Billiards in ellipses and ellipsoids

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22nd Scientific-Professional Colloquium on Geometry and Graphics
Sept. 5–9, 2021, Čiovo/Croatia

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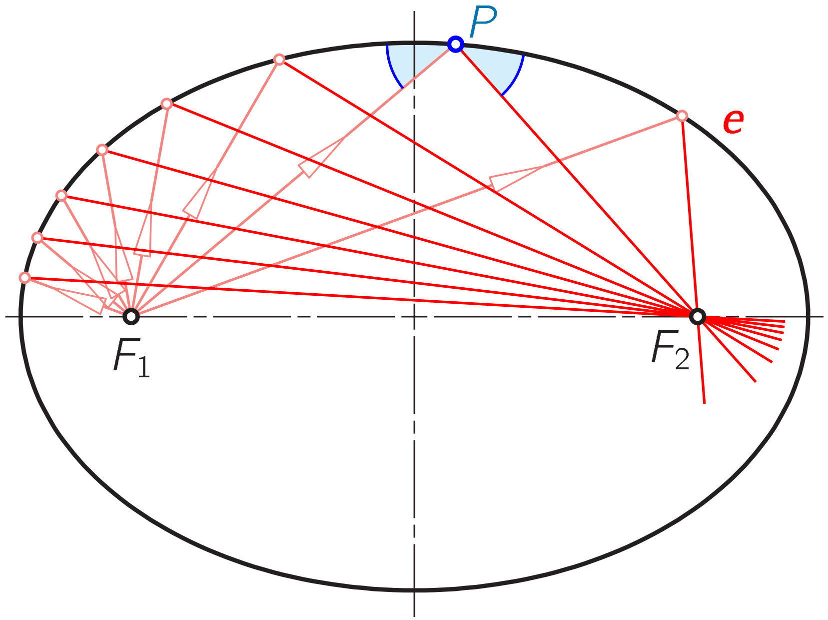
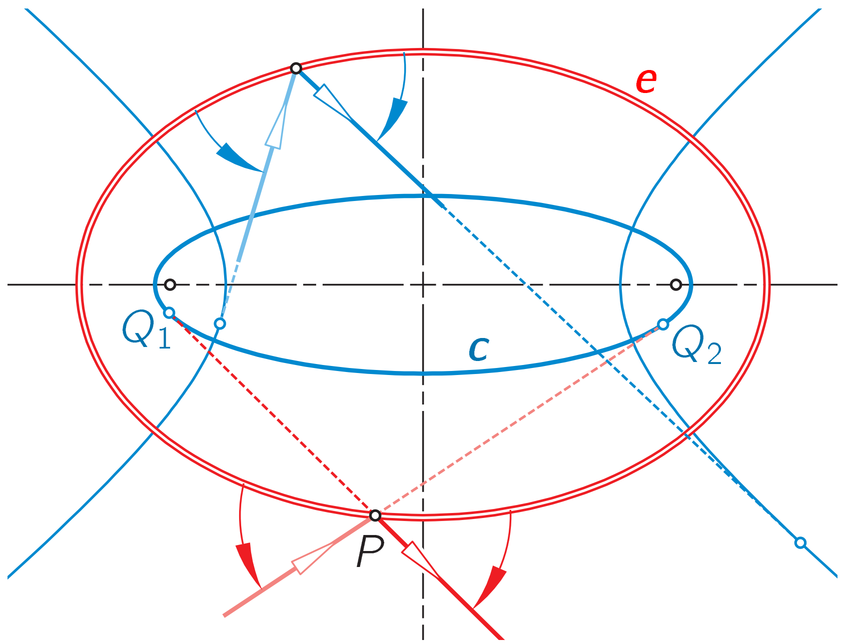
A **billiard** is the trajectory of a mass point within a domain with ideal physical reflections in the boundary — in our case an ellipse e.



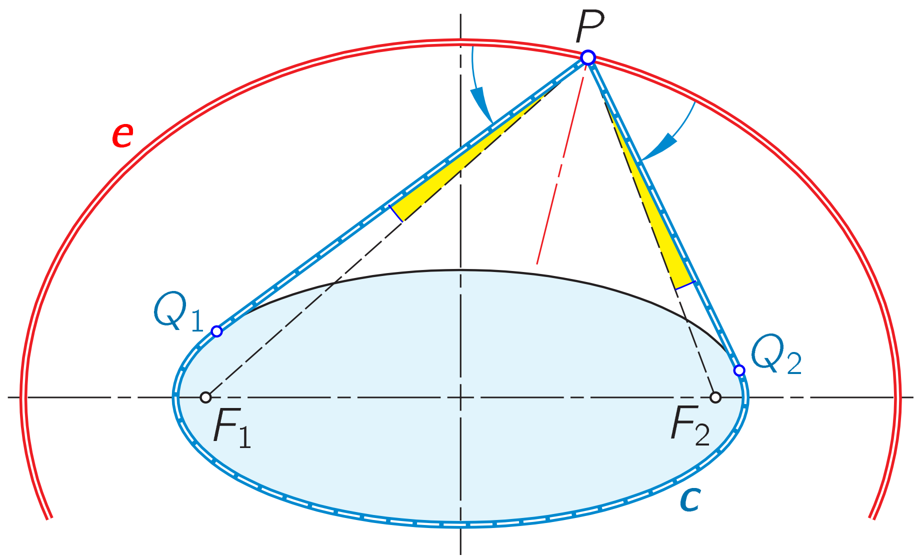
The optical property of ellipses is well known, and also the equivalence:

$$\text{equal angles} \iff \overline{F_1P} + \overline{F_2P} = \text{const.} \iff P \in e\,.$$

1. Billiards in ellipses and billiard motion



If any ray is reflected in a conic e then the incoming and the outgoing ray are tangent to the same confocal conic c, called **caustic** (ellipse or hyperbola).

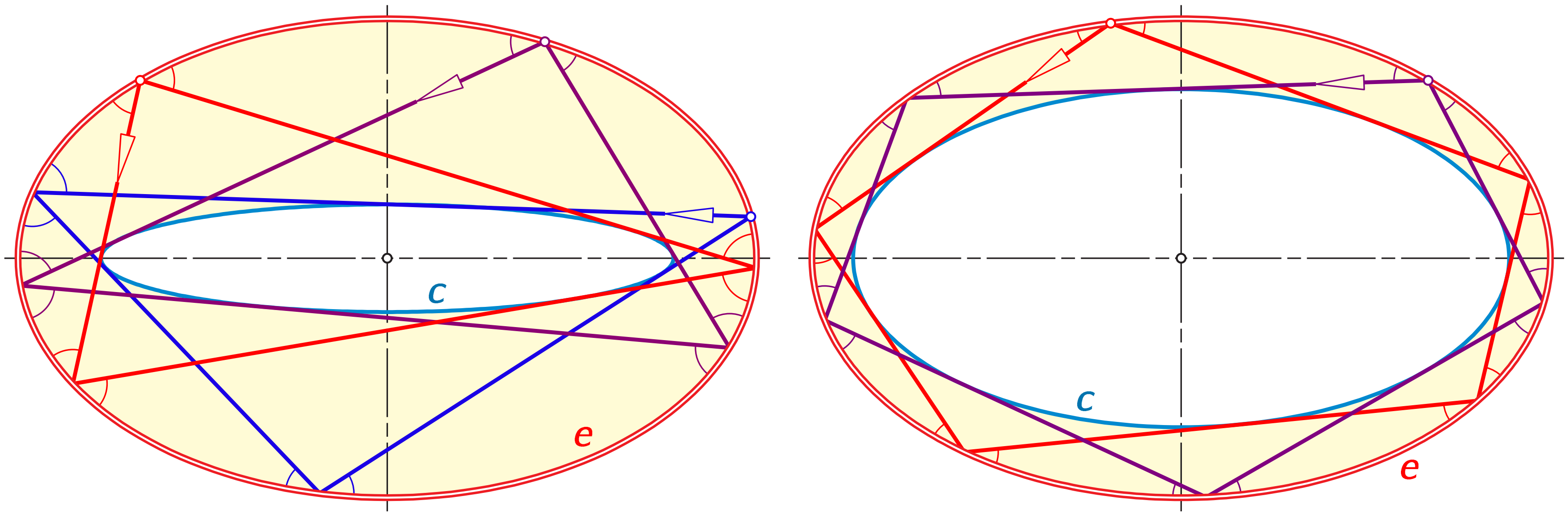


Charles **Graves** (1812-1899), bishop of Limerick and mathematician:

The locus of point P used to pull the string taut around c is a confocal ellipse e.

$$D_e := \overline{PQ_1} + \overline{PQ_2} - \widehat{Q_1Q_2} = \text{const.}$$

1. Billiards in ellipses and billiard motion



Billiards in an ellipse *e* are always tangent to a confocal ellipse or hyperbola.

If one billiard is periodic and closes after *N* reflections, then all billiards close, independent of the initial point on *c* **(Poncelet porism)**, and all these closed loops have the same length – as a consequence of Graves' theorem.

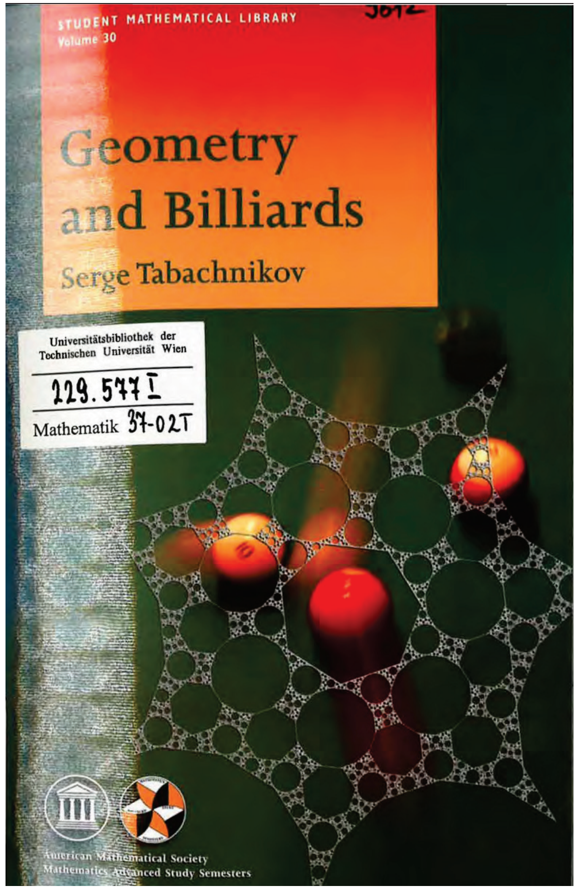
The continuous variation of the billiard in *e* is called **billiard motion**:

1. Billiards in ellipses and billiard motion

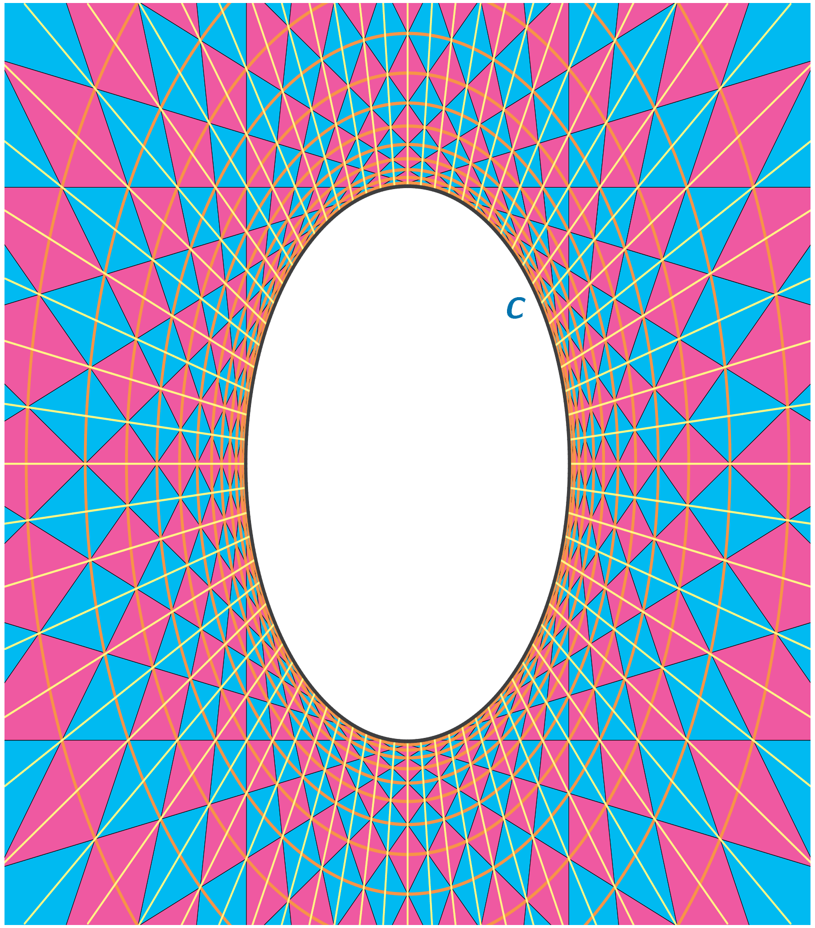
For centuries, billiards (and projectively equivalent polygons with an inscribed and a circumscribed conic) attracted the attention of mathematicians, beginning with J.-V. Poncelet, C.G.J. Jacobi, M. Chasles, A. Cayley, and G. Darboux.

S. Tabachnikov: *Geometry and Billiards.*
American Mathematical Society, 2005

In 2020, Dan Reznik (Brazil) revitalized the interest by computer animations showing the motion of periodic billiards. He identified 80 invariants, e.g., a constant sum of Cosines of interior angles.



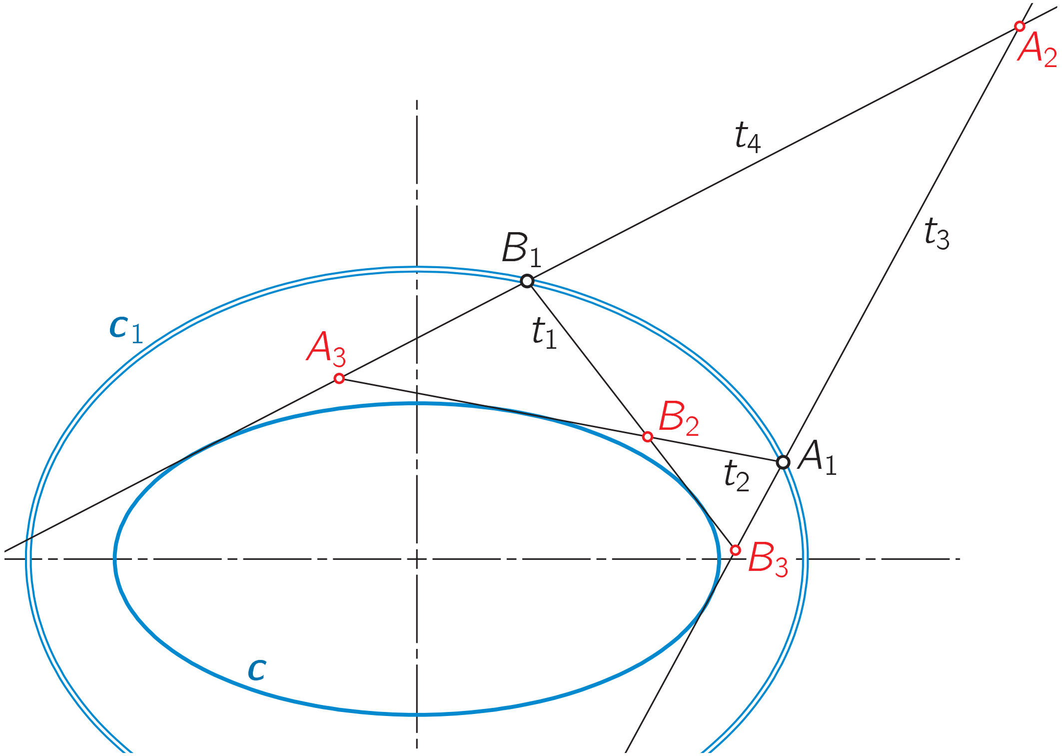
2. Billiard and associated Poncelet grid



The extended sides of a billiard intersect at points of confocal ellipses and hyperbolas and form a **Poncelet grid**.

affinely transformed 72-sided periodic billiard with associated Poncelet grid (G. Glaeser, B. Odehnal, H.S.: *The Universe of Conics*)

2. Billiard and associated Poncelet grid

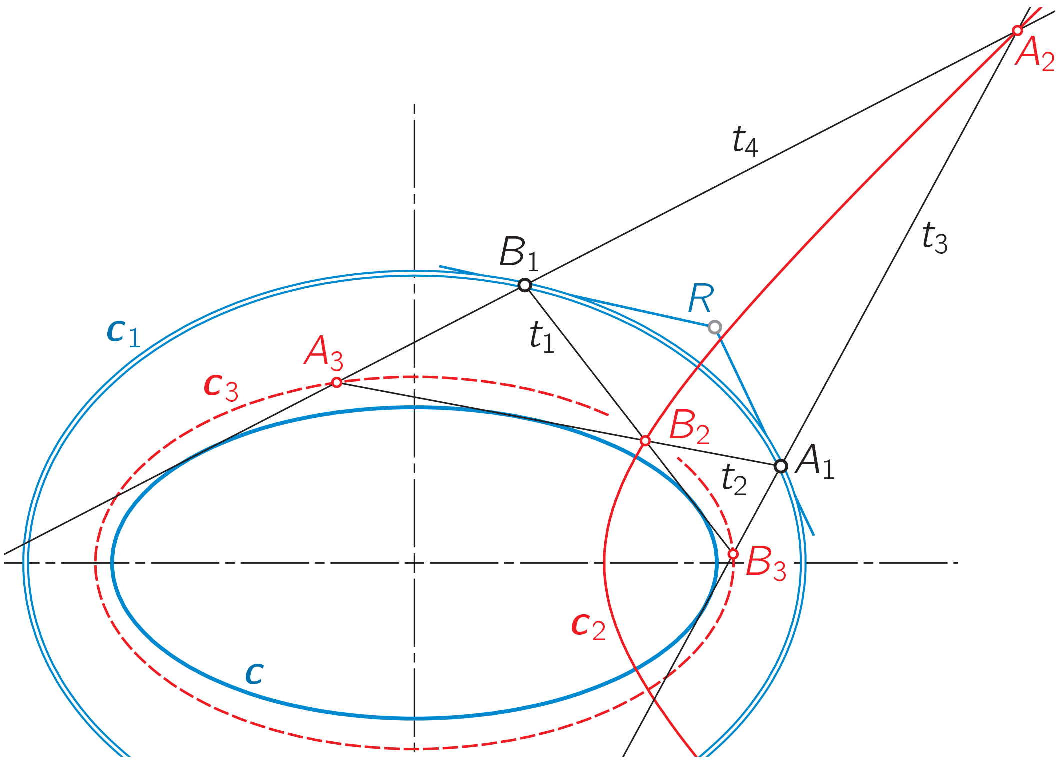


The extended sides of a billiard intersect at points of confocal ellipses and hyperbolas and form a **Poncelet grid**.

Theorem: Given a quadrilateral $t_1, \ldots, t_4$ of tangents to c from $A_1, B_1 \in c_1$. Then the range (= 'dual pencil') $\mathcal{R}_c$ spanned by c and c_1 contains conics c_2, c_3 passing through the remaining pairs of opposite vertices (A_2, B_2) and (A_3, B_3).

M. Chasles (1843), W. Böhm (1961), Izmestiev & Tabachnikov (2016), Akopyan & Bobenko (2017).

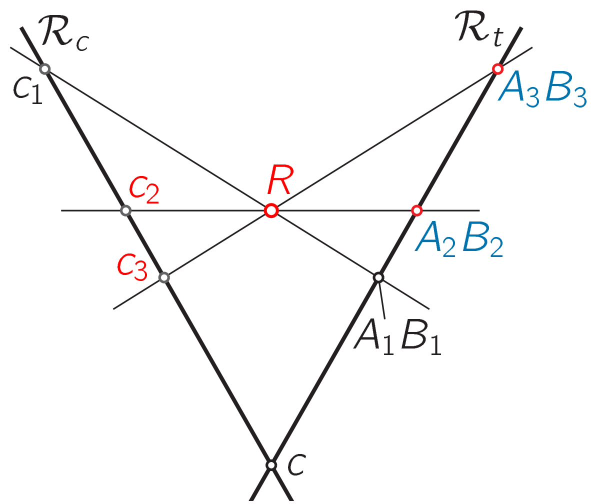
2. Billiard and associated Poncelet grid



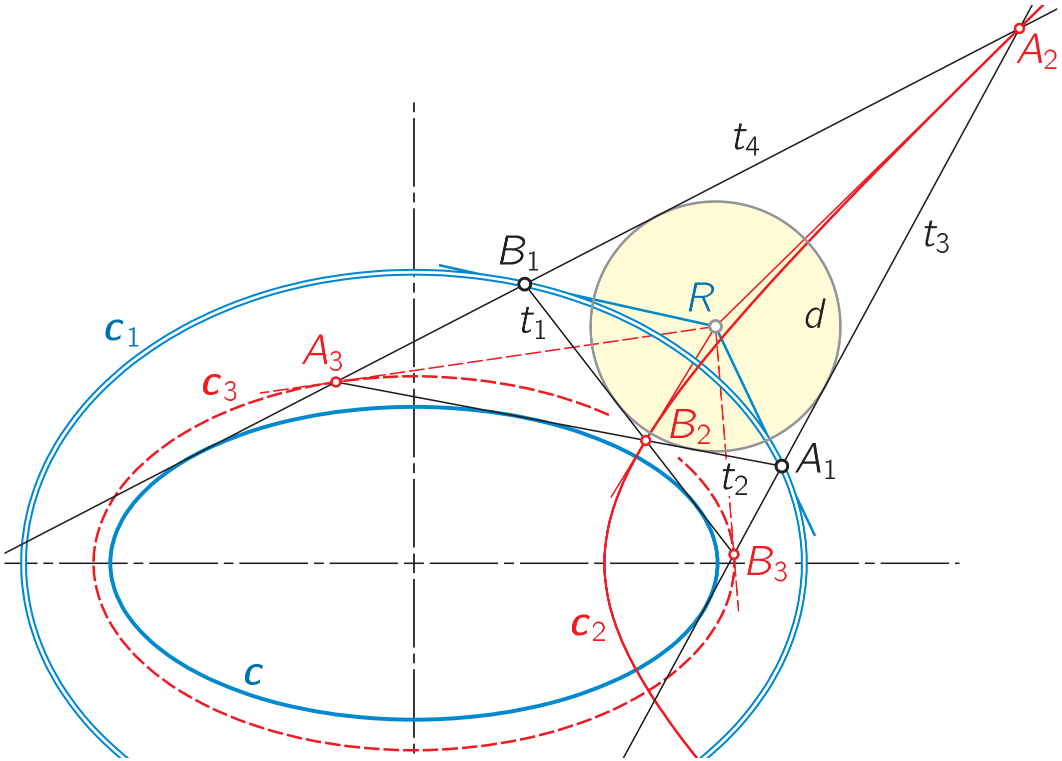
$\mathcal{R}_t := \{\text{conics tangent to } t_1, \ldots, t_4\}$

$\mathcal{R}_c \cap \mathcal{R}_t = \{c\} \implies$ they span a net $\mathcal{N}$ (2-parameter set).

In $\mathcal{N}$, the line elements of c_1 at A_1 and B_1 span a range which contains the rank-1 conic R.



2. Billiard and associated Poncelet grid



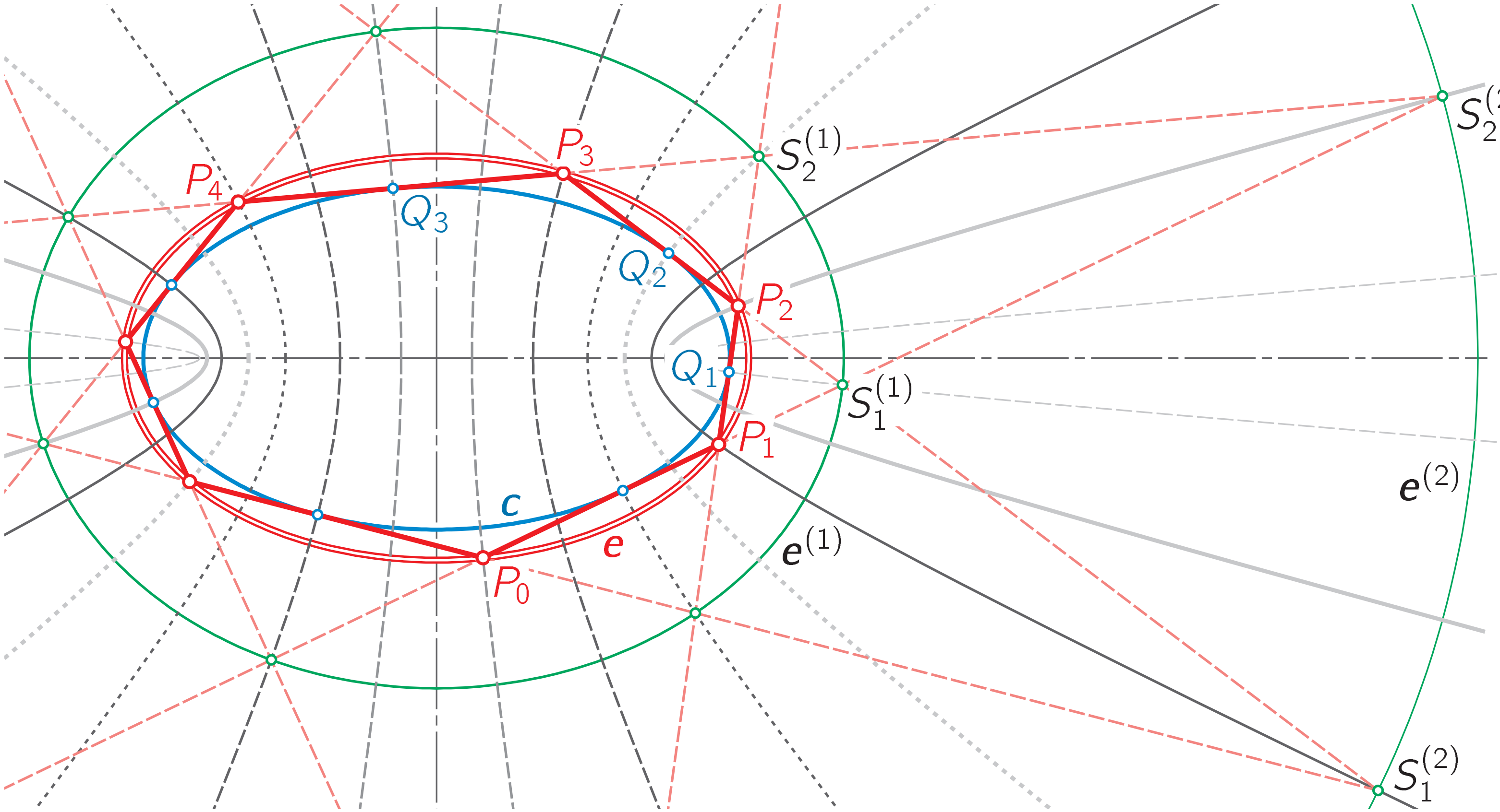
In $\mathcal{N}$, the line pencils A_i, B_i and the pencil R (2-fold) span a range which intersects $\mathcal{R}_c$ at c_i. The range contains conics sharing the line elements at A_i and B_i.

The tangents to c_i at A_i and B_i pass through R.

c, c_1 confocal $\Longrightarrow$ concyclic quadrilateral.

This holds also when $B_2 \in c$ $(t_1 = t_2)$.

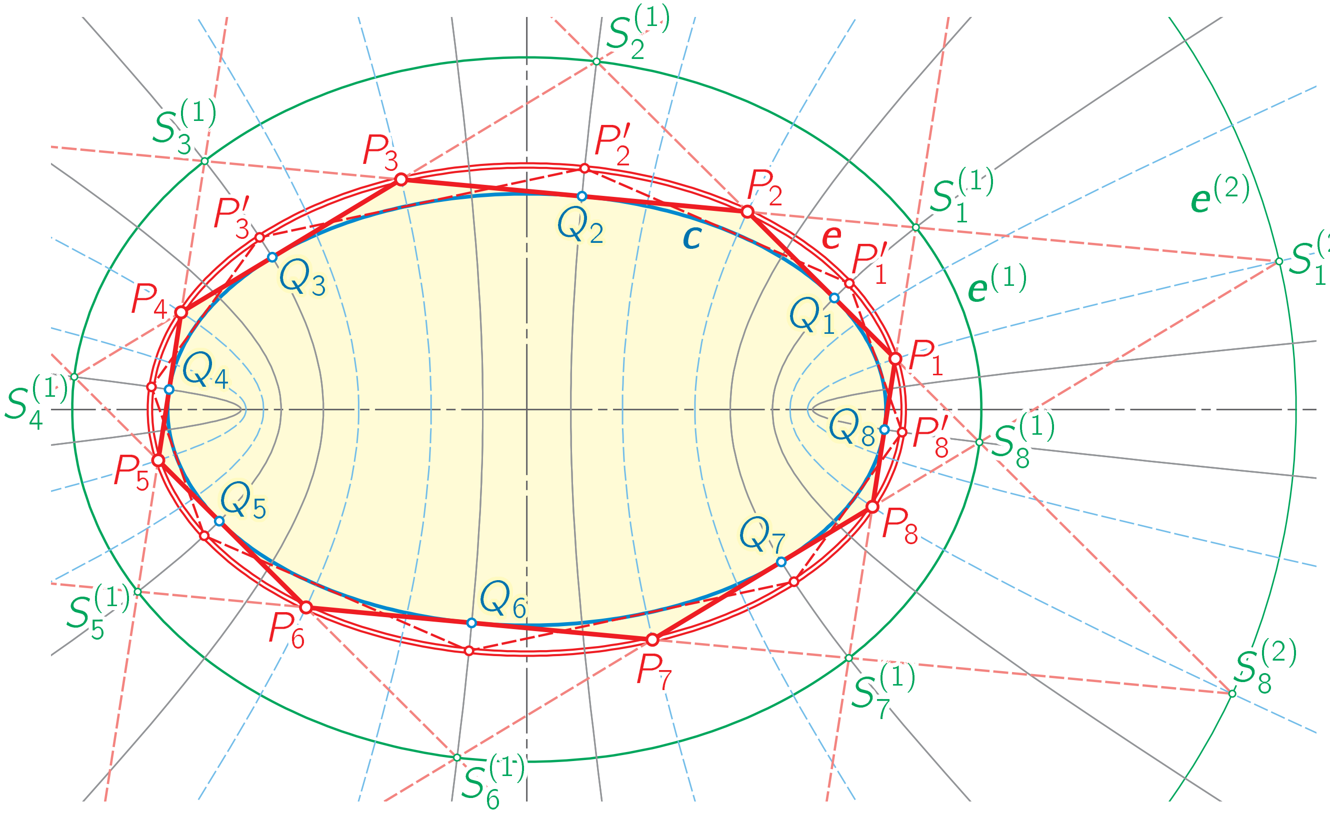
2. Billiard and associated Poncelet grid



$S_2^{(1)} := [P_1, P_2] \cap [P_3, P_4]$ on the confocal hyperbola through Q_2,

$S_2^{(2)} := [P_0, P_1] \cap [P_3, P_4] \in e^{(2)}$ on the confocal hyperbola through P_2.

2. Billiard and associated Poncelet grid

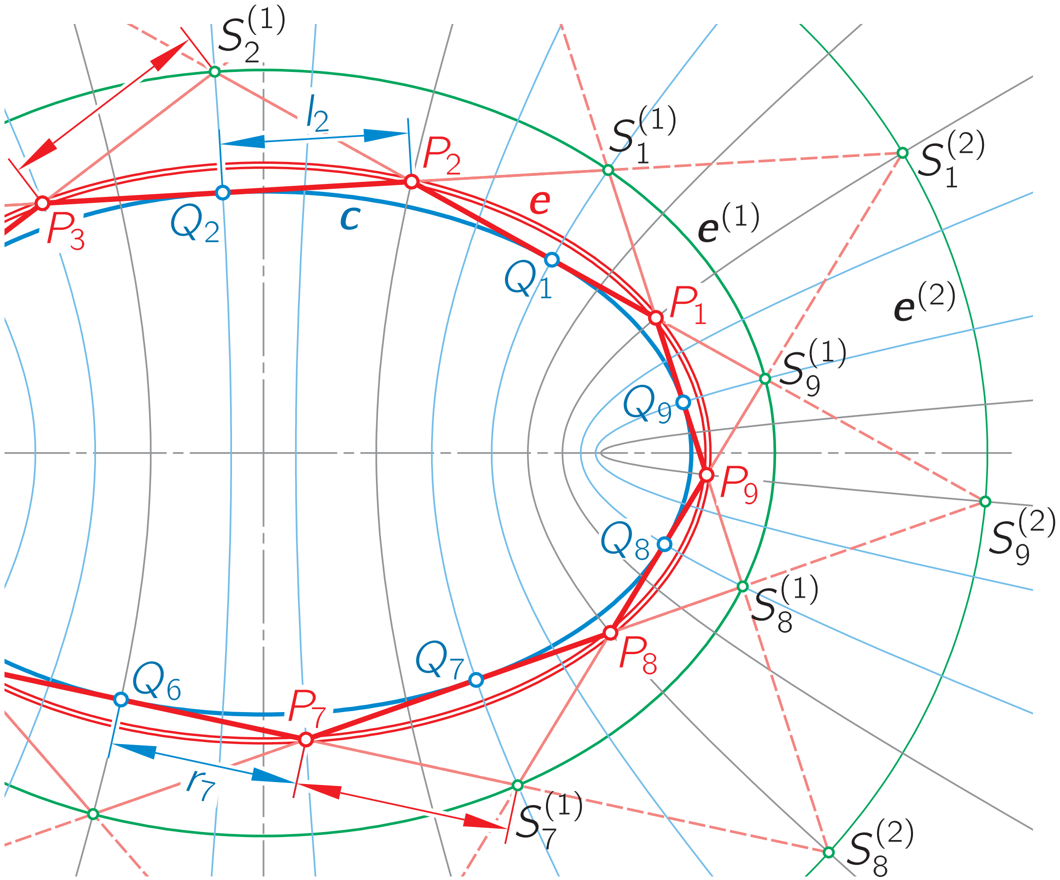


For each billiard $P_1P_2\ldots$ exists a **conjugate billiard** $P_1'P_2'\ldots$

2. Billiard and associated Poncelet grid

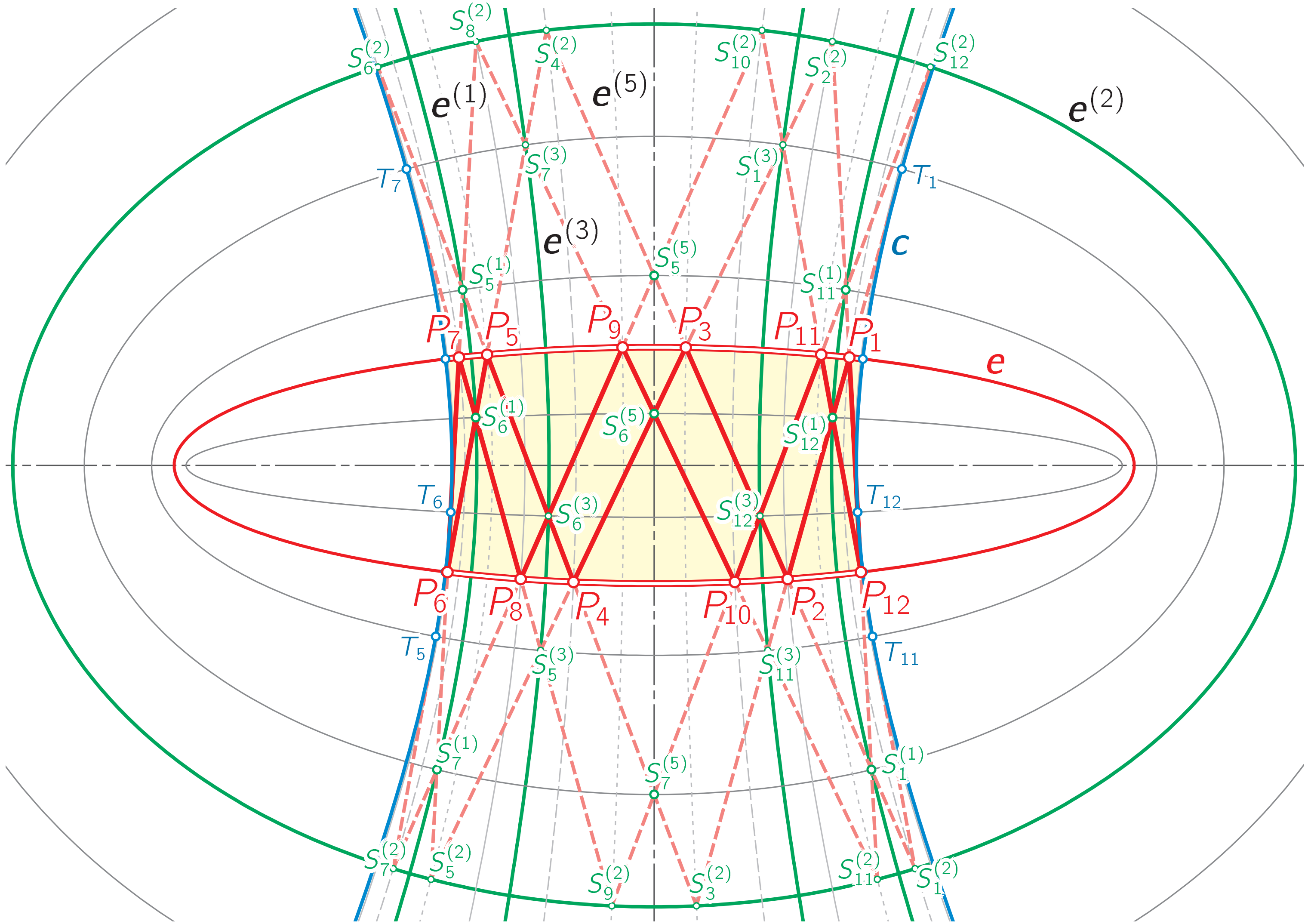
Clearly, a billiard motion induces a variation of the complete Poncelet grid and the conjugate billiard.

2. Billiard and associated Poncelet grid



Ivory's theorem implies for **odd** $N = 2n + 1$: the length l_i equals l'_{i+n} of the conjugate billiard and the symmetric r_{i+n+1}.

Theorem: $\sum l_i = \sum r_i = L_e/2.$



Billiards in ellipses e with a hyperbola as caustic c look different (N even).

3. Isometric elliptic and hyperbolic billiards

Theorem:

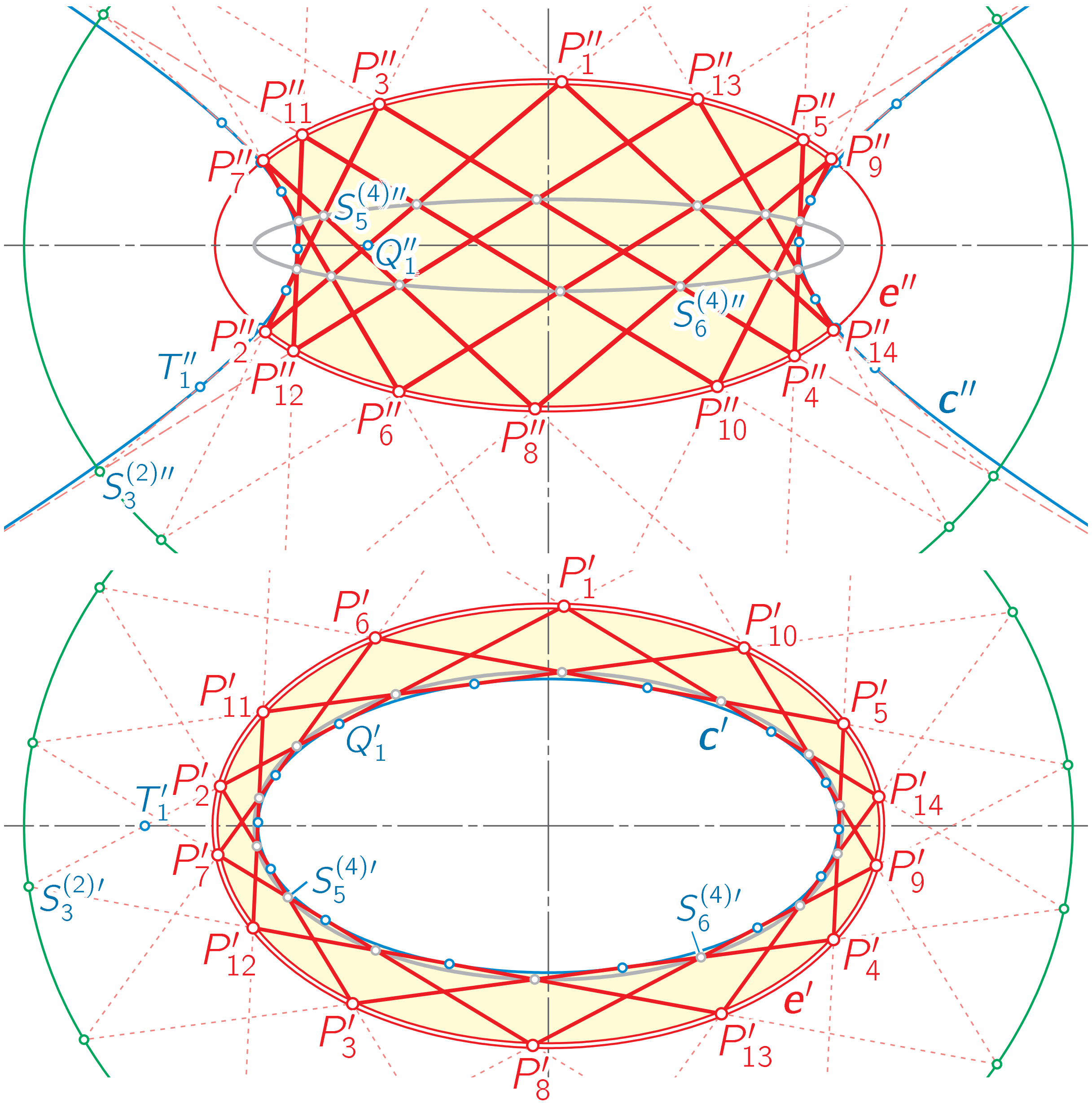
1) For each billiard $P_1'P_2'\ldots$ in an ellipse e' with an ellipse c' as caustic there exists an isometric billiard $P_1''P_2''\ldots$ in an ellipse e'' with a hyperbola c'' as caustic, i.e.,

$$\overline{P_i''P_{i+1}''} = \overline{P_i'P_{i+1}'}.$$

2) Conversely, for each billiard with a hyperbola c'' as caustic there exists an isometric billiard with an ellipse c' as caustic, provided that in the case of an N-periodic billiard with $N \equiv 2 \pmod 4$, we traverse the elliptic billiard twice.

$$a_c''^2 = a_c'^2 - b_c'^2, \quad b_c'' = b_c', \quad a_c''^2 - b_c''^2 = a_c'^2, \quad a_e'' = a_e', \quad b_e''^2 = a_e'^2 - b_e'^2,$$

$$a_c'^2 = a_c''^2 - b_c''^2, \quad b_c' = b_c'', \quad a_c'^2 - b_c'^2 = a_c''^2, \quad a_e' = a_e'', \quad b_e'^2 = a_e''^2 - b_e''^2.$$

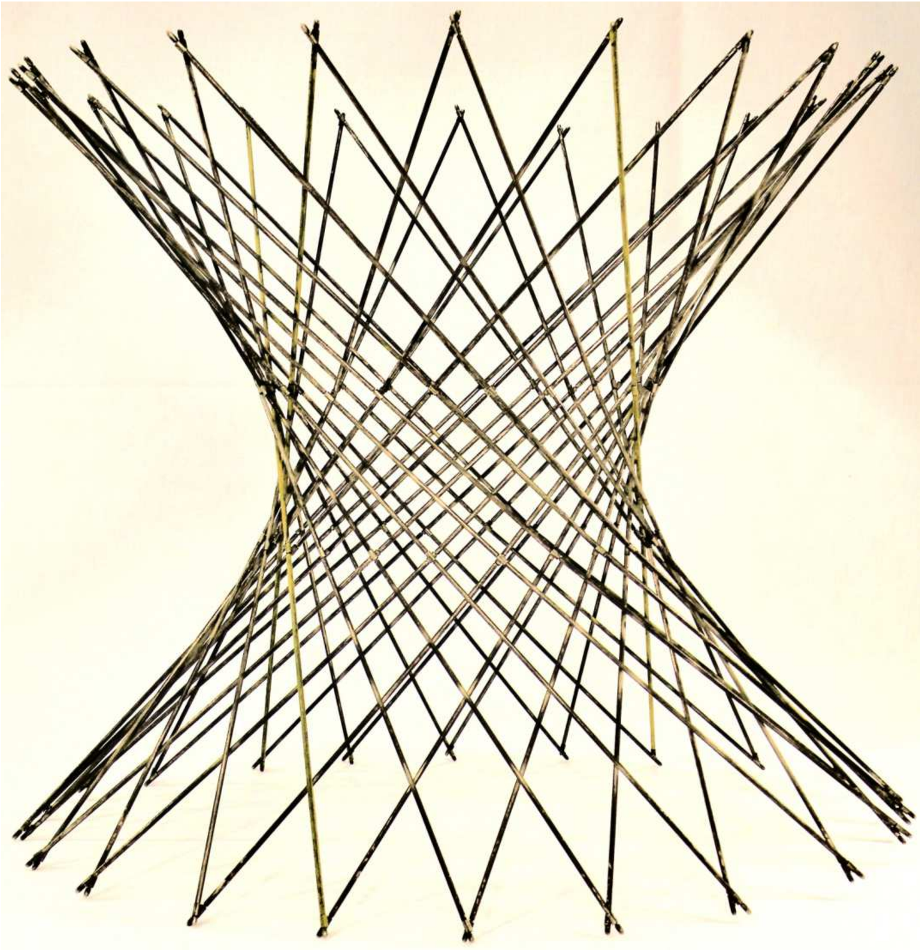


Moreover, contact points with the caustic correspond to intersection points with the principal axis, i.e.,

$$\overline{P'_1Q'_1} = \overline{P''_1Q''_1}, \quad \overline{P'_2T'_1} = \overline{P''_2T''_1}$$

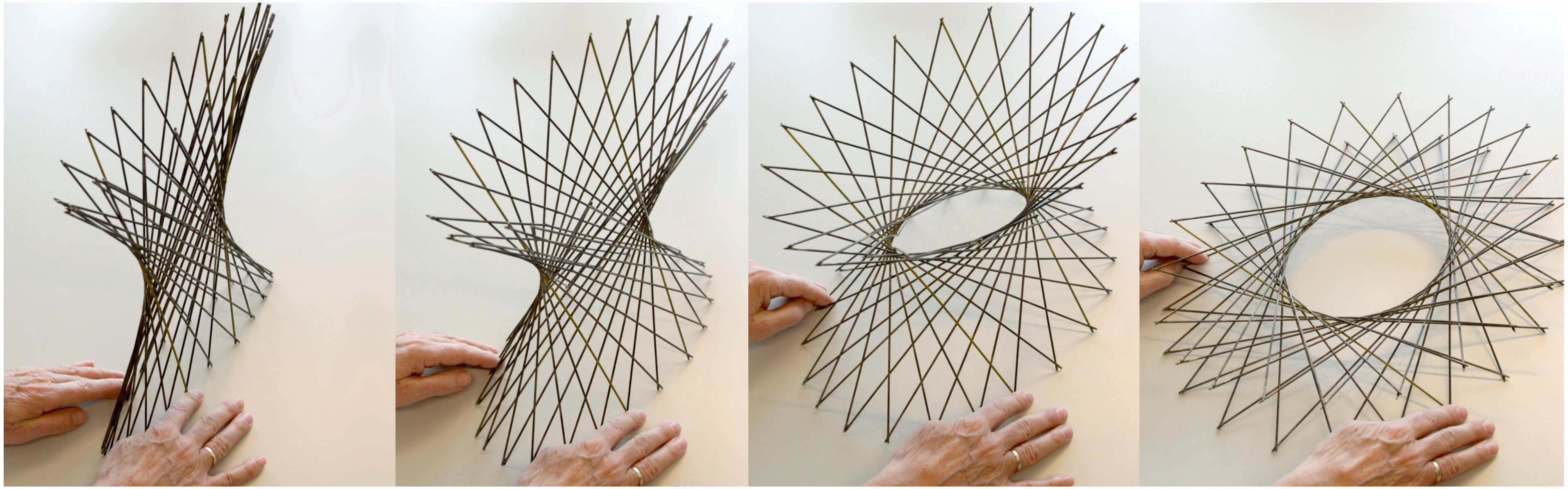
$$\overline{P'_8S^{(4)\prime}_6} = \overline{P''_8S^{(4)\prime\prime}_6}.$$

3. Isometric elliptic and hyperbolic billiards



The proof of the isometry is based on **Henrici's flexible hyperboloid** with its flat limiting poses.

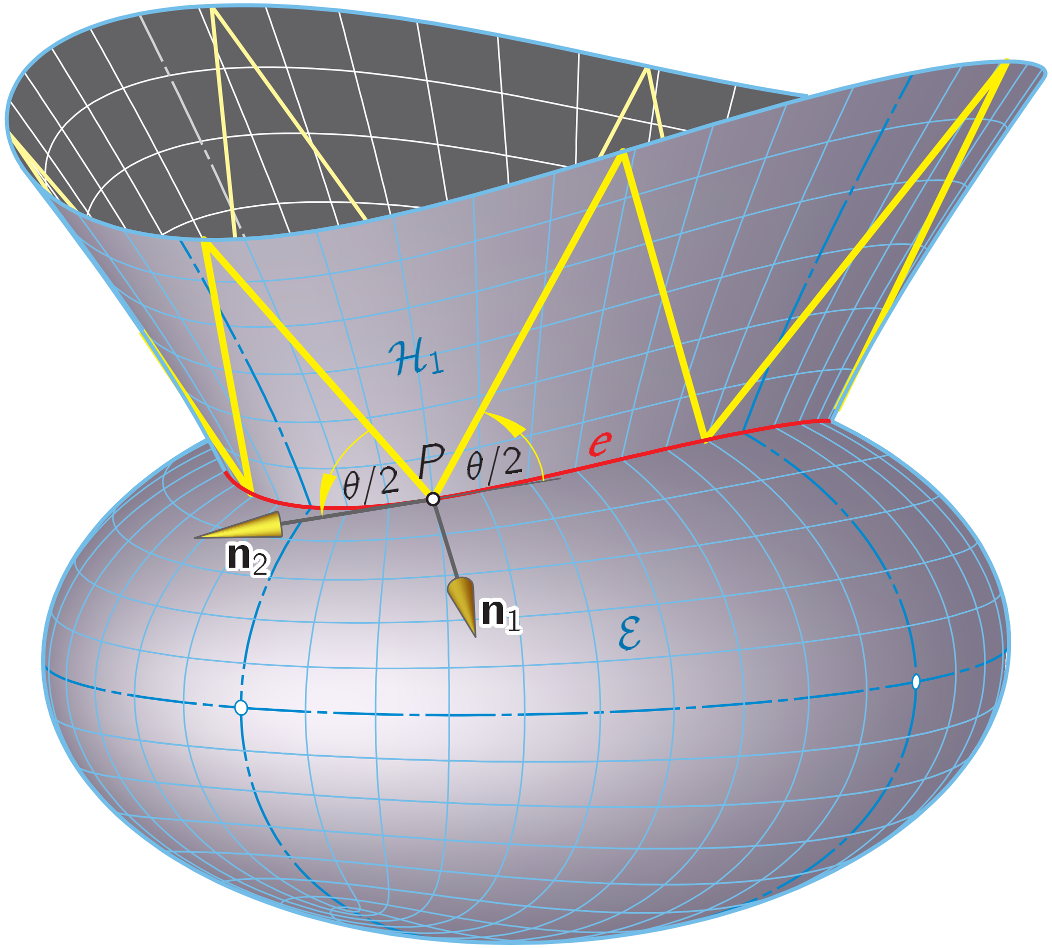
3. Isometric elliptic and hyperbolic billiards



If the axes of symmetry of the one-sheeted hyperboloids remain fixed, then the hyperboloid remains confocal.

The vertices trace orthogonal trajectories of the hyperboloids which are lines of curvature on the confocal ellipsoids and two-sheeted hyperboloids. Hence, each vertex remains on a confocal ellipsoid.

4. Focal billiards in ellipsoids



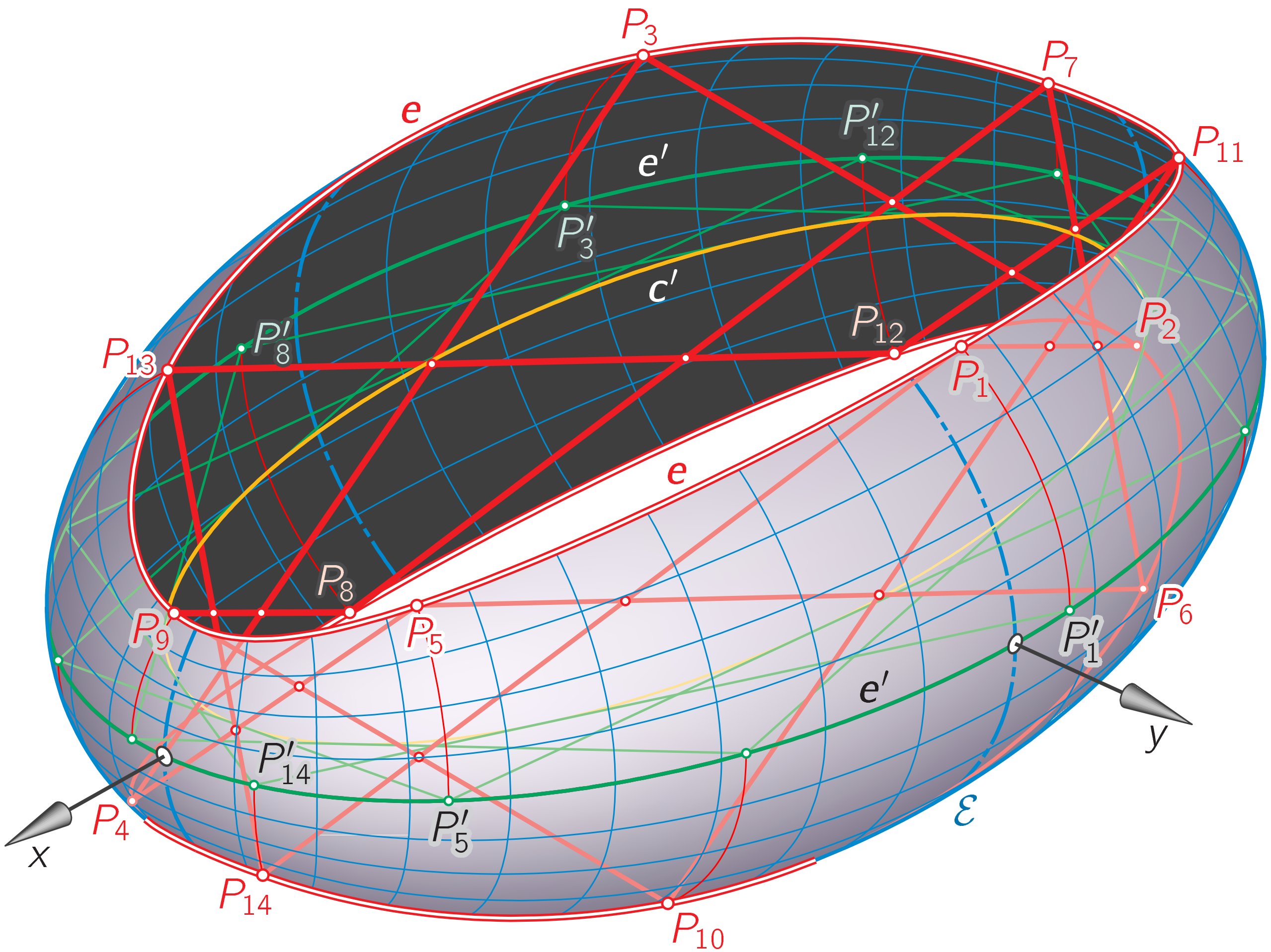
The generators of the one-sheeted hyperboloids in a confocal family are called **focal lines** of the confocal ellipsoids.

The reflection in an ellipsoid maps focal lines again on focal lines since they are asymptotic curves on the one-sheeted hyperboloid while the intersection curves with ellipsoid are lines of curvature.

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{k_0 - k_1}{k_1 - k_2}}$$

$k_0,\ k_1,\ k_2$ are elliptic coordinates of P (e.g., $k_0 = a_e^2 - a_c^2$).

4. Focal billiards in ellipsoids

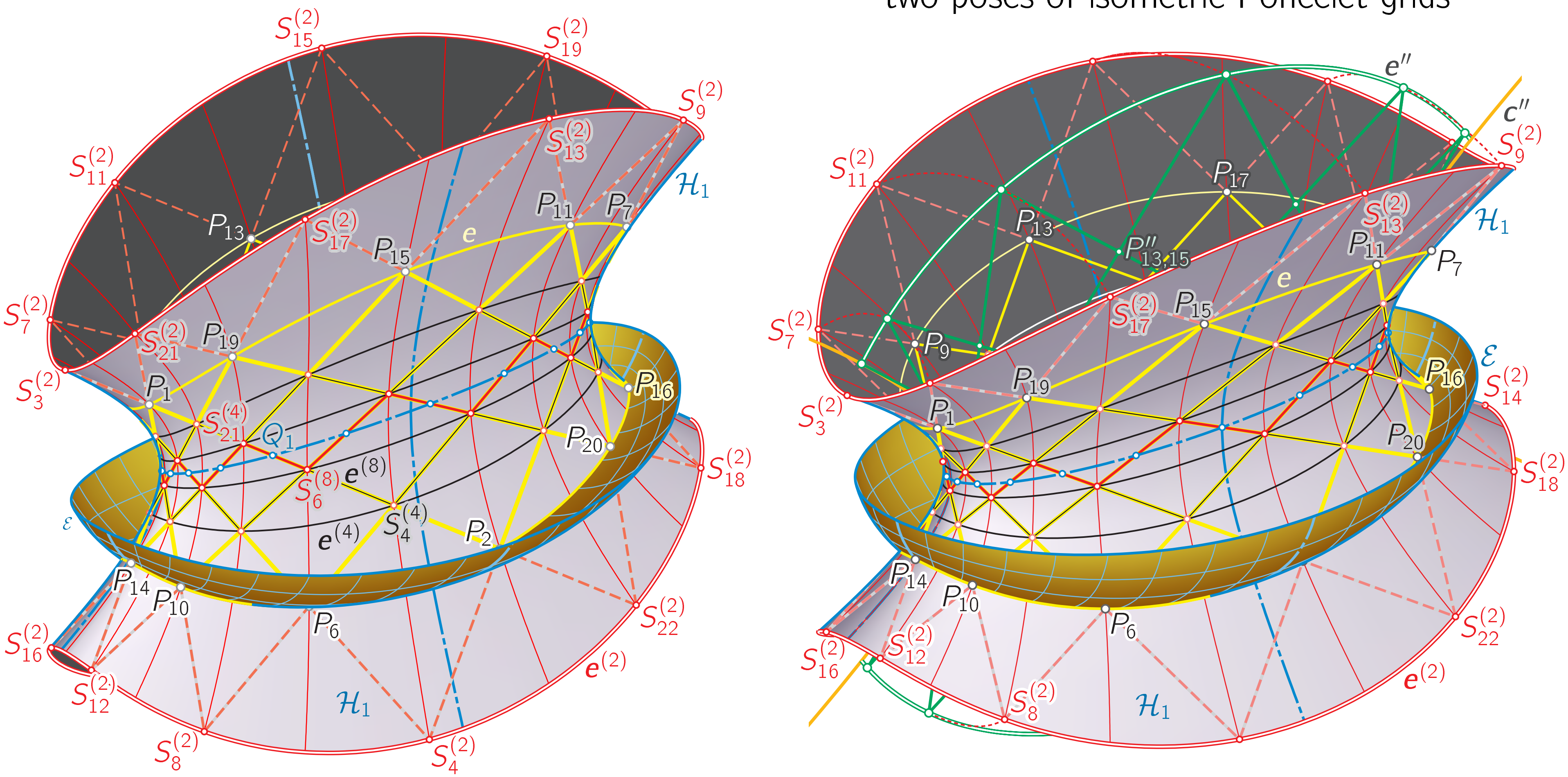


Let the generators on a Henrici hyperboloid end on a confocal ellipsoid $\mathcal{E}$. Then they remain on $\mathcal{E}$ during the flex.

At the flat limits the generators are tangents of the focal ellipse c' or hyperbola and end on principal sections of $\mathcal{E}$.

4. Focal billiards in ellipsoids

two poses of isometric Poncelet grids



4. Focal billiards in ellipsoids

Here are some invariants of N-periodic focal billiards in $\mathcal{E}$ w.r.t. billiard motions along curvature lines e:

Theorem: All focal billiards in $\mathcal{E}$ are N-periodic and have the same length $L_{\mathcal{E}}$.

Theorem: $\sum_{i=1}^{N} \cos\theta_i = N - \dfrac{k_0 - k_1}{a_{\mathcal{E}} b_{\mathcal{E}} c_{\mathcal{E}}} L_{\mathcal{E}}$ with $a_{\mathcal{E}}, b_{\mathcal{E}}, c_{\mathcal{E}}$ as semiaxes of $\mathcal{E}$.

Theorem: If N is even, then

$$\prod \overline{P_i Q_i} = \prod l_i = \prod r_i = \prod \overline{P_i Q_{i-1}} = k_0^{N/2} = c_{\mathcal{E}}^N.$$

Slatine Beach, Čiovo/Croatia

Thank you for your attention!

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