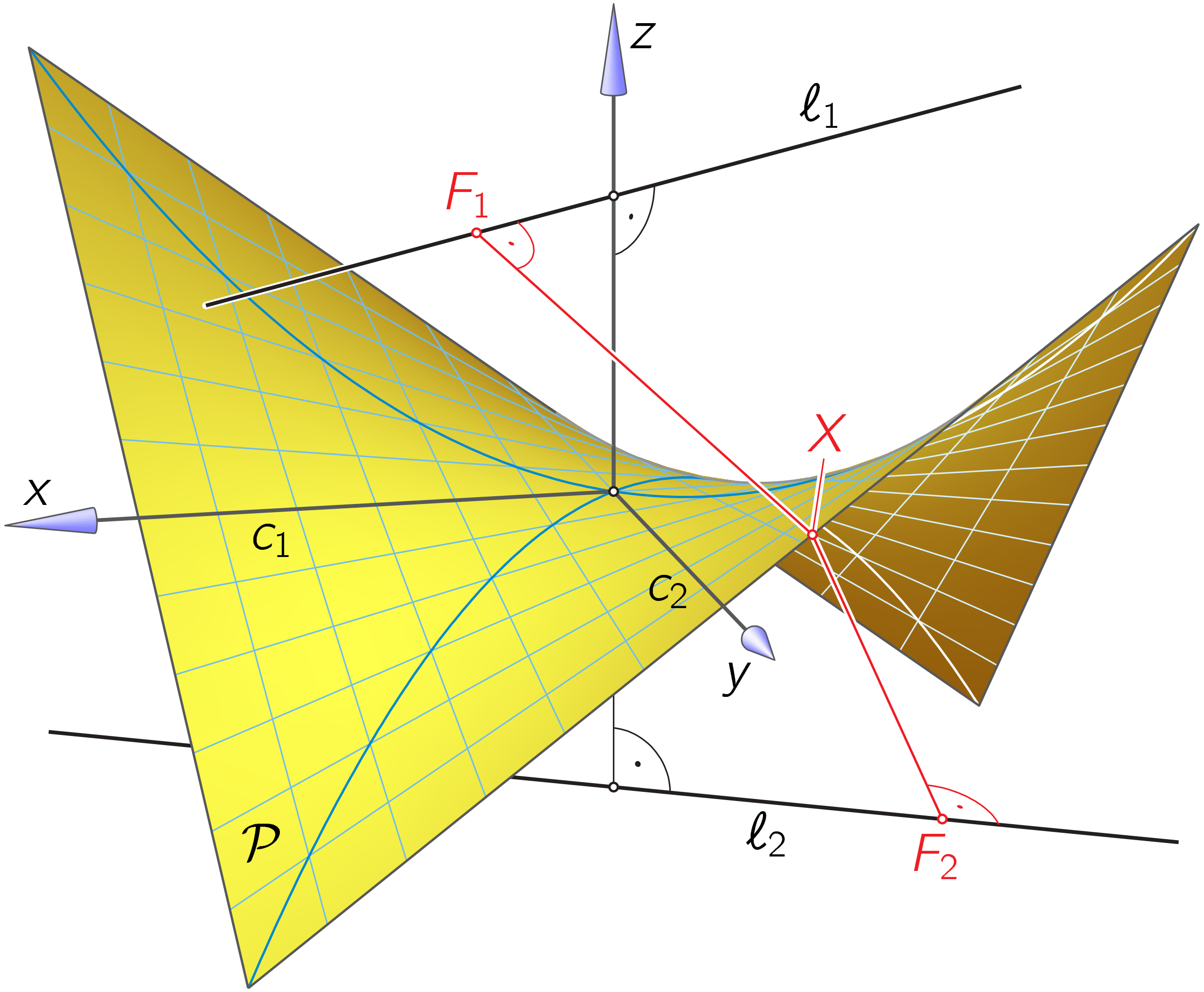
$\mathcal{C}$ contains only planar pedal curves, and e is one of them.

1. Plücker's conoid



Plücker's conoid $\mathcal{C}$ with several ellipses

2. Bisector of two skew lines



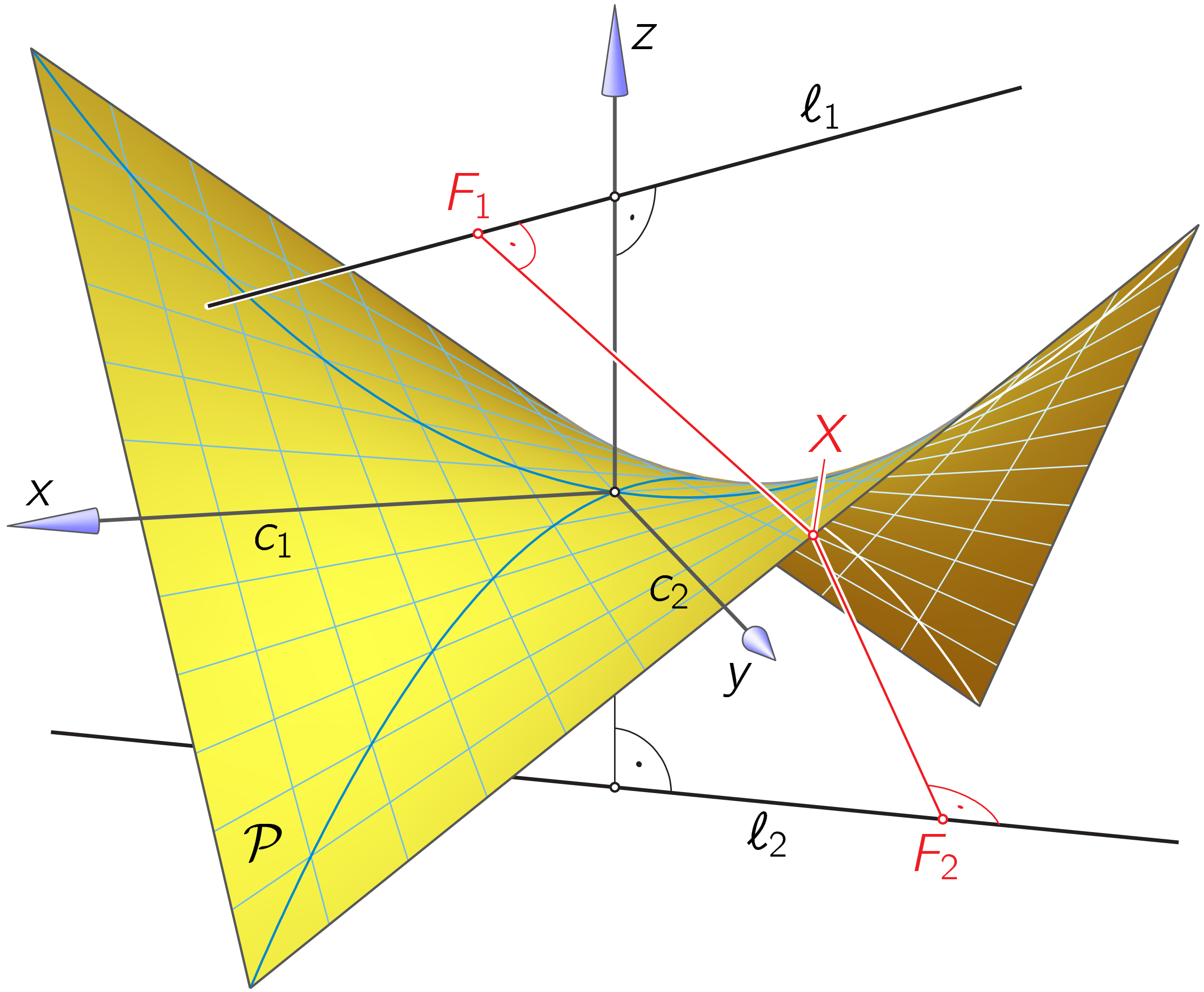
Given two skew lines ℓ_1, ℓ_2, the bisector, i.e., $\{X \mid \overline{X\ell_1} = \overline{X\ell_1}\}$, is an orthogonal parabolic hyperboloid $\mathcal{P}$.

$2d := \overline{\ell_1\ell_2}$, $2\varphi := \sphericalangle \ell_1\ell_2$

$$\mathcal{P}\colon\ z + \frac{\sin 2\varphi}{2d}\,xy = 0\,.$$

The axes of symmetry c_1, c_2 of ℓ_1 and ℓ_2 are the vertex generators of $\mathcal{P}$.

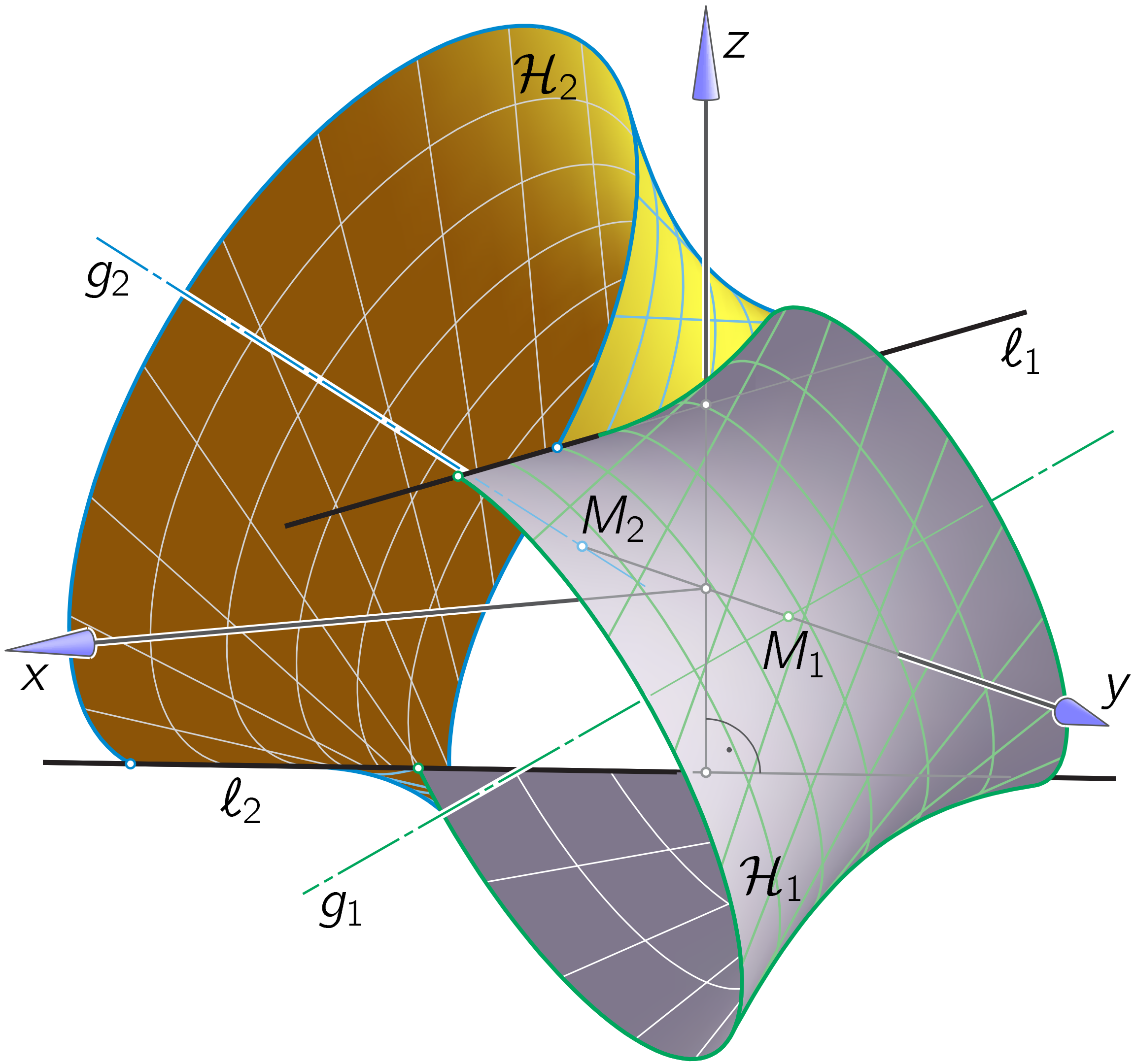
2. Bisector of two skew lines



The tangent plane to $\mathcal{P}$ at X is the plane of symmetry of the two pedal points F_1, F_2 of X on ℓ_1, ℓ_2.

The generators of $\mathcal{P}$ are the axes of rotations which send ℓ_1 to ℓ_2. Hence, they are axes of one-sheeted hyperboloids of revolution through ℓ_1 and ℓ_2.

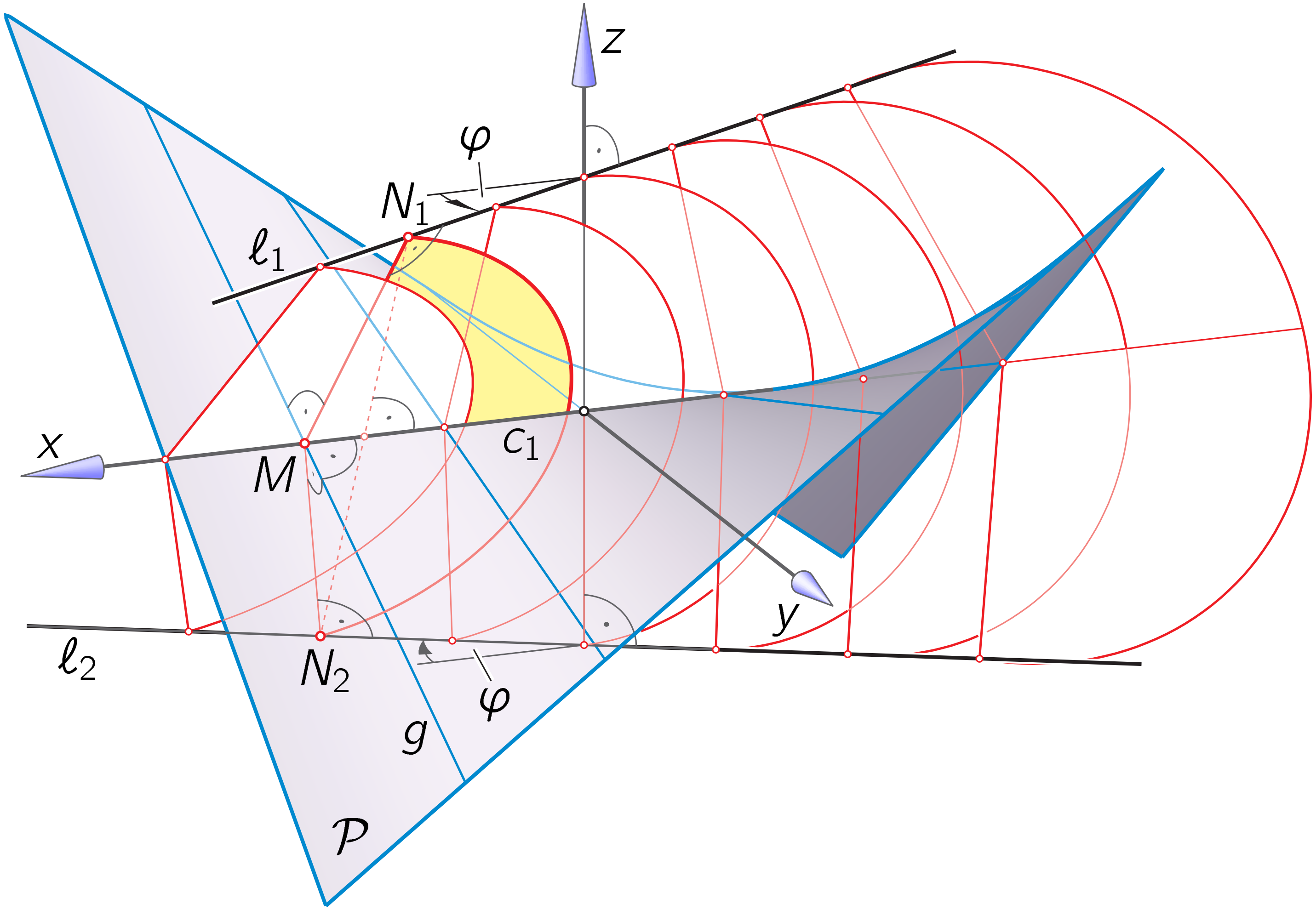
2. Bisector of two skew lines



Two hyperboloids of revolution $\mathcal{H}_1$, $\mathcal{H}_2$ through the given skew lines ℓ_1 and ℓ_2. Both hyperboloids have the same secondary axis b.

If two hyperboloids $\mathcal{H}_1$, $\mathcal{H}_2$ share two skew generators, then their complete intersection contains two other (not necessarily real) generators of the complementary regulus.

2. Bisector of two skew lines

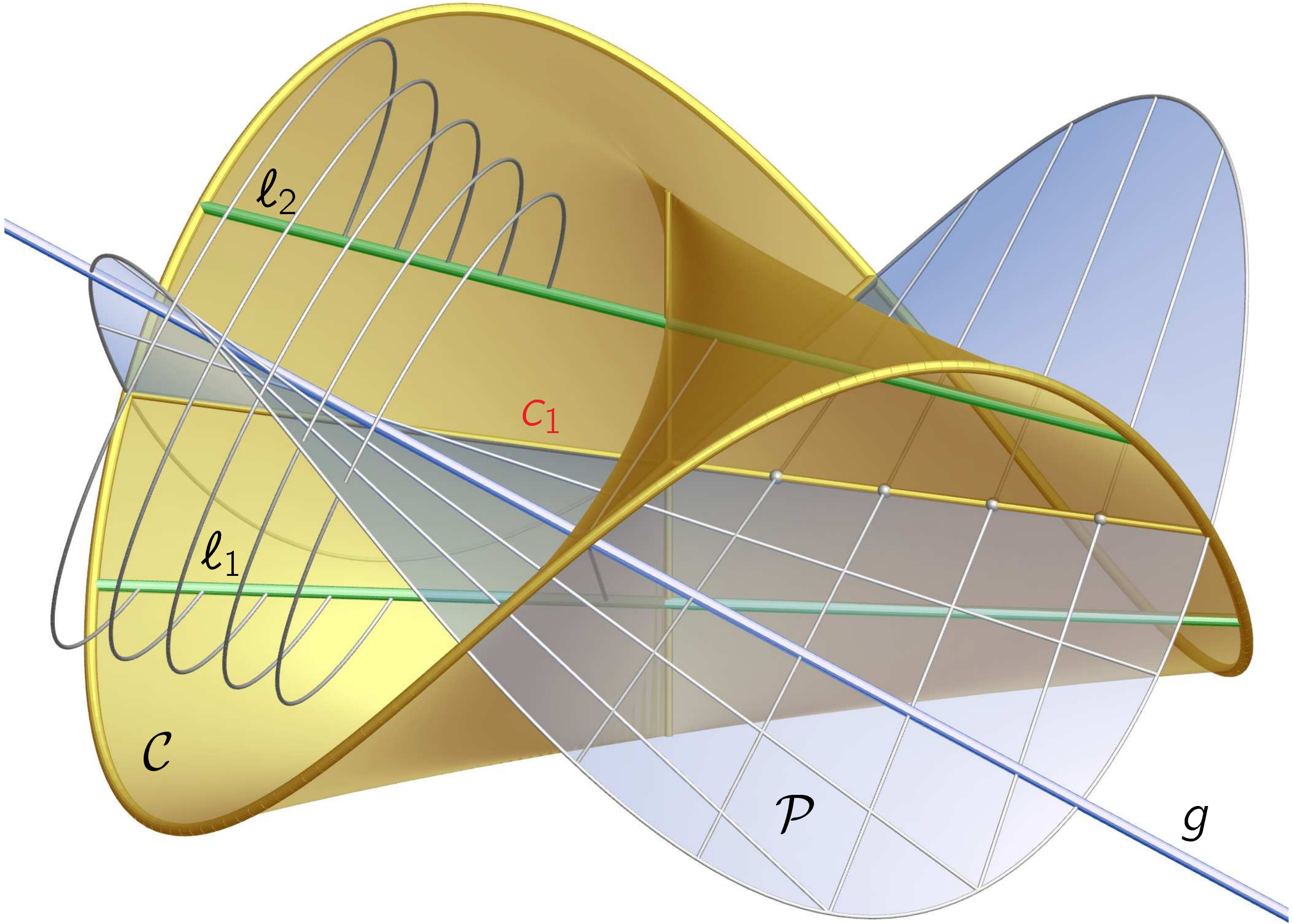


Gorge circles of hyperboloids of revolution passing through ℓ_1 and ℓ_2.

Their axes g form one regulus of the bisecting hyp. paraboloid $\mathcal{P}$.

Their centers M lie on c_1.

2. Bisector of two skew lines



All pairs of skew lines (ℓ_1, ℓ_2) sharing the bisector

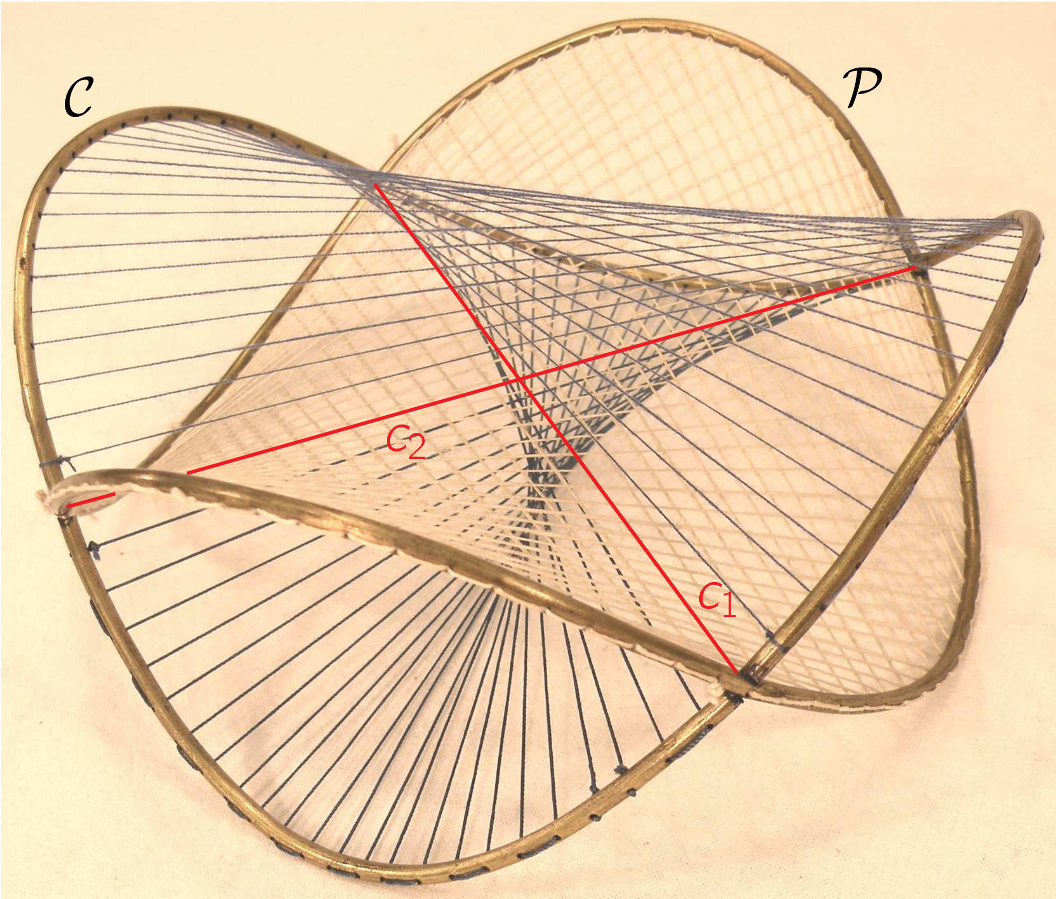
$$\mathcal{P}:\ z + \frac{\sin 2\varphi}{2d}\,xy = 0$$

are located on Plücker's conoid

$$\mathcal{C}:\ z = c \sin 2\varphi\,.$$

Generators g of $\mathcal{P}$ are axes of hyperboloids $\mathcal{H}$ which intersect $\mathcal{C}$ in two symmetric lines (ℓ_1, ℓ_2).

2. Bisector of two skew lines



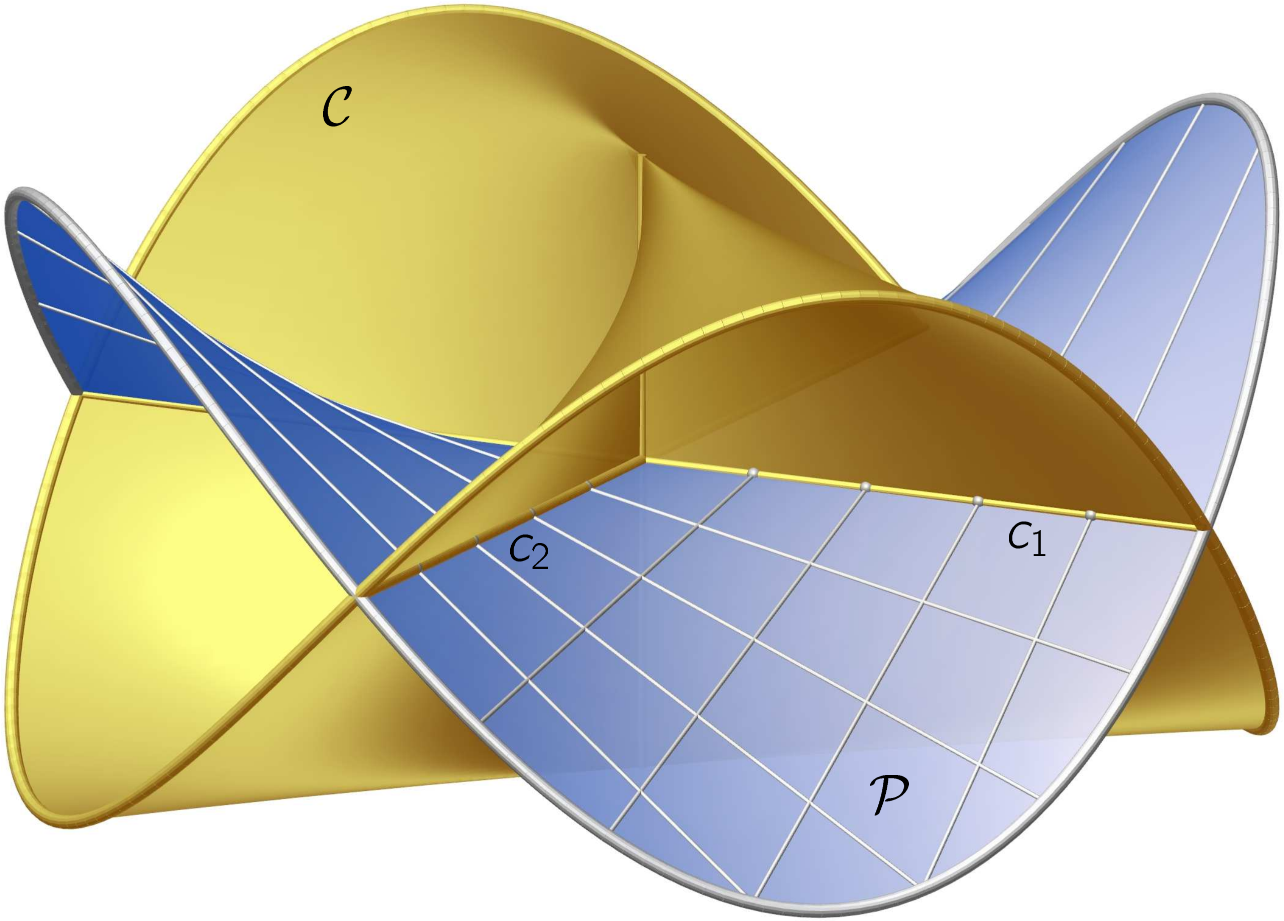
If ℓ_1, ℓ_2 coincide in c_1, the hyperboloid $\mathcal{H}$ contacts $\mathcal{C}$.

Normals of $\mathcal{C}$ along c_1 are generators of $\mathcal{P}$.

String model of Plücker's conoid $\mathcal{C}$ together with $\mathcal{P}$, the normal surface of $\mathcal{C}$ along c_1 and c_2.

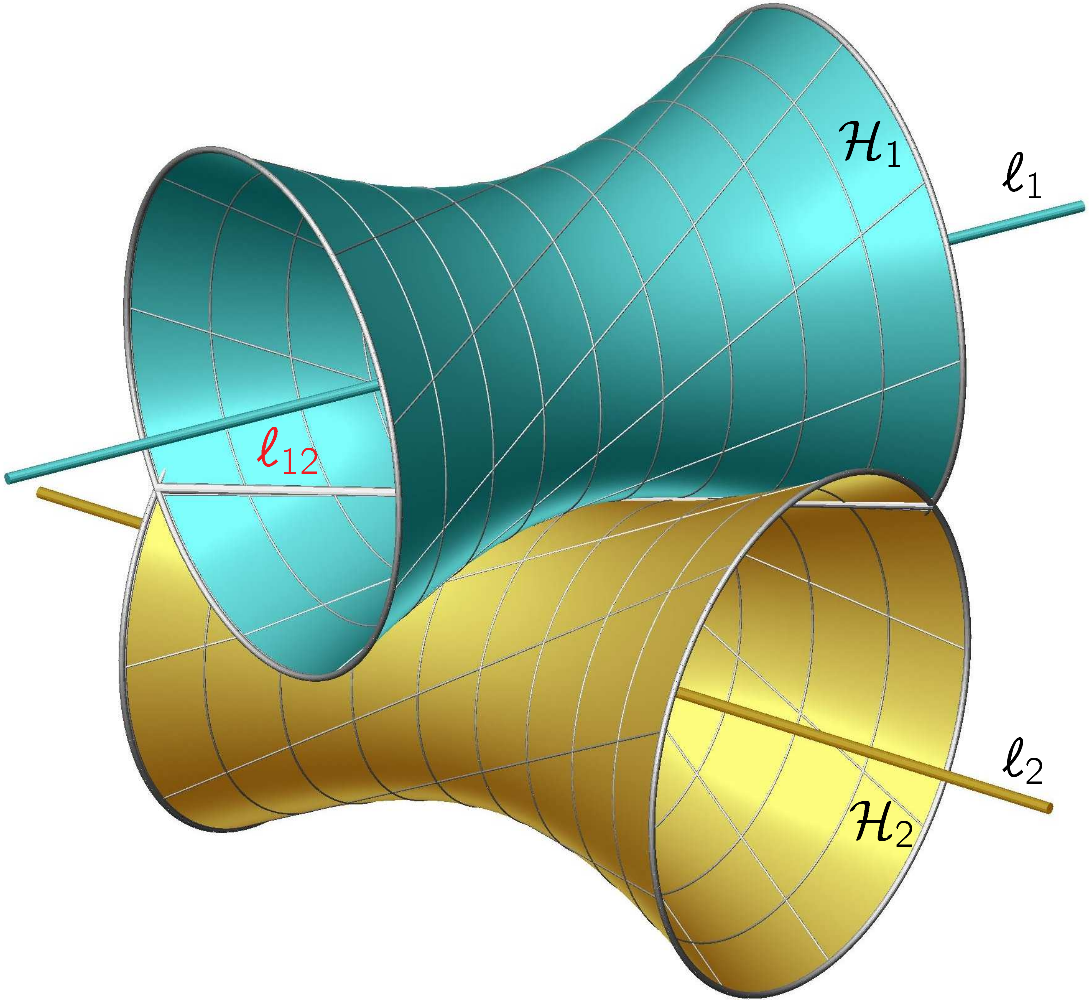
This is model XXIII, no. 10, of Schilling's famous collection of mathematical models.

2. Bisector of two skew lines



Plücker's conoid $\mathcal{C}$ together with $\mathcal{P}$, the normal surface along c_1 and c_2.

3. Hyperboloids of revolution with line-contact

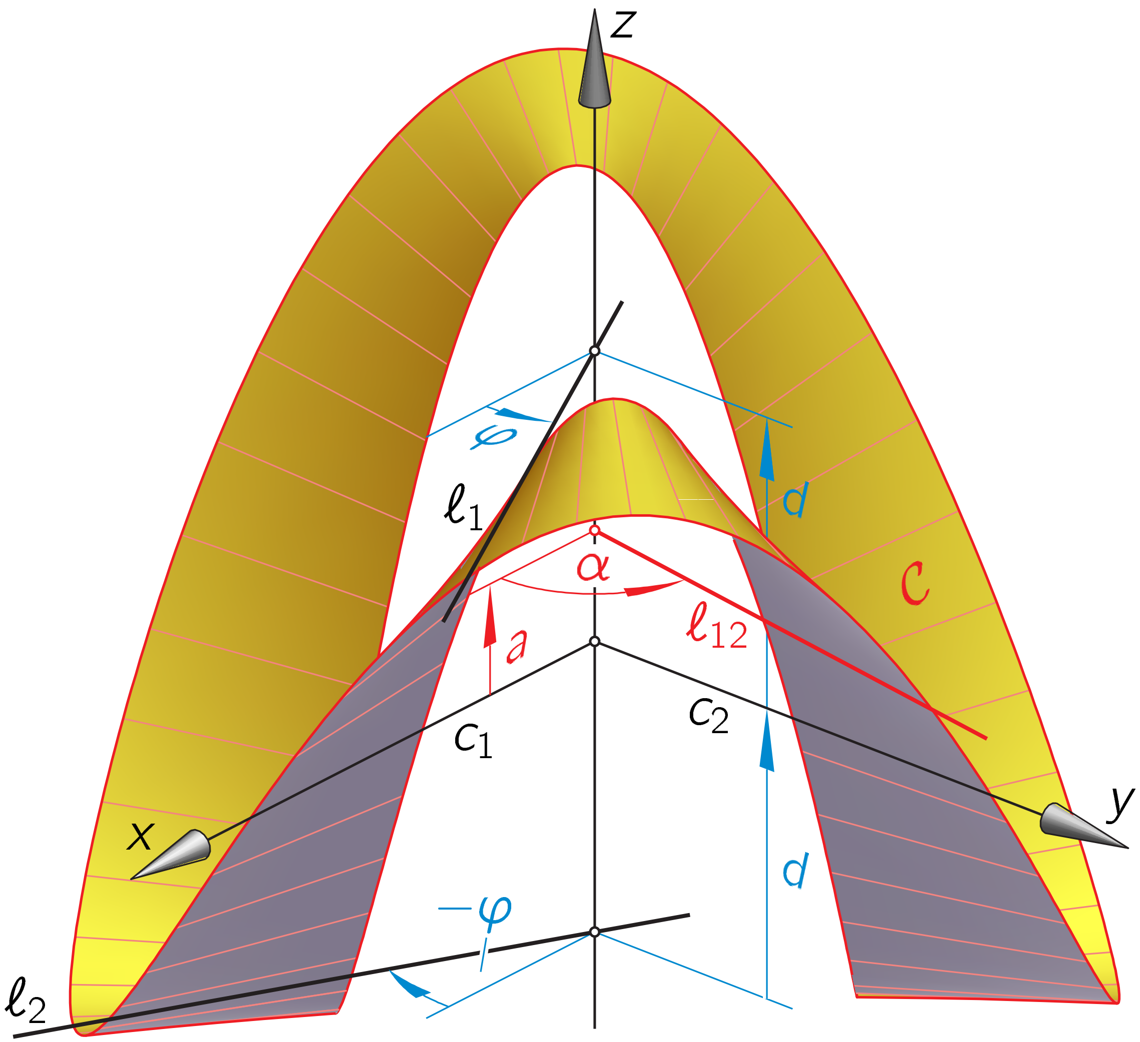


For given skew axes ℓ_1, ℓ_2, find pairs of hyperboloids $\mathcal{H}_1, \mathcal{H}_2$ with contact along a line ℓ_{12}.

Kinematics: the hyperboloids are the axodes of the relative motion of two bodies rotating about ℓ_1 and ℓ_2 with constant velocities ω_1, ω_2.

The common normal lines along ℓ_{12} must meet ℓ_1 and ℓ_2. They form an orthogonal hyperbolic paraboloid with vertex generator ℓ_{12} $\Longrightarrow$

3. Hyperboloids of revolution with line-contact



A result well-known in spatial gearing:

The locus of the lines of contact ℓ_{12} (instant screw axes) for variable ratio $\omega_1 : \omega_2$ is a Plücker conoid $\mathcal{C}$ through ℓ_1 and ℓ_2 with the axes of symmetry c_1, c_2 of ℓ_1, ℓ_2 as central axes.

On $\mathcal{C}$, every symmetric choice of ℓ_1, ℓ_2 and of ℓ_{12} yields contacting hyperboloids of revolution.

Schönbrunn Castle, Vienna

Thank you for your attention!

References

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