The 19th International Conference on Geometry and Graphics (ICGG 2020)

STRING CONSTRUCTIONS OF QUADRICS REVISITED

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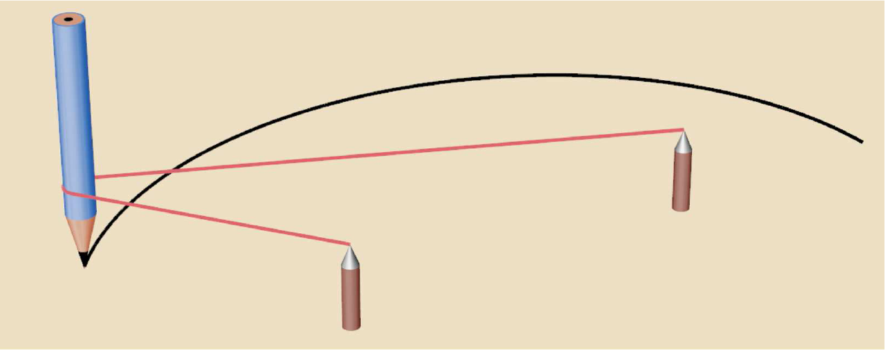
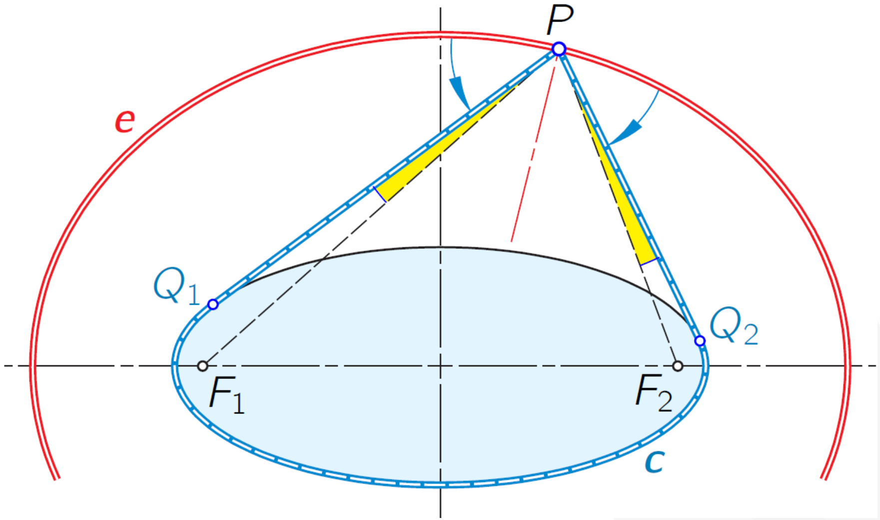


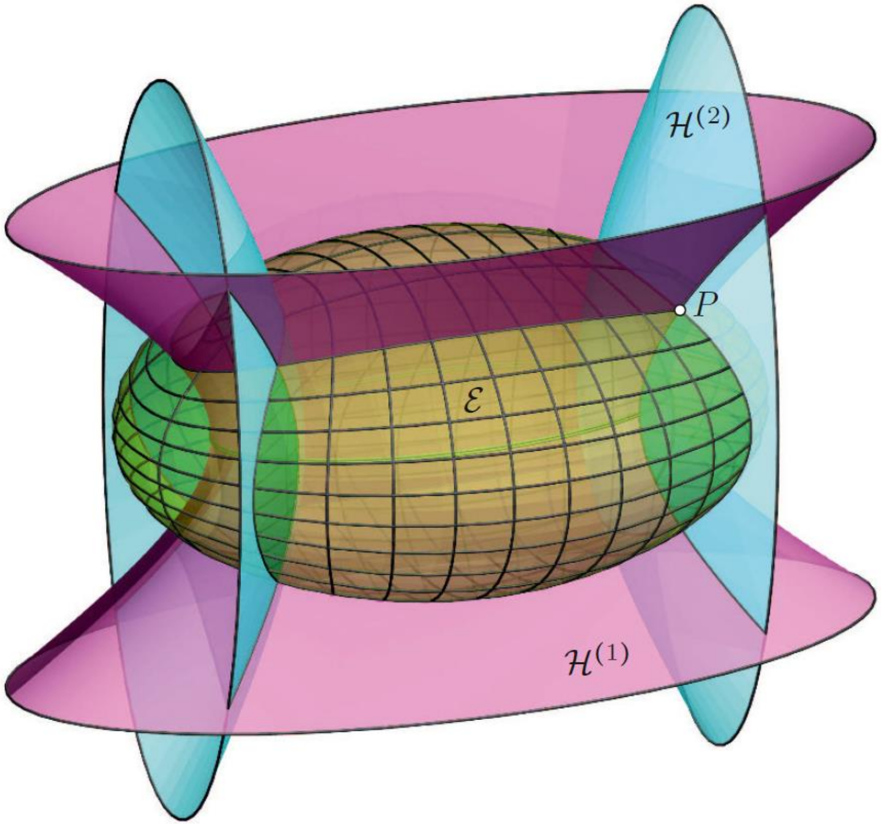


Wellknown string constructions of ellipses are the gardener's construction and Graves's construction based on the confocal smaller ellipse *c*

1. Introduction

- It is natural to ask for string constructions of quadrics.
- The first solution, given in 1882 by Otto Staude, is based on an ellipse e and its focal hyperbola h.
- Later in 1896, Staude presented a second type of string constructions where e and h are replaced by an ellipsoid $\mathcal{E}_0$ and a confocal hyperboloid $\mathcal{H}_0$.
- We present new proofs for these constructions.

2. Confocal central quadrics

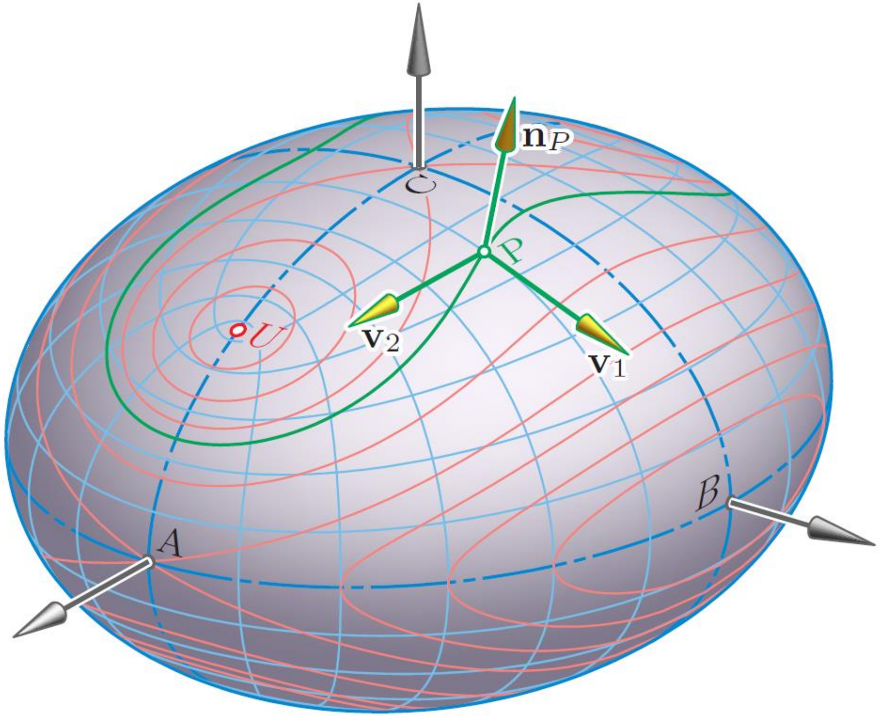


The family of confocal quadrics

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,\quad k\in\mathbb{R}\setminus\{-a^2,b^2,c^2\}$$

(ellipsoids, one- and two-sheeted hyperboloids) sends through each point *P* three mutually orthogonal surfaces, one of each type.

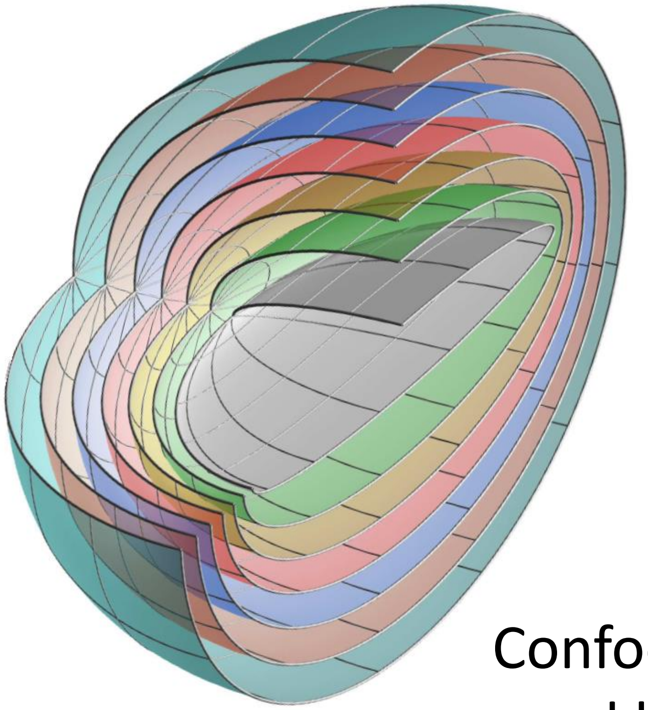
2. Confocal central quadrics

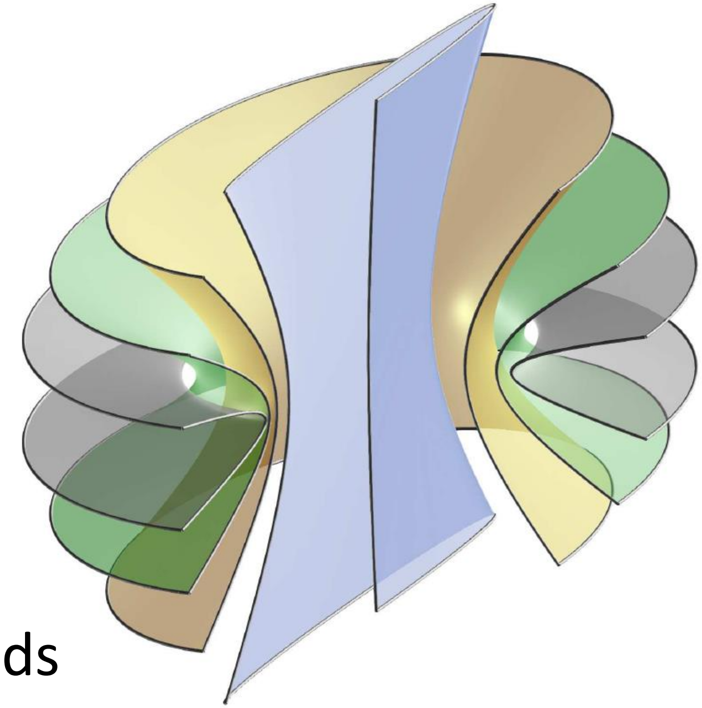


The three surface normals at any point *P* determine an orthogonal frame.

The **tangent cones** from any point *P* to the confocal quadrics are confocal, too. The said orthogonal frame determines the common axes of these cones

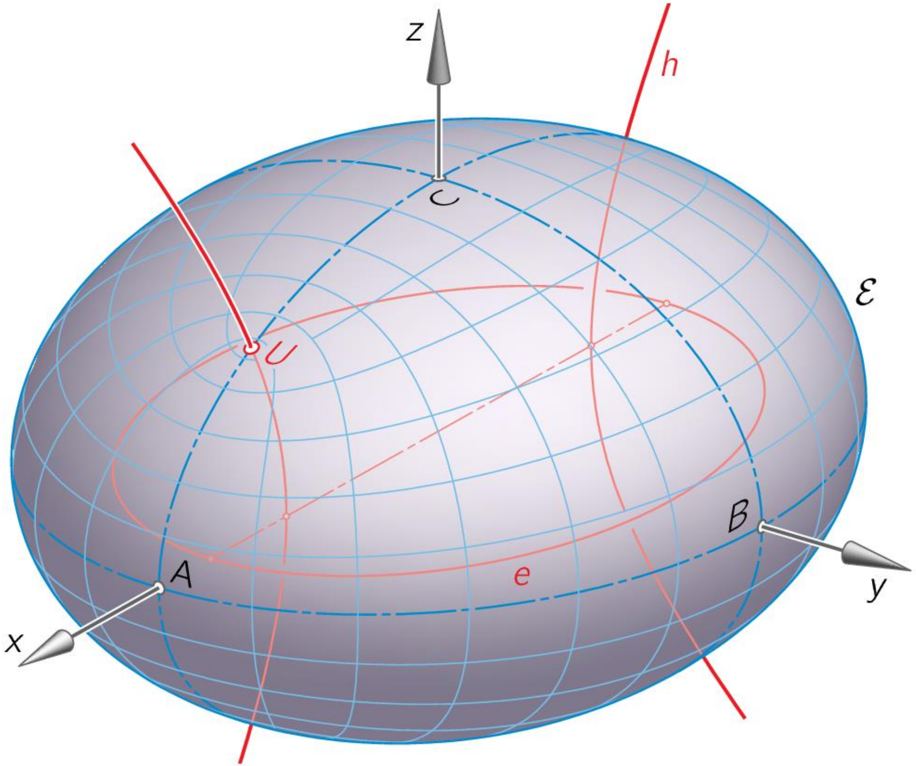
2. Confocal central quadrics





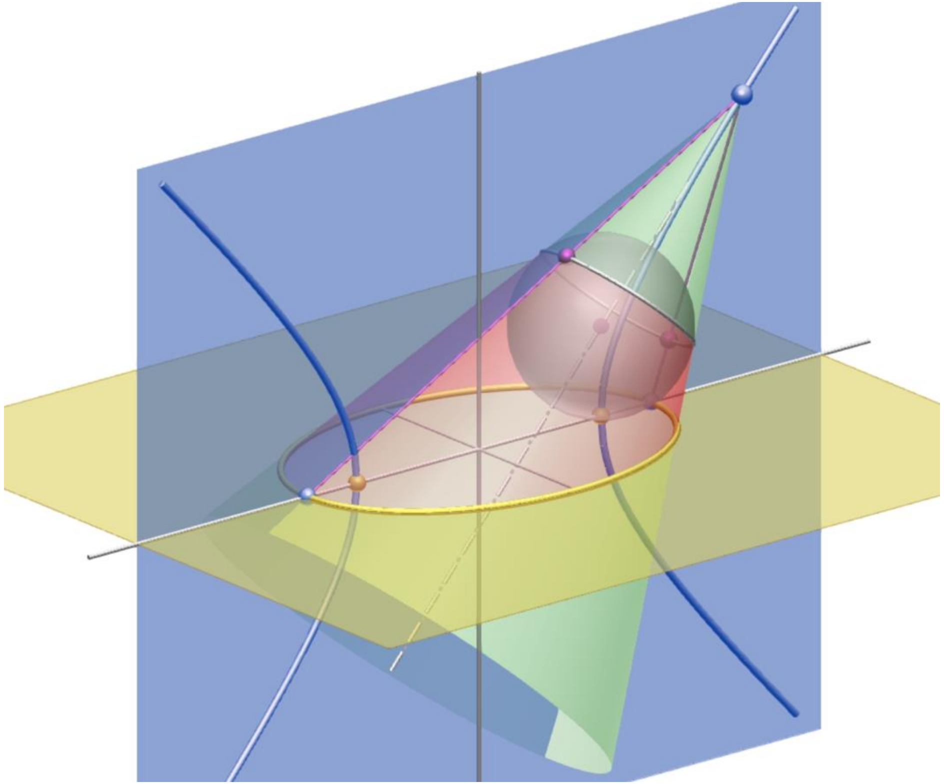
Confocal ellipsoids
and hyperboloids

2. Confocal central quadrics



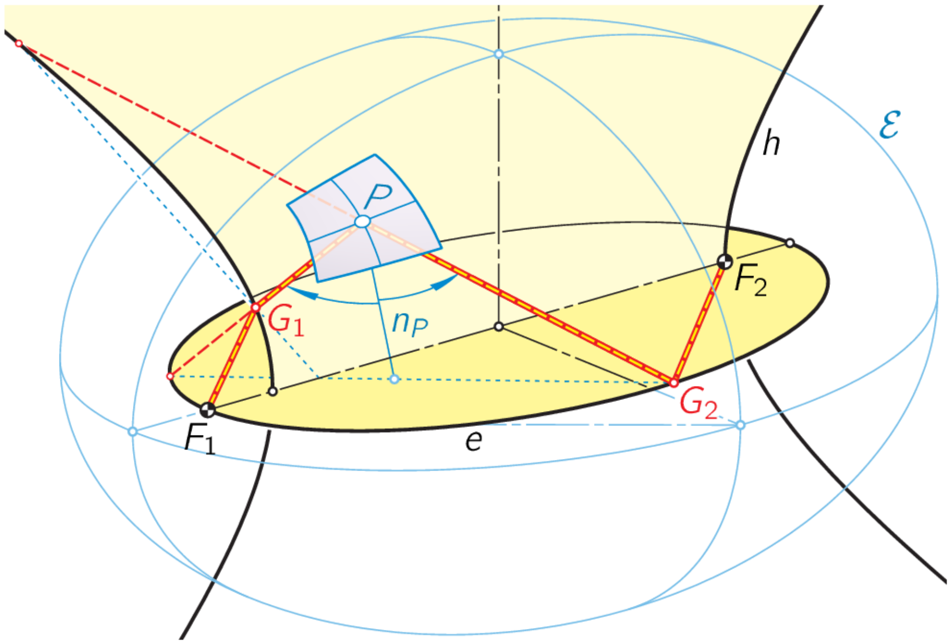
The ‚flat' surfaces in the confocal family determine the two focal conics of the quadrics, the *focal ellipse e* and the *focal hyperbola h.*

2. Confocal central quadrics



The connecting cone of any point *P* on *h* with the focal conic *e* is a cone of revolution with the tangent to *e* at *P* as axis, and vice versa.

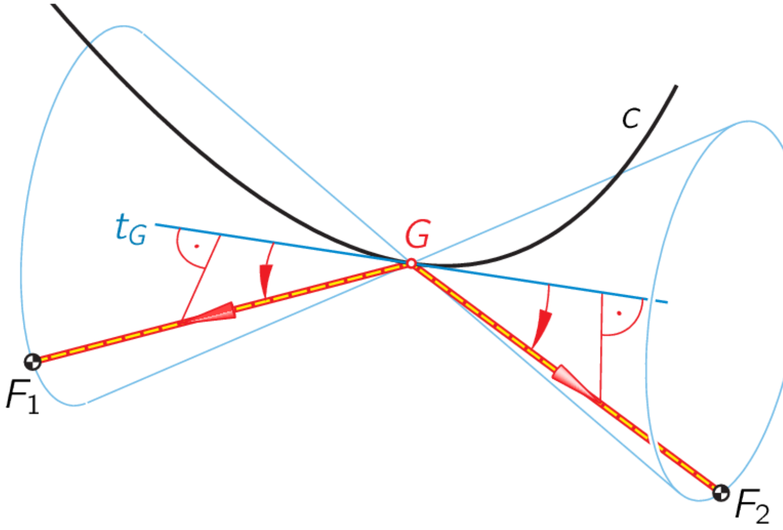
3. String constructions of ellipsoids



O. Staude's first string construction of the ellipsoid $\mathcal{E}$ (1882) is based on the *focal ellipse e* and the *focal hyperbola h* of $\mathcal{E}$.

The strengthened string forces *P* to move locally on the ellipsoid $\mathcal{E}$.

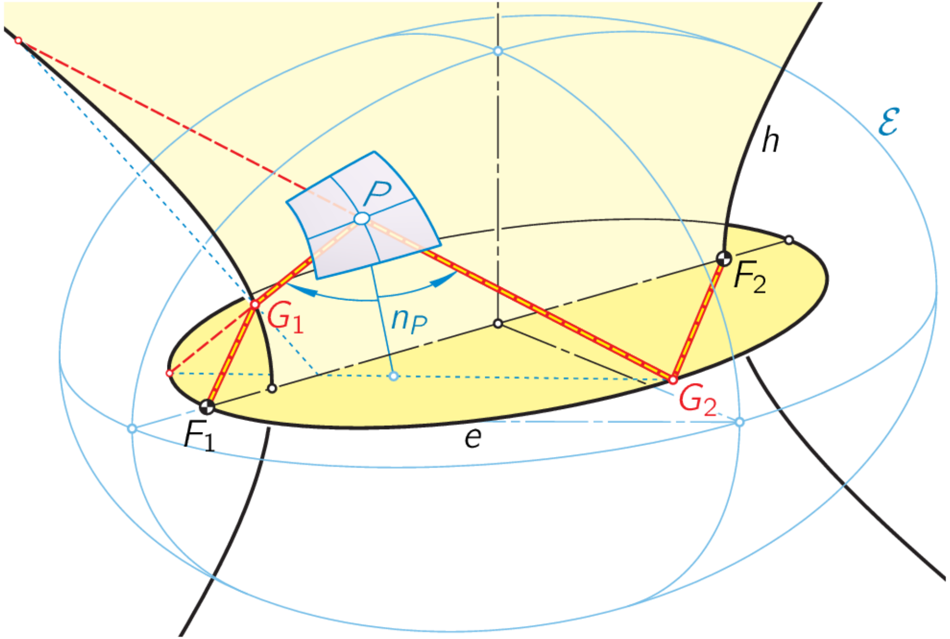
3. String constructions of ellipsoids



Lemma 1:
If the string from F_1 to F_2 is strengthened at G over the curve c, then the two segments F_1G and GF_2 belong to the same cone of revolution with the tangent at G as axis.

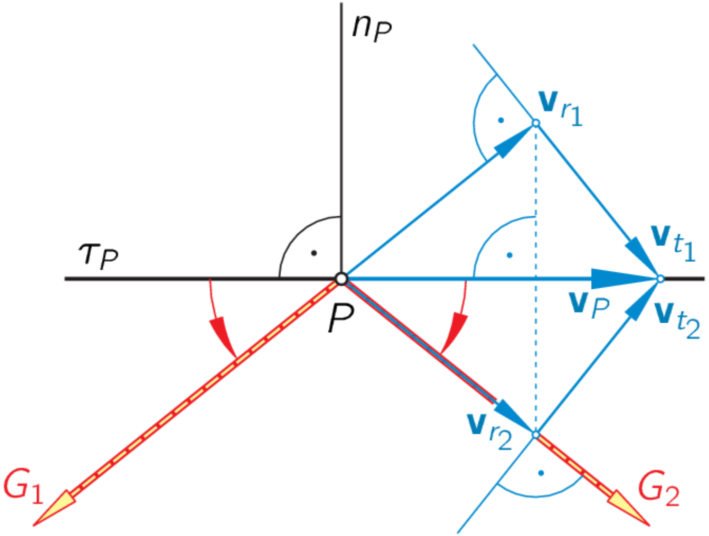
Moreover, the normal plane of c at G must separate F_1 from F_2

3. String constructions of ellipsoids



The cone of revolution through P and F_1 with apex G_1 passes through e. Hence, the extensions of PG_1 and PG_2 meet both conics e and h and are symmetric w.r.t. the normal n_P at P.

3. String constructions of ellipsoids



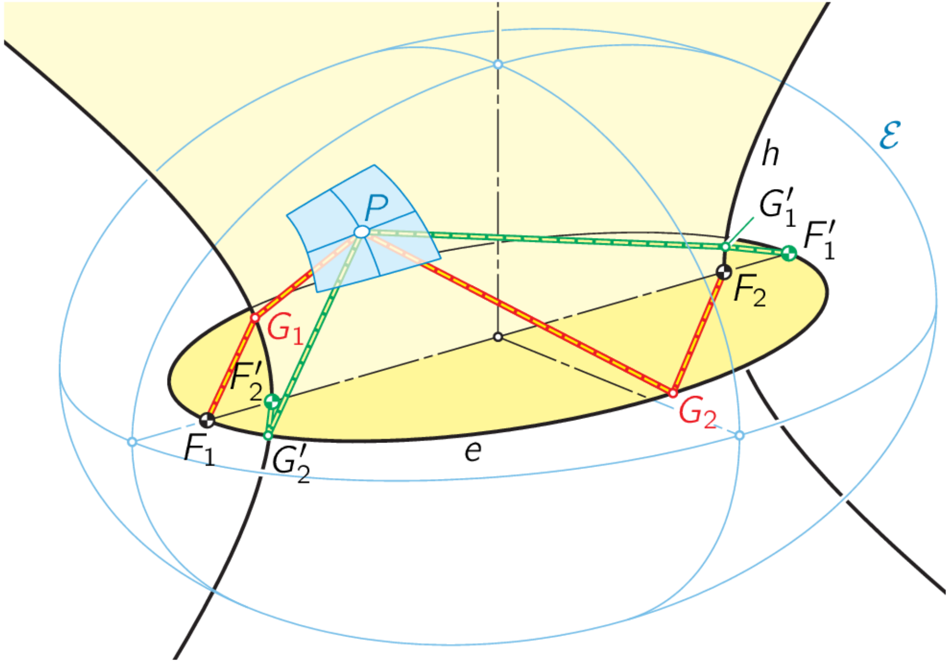
Lemma 2:

A point P, which is fixed on the left part of the string with endpoint F_1, moves orthogonal to G_1P.

Due to the constant length, if for P the distance to F_1 along the string increases, then that to F_2 decreases by the same amount.

$\Rightarrow$ *P moves orthogonal to* n_P

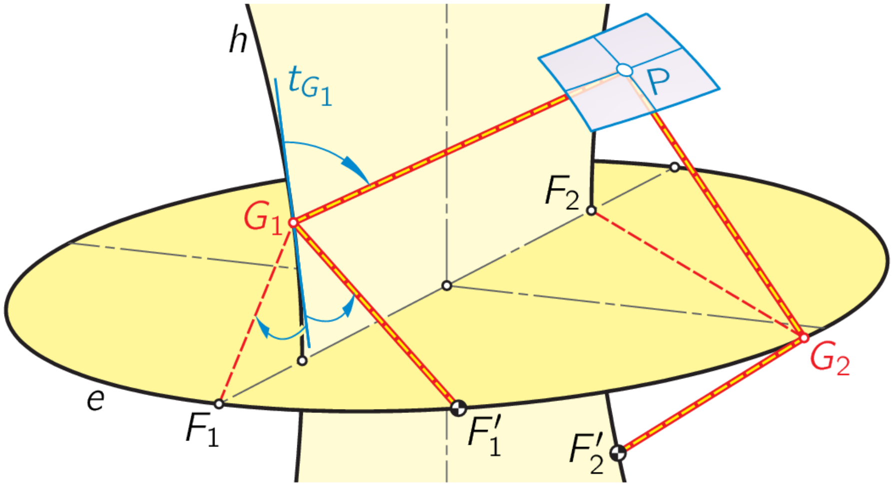
3. String constructions of ellipsoids



The cones from *P* to *e* and *h* share four generators.

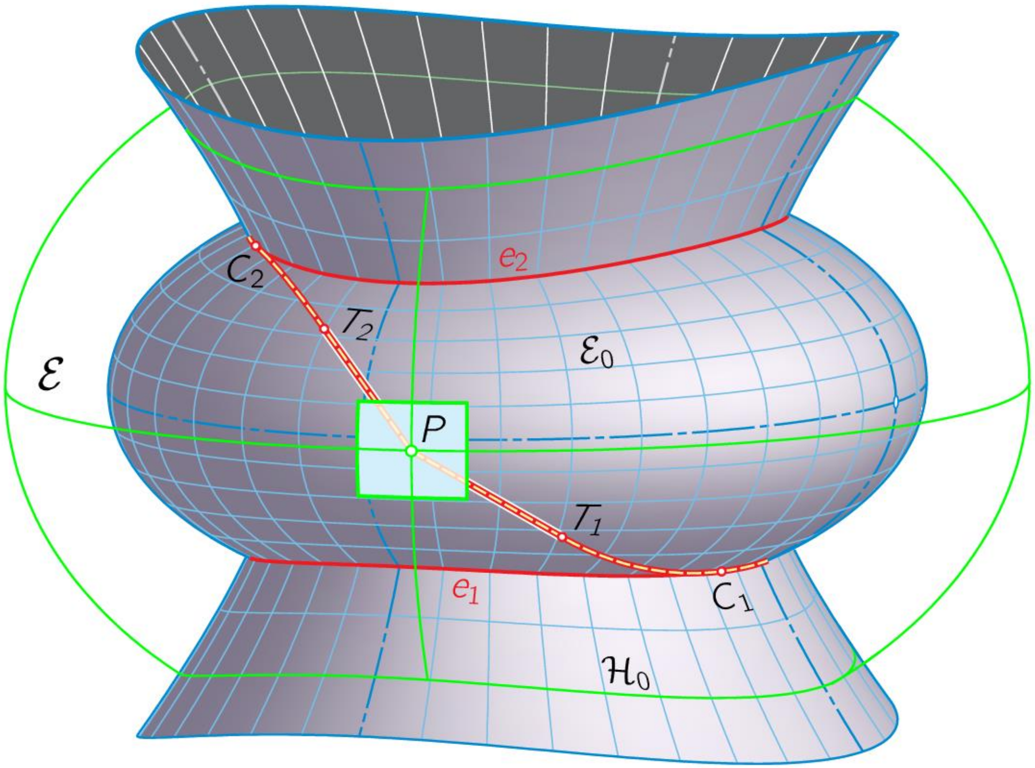
At Staude‘s first string construction for the ellipsoid $\mathcal{E}$, simultaneously two strings can be used, the red and the green one.

3. String constructions of ellipsoids



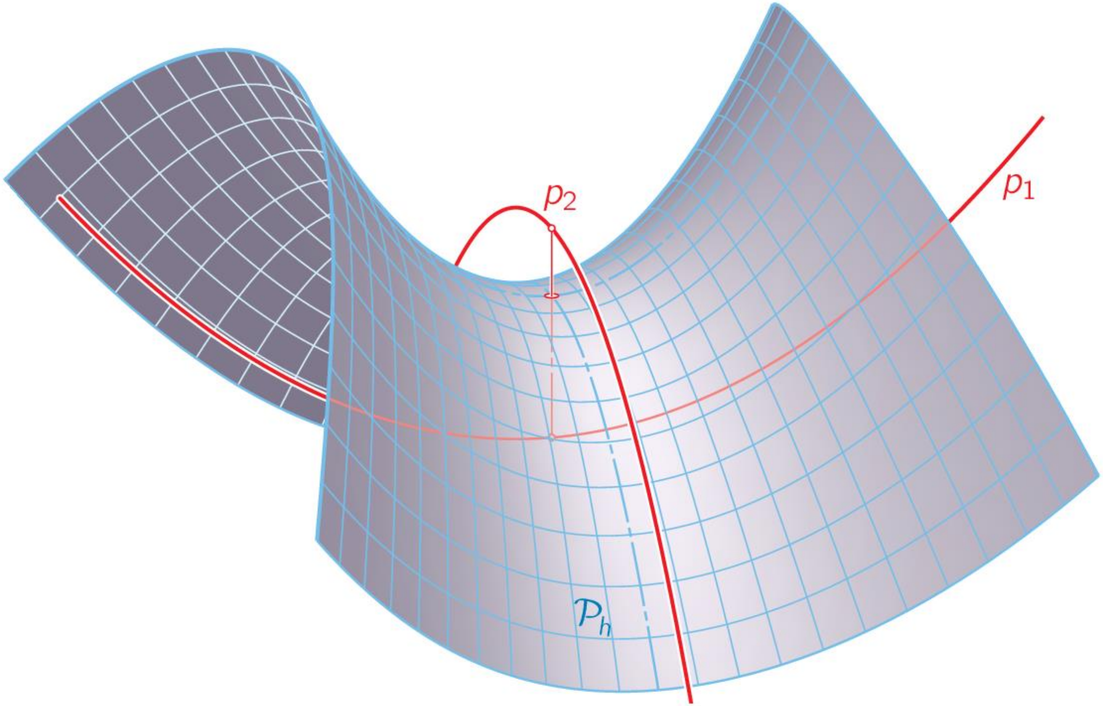
Finally, at Staude‘s first string construction for the ellipsoid $\mathcal{E}$, the endpoints F_1 on e and F_2 on h can vary on the respective conics.

3. String constructions of ellipsoids



Staude's string construction of **type 2** (1896) for the ellipsoid $\mathcal{E}$ is based on the two components e_1, e_2 of the line of curvature $\mathcal{E}_0 \cap \mathcal{H}_0$

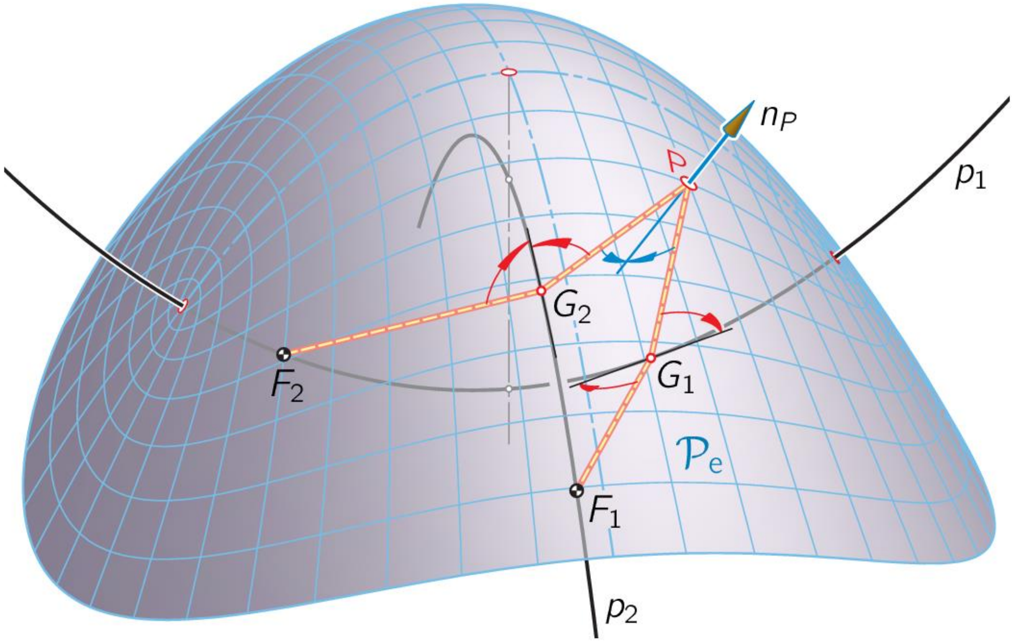
4. String constructions of paraboloids



Elliptic paraboloid with its two focal parabolas on p_1 and p_2.

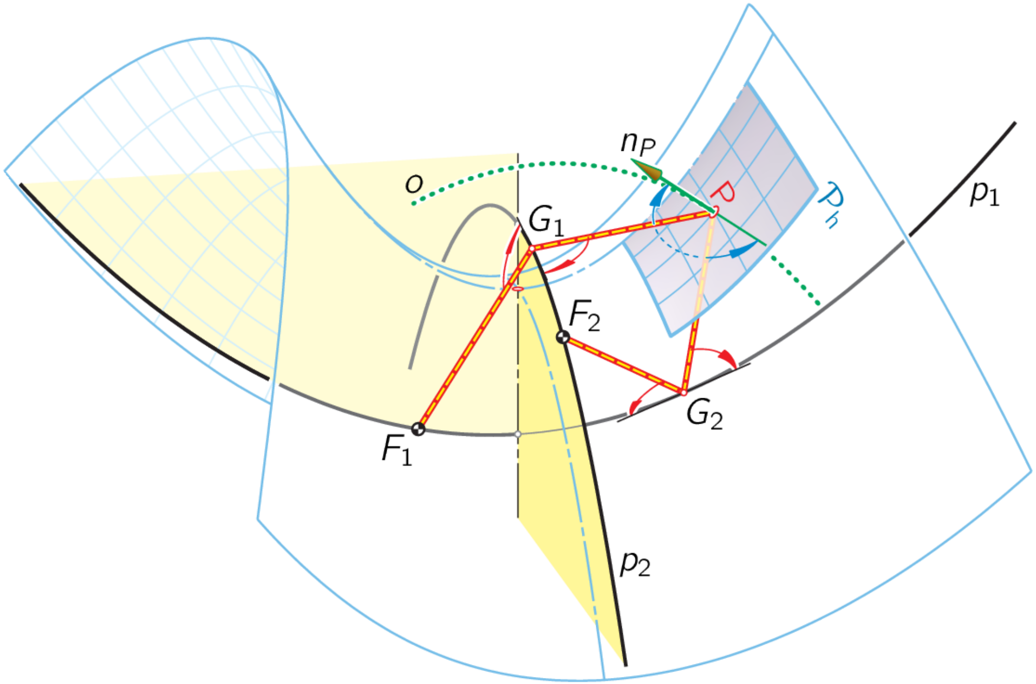
If Staude‘s string construction of **type 1** works also for paraboloids, then it must be based on p_1 and p_2.

4. String constructions of paraboloids



The construction fails for the elliptic paraboloid, since the normal plane of p_2 at G_2 **does not** separate P from F_2

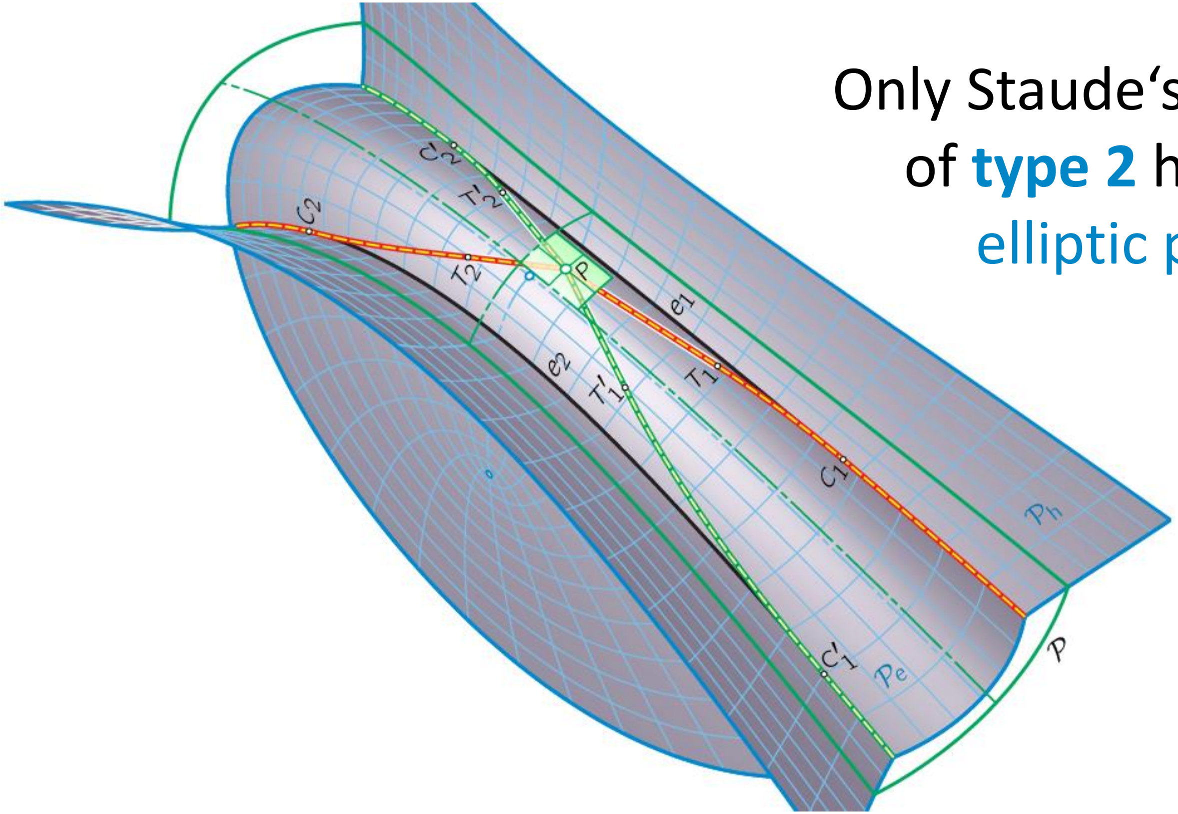
4. String constructions of paraboloids



The construction fails for the hyperbolic paraboloid, since n_P **is not** the interior angle bisector of G_1PG_2.

However, the difference of lengths F_1G_1P and F_2G_2P is constant.

4. String constructions of paraboloids



Only Staude‘s string construction of **type 2** holds also for the elliptic paraboloid $\mathcal{P}$

Schönbrunn Castle, Vienna

Thank you for your attention !

References

D. Hilbert, St. Cohn-Vossen: *Anschauliche Geometrie*, 2nd ed., Springer, Berlin 1996. English translation: *Geometry and the Imagination*. Chelsea Publ., reprinted by American Mathematical Society, Providence, RI, 1999.

B. Odehnal, H. Stachel, G. Glaeser: *The Universe of Quadrics*. Springer Verlag Berlin Heidelberg 2020.

O. Staude: *Über Fadenconstructionen des Ellipsoides*. Mathematische Annalen **20**, 147-185 (1882).

O. Staude: *Die Focaleigenschaften der Flächen zweiter Ordnung*. B.G. Teubner, Leipzig 1896.