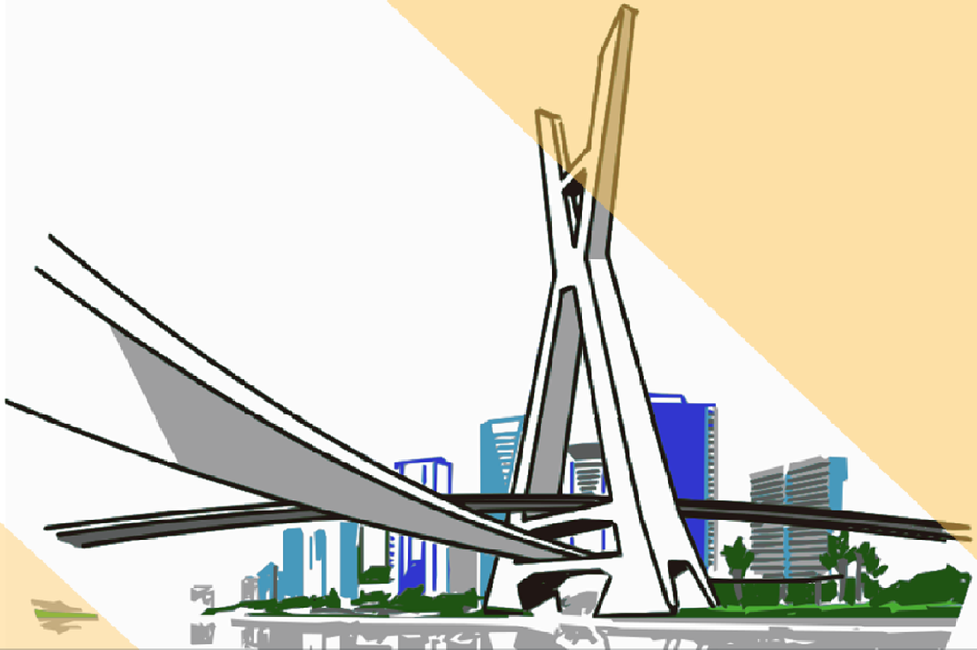
ICGG2022
Virtual, 15-19 AUG

THE 20TH INTERNATIONAL CONFERENCE
ON GEOMETRY AND GRAPHICS



On the Diagonals of Billiards

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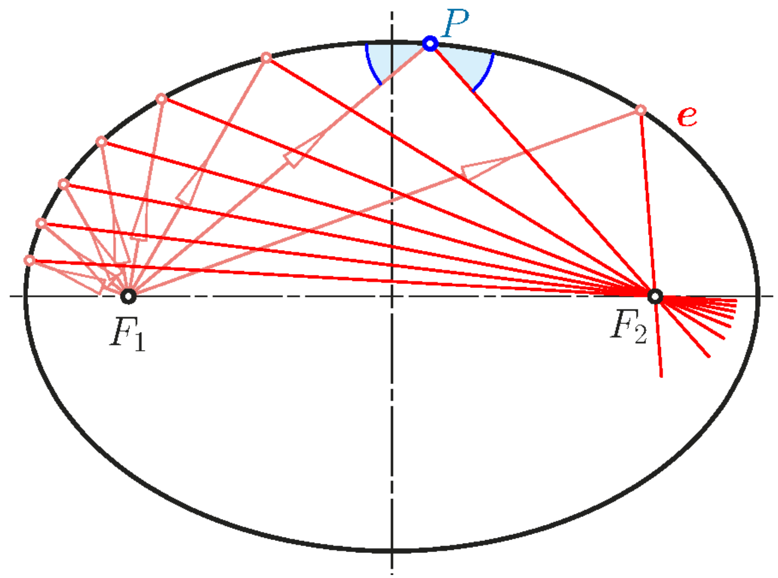
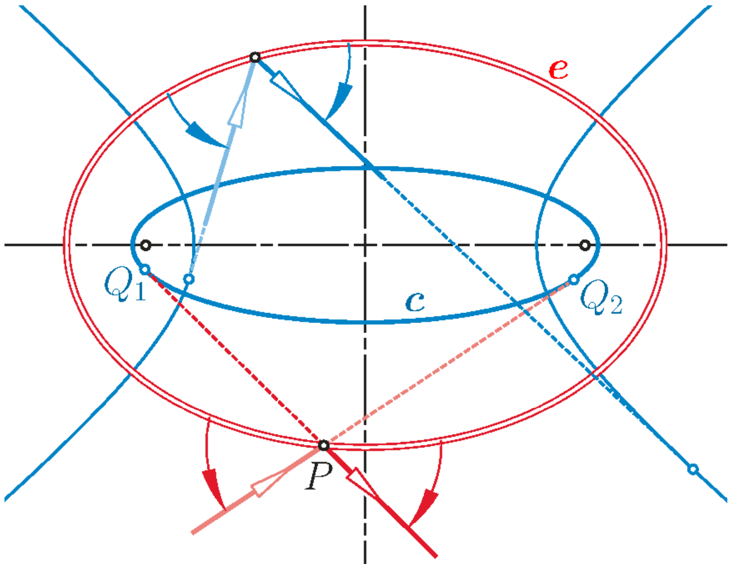
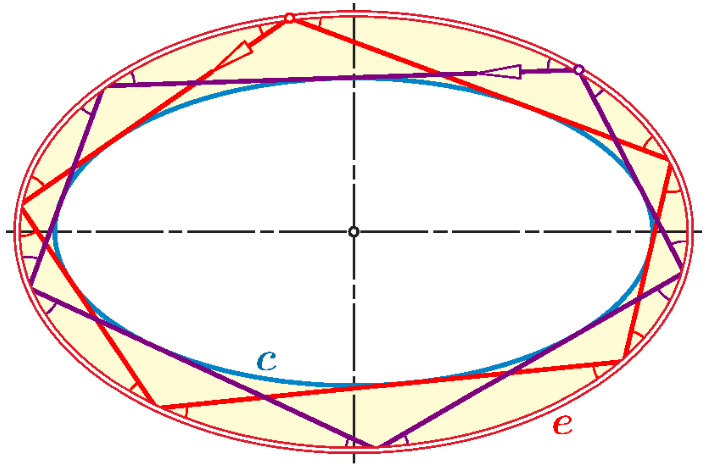
A ***billiard*** is the *trajectory of a mass point* within a domain with ideal physical reflections in the boundary which in our case an ellipse *e*.



The optical property of ellipses is well known. We generalize:

1. Billiards in ellipses and billiard motion

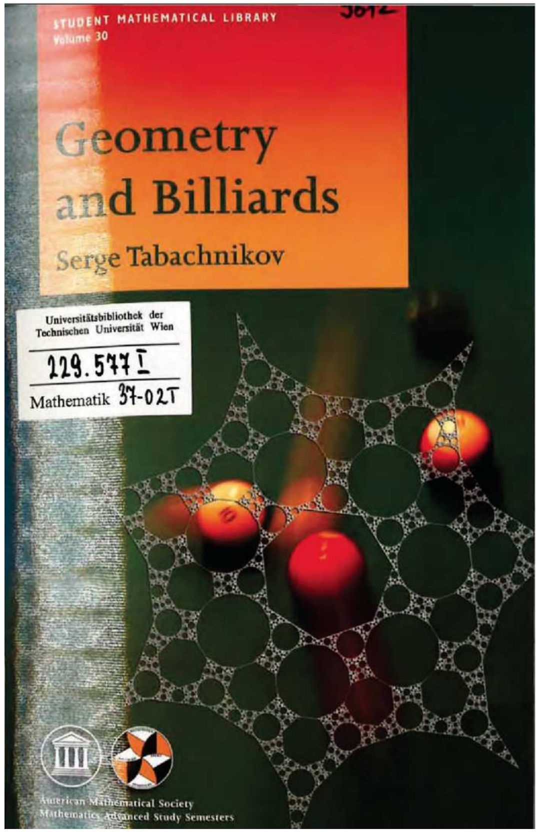




After reflection in a conic *e* the incoming and outgoing ray are tangent to the same confocal conic *c*, called **caustic** (ellipse or hyperbola).

If one billiard closes after *N* reflections, then all billiards in *e* with caustic *c* close (**Poncelet porism**). The continuous variation of the billiard in *e* is called **billiard motion**.

1. Billiards in ellipses and billiard motion

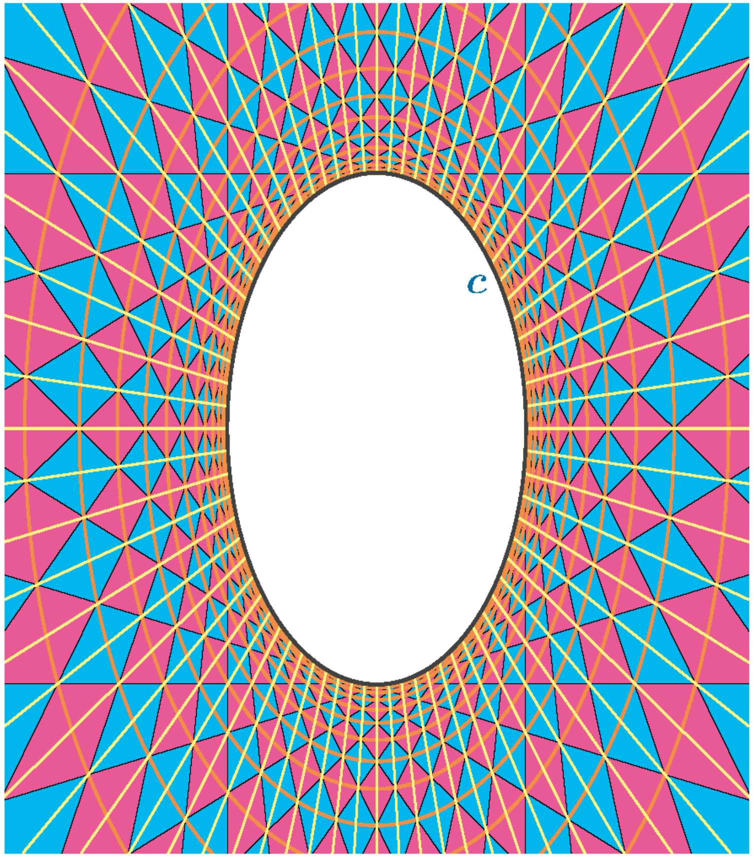


For centuries, billiards (and projectively equivalent polygons with an inconic and circumconic) attracted the attention of mathematicians, beginning with J.-V. Poncelet, C.G.J. Jacobi, M. Chasles, A. Cayley, and G. Darboux.

S. Tabachnikov: *Geometry and Billiards*. American Mathematical Society, 2005

In 2020, Dan Reznik (Brazil) revitalized the interest by computer animations showing the motion of periodic billiards. He identified 50 invariants, e.g., a constant sum of Cosines of interior angles.

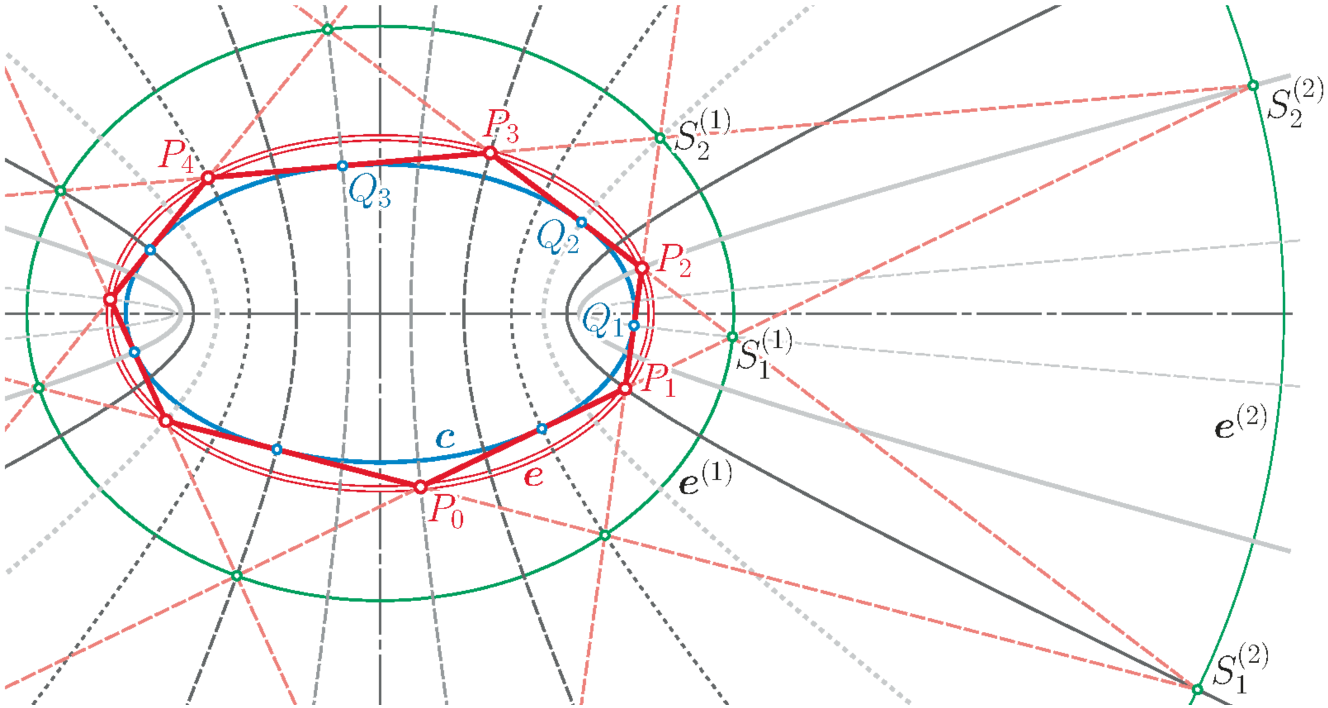
1. Billiards in ellipses and billiard motion



The extended sides of a billiard intersect at points of confocal ellipses and hyperbolas and form the associated **Poncelet grid**.

Left:
affinely transformed 72-sided periodic billiard with associated Poncelet grid (G. Glaeser, B. Odehnal, H.S.: *The Universe of Conics*)

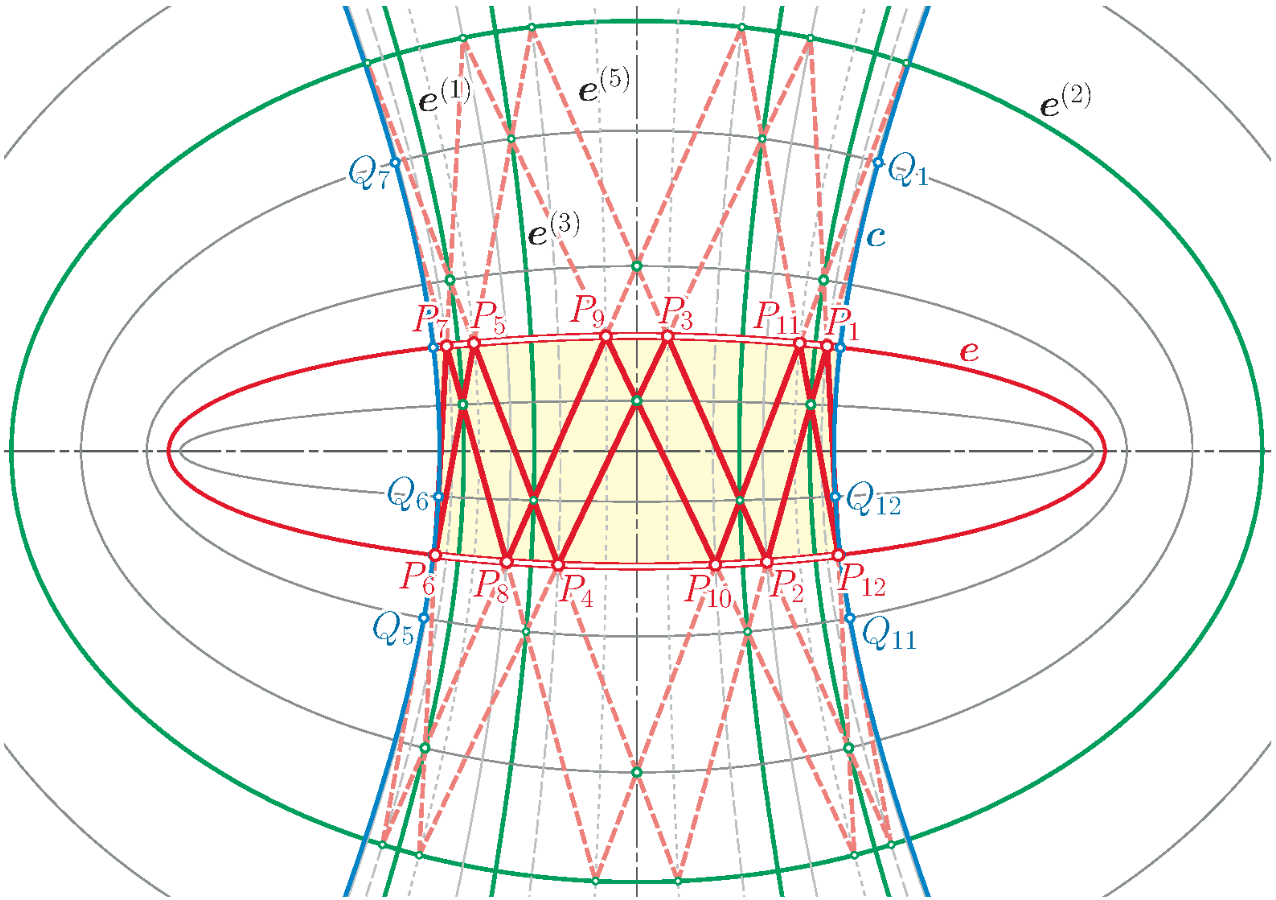
1. Billiards in ellipses and billiard motion



$$S_2^{(1)} := [P_1, P_2] \cap [P_3, P_4] \in e^{(1)}, \; S_2^{(2)} := [P_0, P_1] \cap [P_3, P_4] \in e^{(2)}$$

The ellipses $e^1, e^2, \ldots$ of the Poncelet grid are motion invariant.

1. Billiards in ellipses and billiard motion



If the caustic *c* is a hyperbola, then we obtain a zig-zag billiard (red).

Left: 12-periodic billiard with hyperbola as caustic

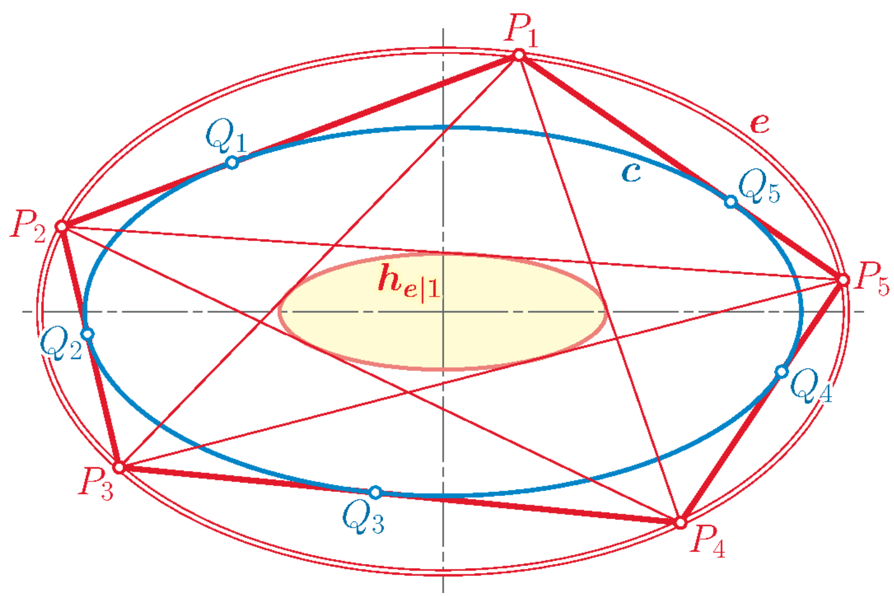
2. Diagonals of billiards in ellipses

Extension of a result of Poncelet (1822) and Jacobi (1828):

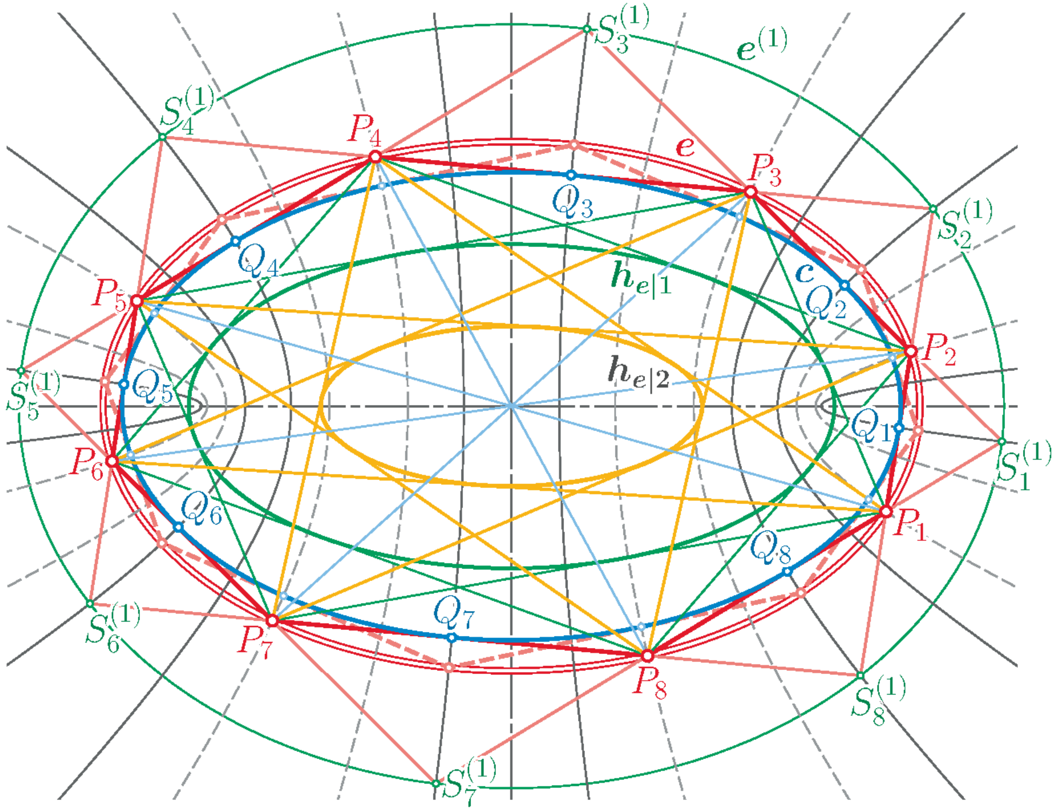
Theorem.
The envelope of the j-th diagonals $P_i P_{i+j+1}$ is a coaxial ellipse $h_{e|j}$ with the semiaxes

$$a_j = \frac{a_e a_c}{a_{e|j}}, \quad b_j = \frac{b_e b_c}{b_{e|j}}.$$

The ellipses $h_{e|1}, h_{e|2}, \ldots$ belong to the pencil spanned by *c* and ***e***.



2. Diagonals of billiards in ellipses

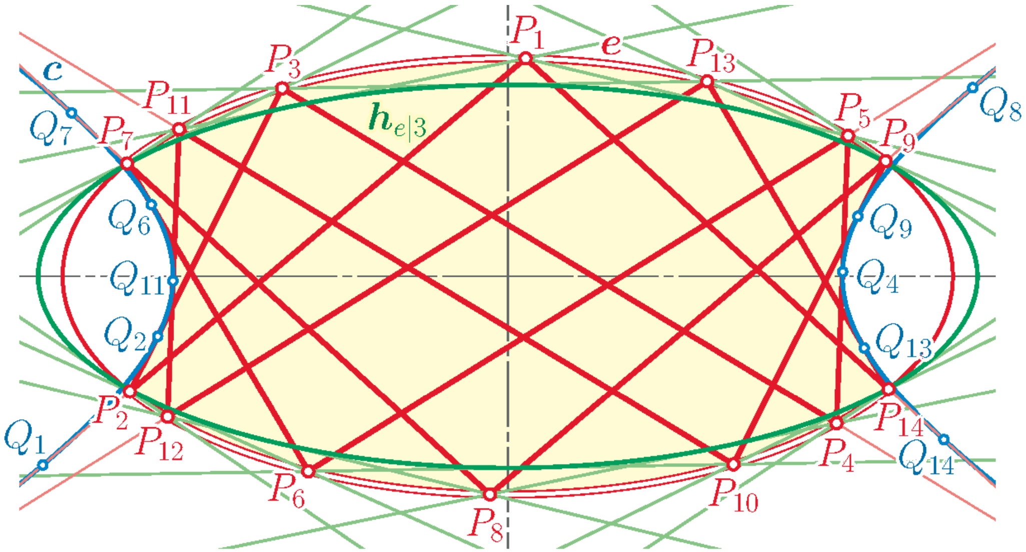


Left: Envelopes $h_{e|1}$, $h_{e|2}$ of diagonals of the billiard $P_1\, P_2$ …

Proof: The polar line of $S_i^{(j)}$ w.r.t. c is a j-th diagonal of the polygon $Q_1\, Q_2$ … of contact points.

The affine scaling with $c \rightarrow e$ takes these diagonals to diagonals of $P_1\, P_2$ … .

2. Diagonals of billiards in ellipses



The same formulas hold for the semiaxes a_j, b_j of the envelope $h_{e|j}$ when the caustic is a hyperbola. For odd j the envelope $h_{e|j}$ is an ellipse, otherwise a hyperbola.

The proof must be modified since there is no affine scaling with $c \to e$.

2. Diagonals of billiards in ellipses

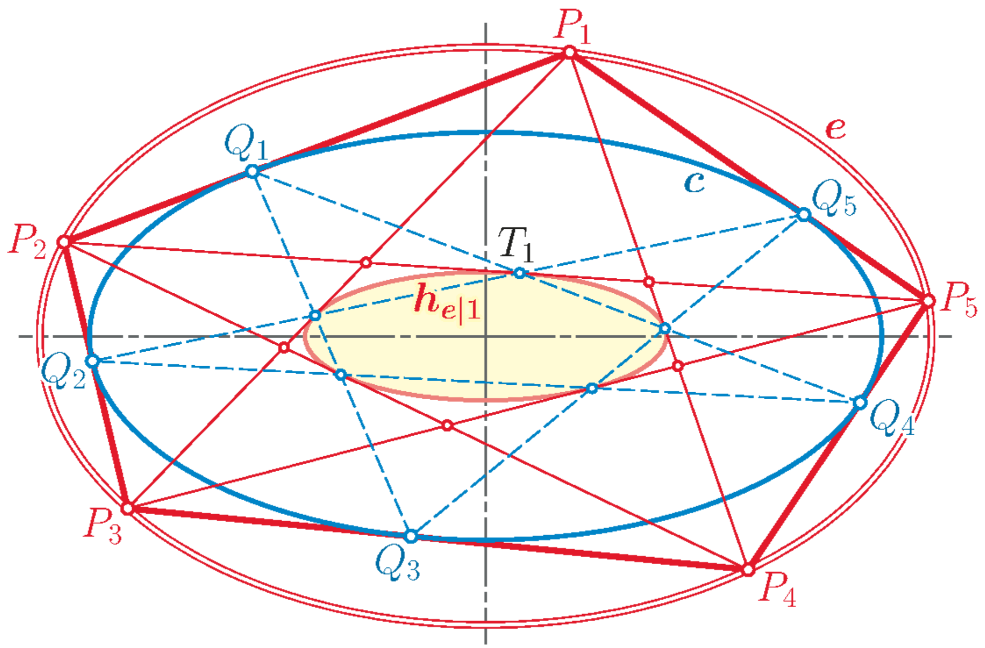
Right: Construction of contact points of h_{e|j}.

Theorem. The j-th diagonal $P_i P_{i+j+1}$ contacts the envelope $h_{e|j}$ at the intersection

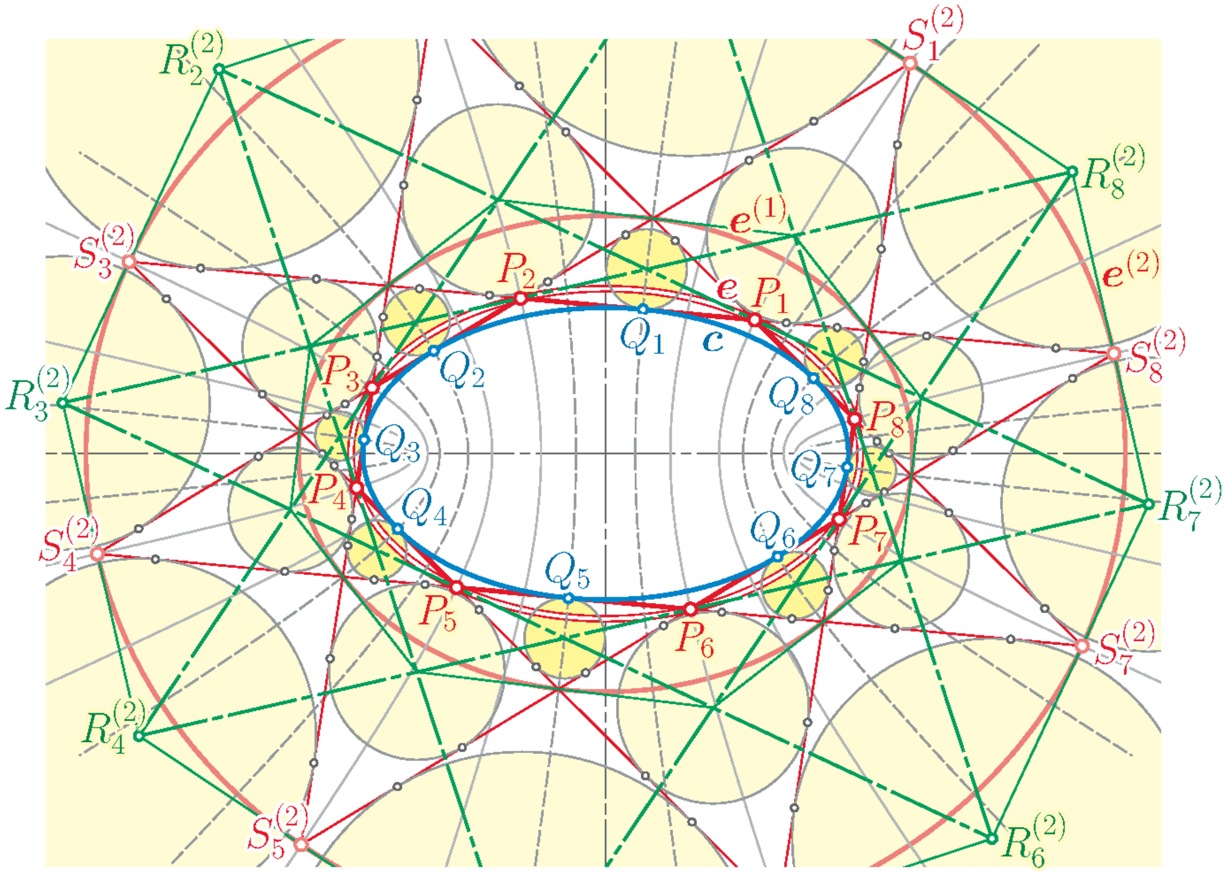
$$[Q_{i-1}, Q_{i+j}] \cap [Q_i, Q_{i+j+1}]$$

of neighbouring j-th diagonals of the polygon $Q_1\, Q_2\, Q_3 \ldots$.

Proof: The affine scaling $e^{(j)} \to c$ takes *e* to the envelope $h_{e|j}$.



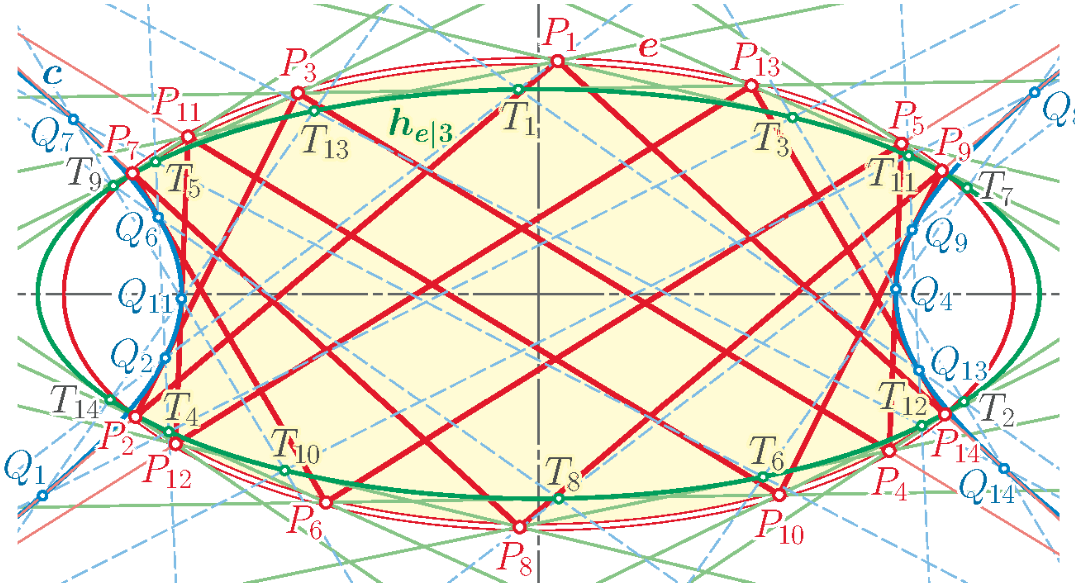
2. Diagonals of billiards in ellipses



Proof: The affine scaling $e^{(j)} \rightarrow c$ takes e to the envelope $h_{e|j}$.

The tangents to $e^{(j)}$ pass through the intersection points $R_i^{(j)}$ of the tangents to *e* at the vertices *P* .

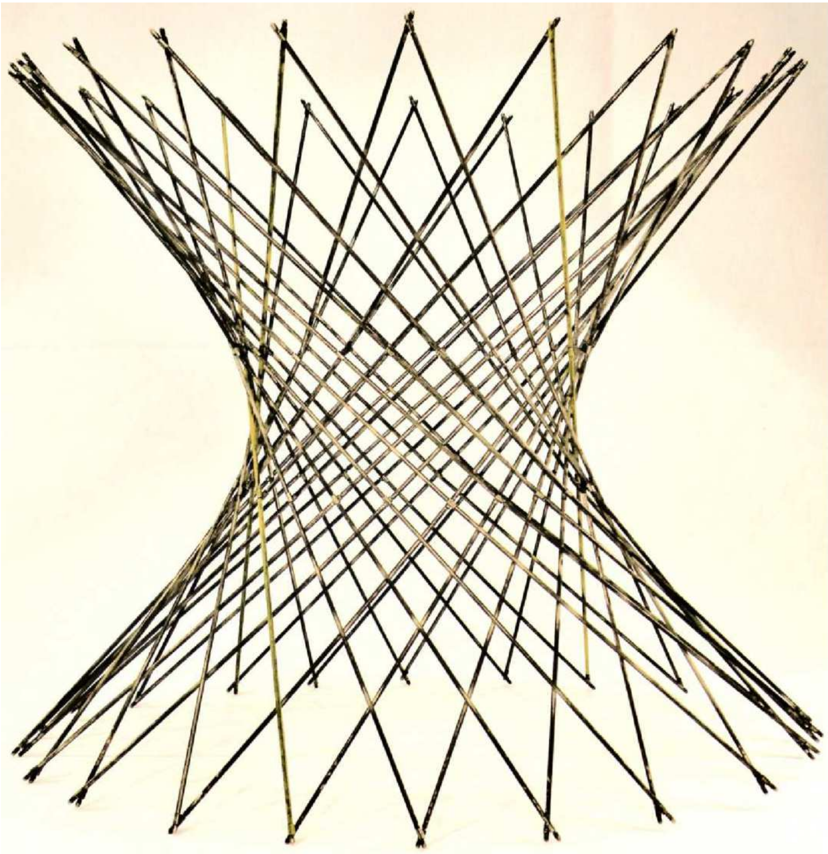
2. Diagonals of billiards in ellipses



The construction of the contact points of j-th diagonals with the envelope $h_{e|j}$ is also valid when the caustic is a hyperbola.

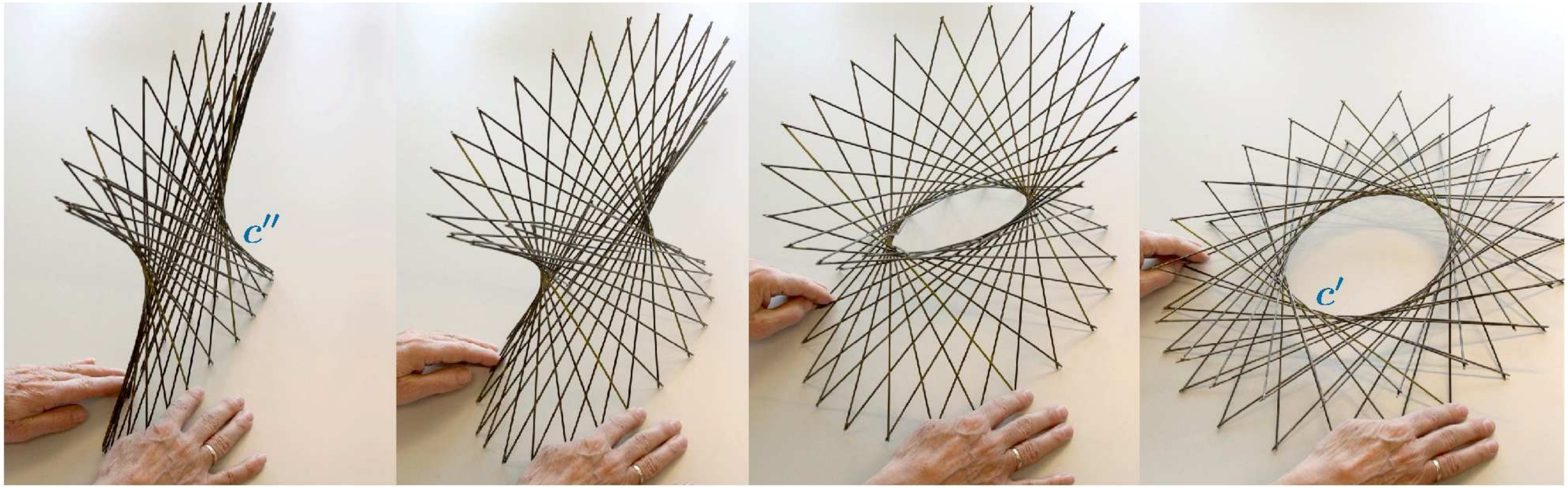
However, the proof must be modified.

3. Diagonals of focal billiards



Henrici's flexible hyperboloid brings about a continuous transition from a billiard with an ellipse *c'* a caustic via spatial **focal billiards** to a billiard with a hyperbola *c''* as caustic.

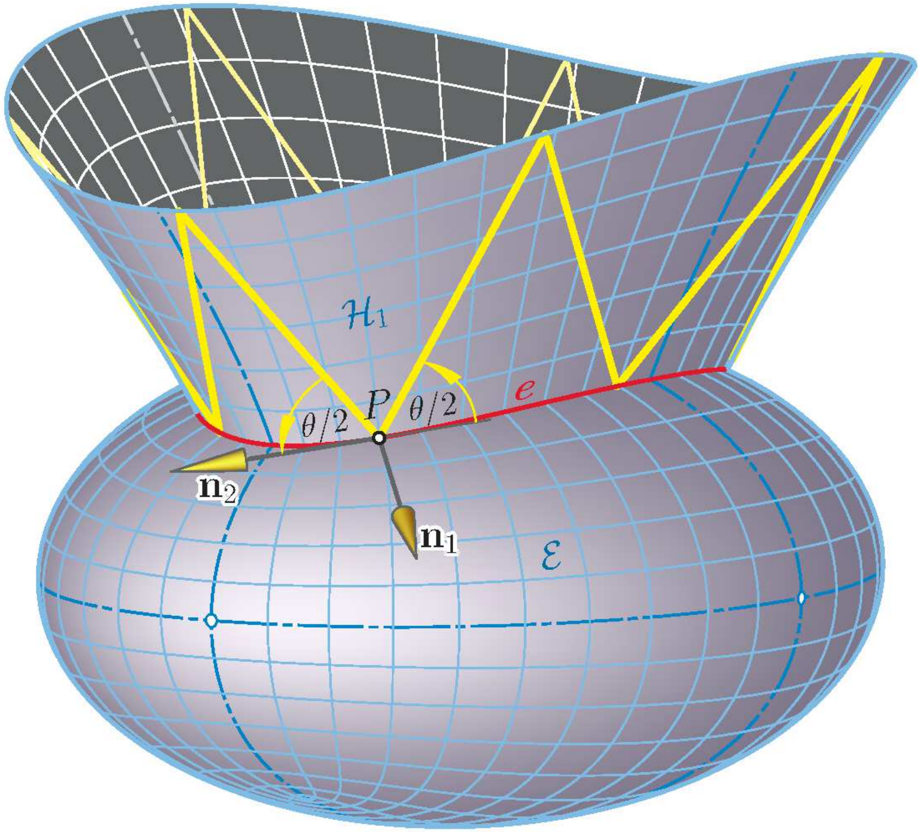
3. Diagonals of focal billiards



If the axes of symmetry of the hyperboloids are fixed, then the varying hyperboloids remain confocal. All confocal ellipsoids remain fixed.

The flexion is terminated by flat poses where generators are tangents of the focal ellipse *c*' (right) or the focal hyperbola *c*" (left).

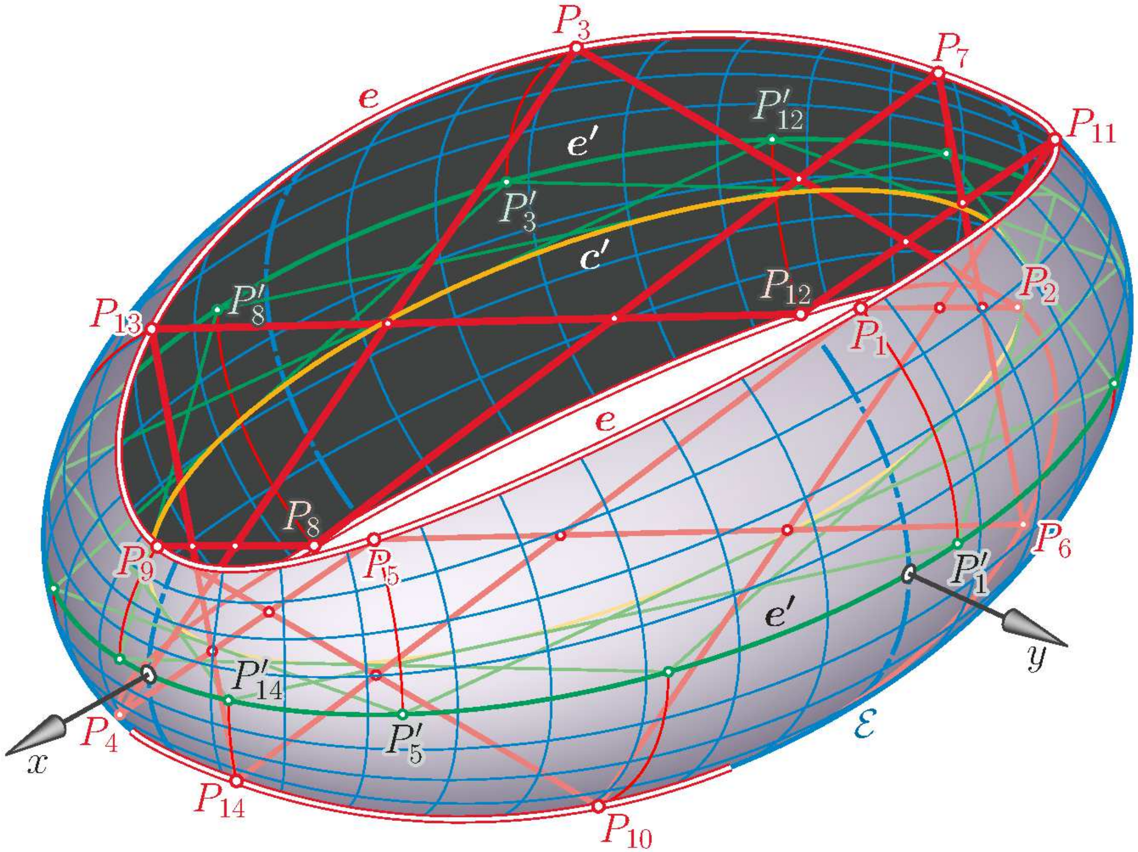
3. Diagonals of focal billiards



The generators of the one-sheeted hyperboloids in a confocal family are called **focal lines** of the confocal ellipsoids.

The reflection in an ellipsoid maps focal lines again on focal lines since they are asymptotic curves on the one-sheeted hyperboloid while the intersection curves with ellipsoid are lines of curvature.

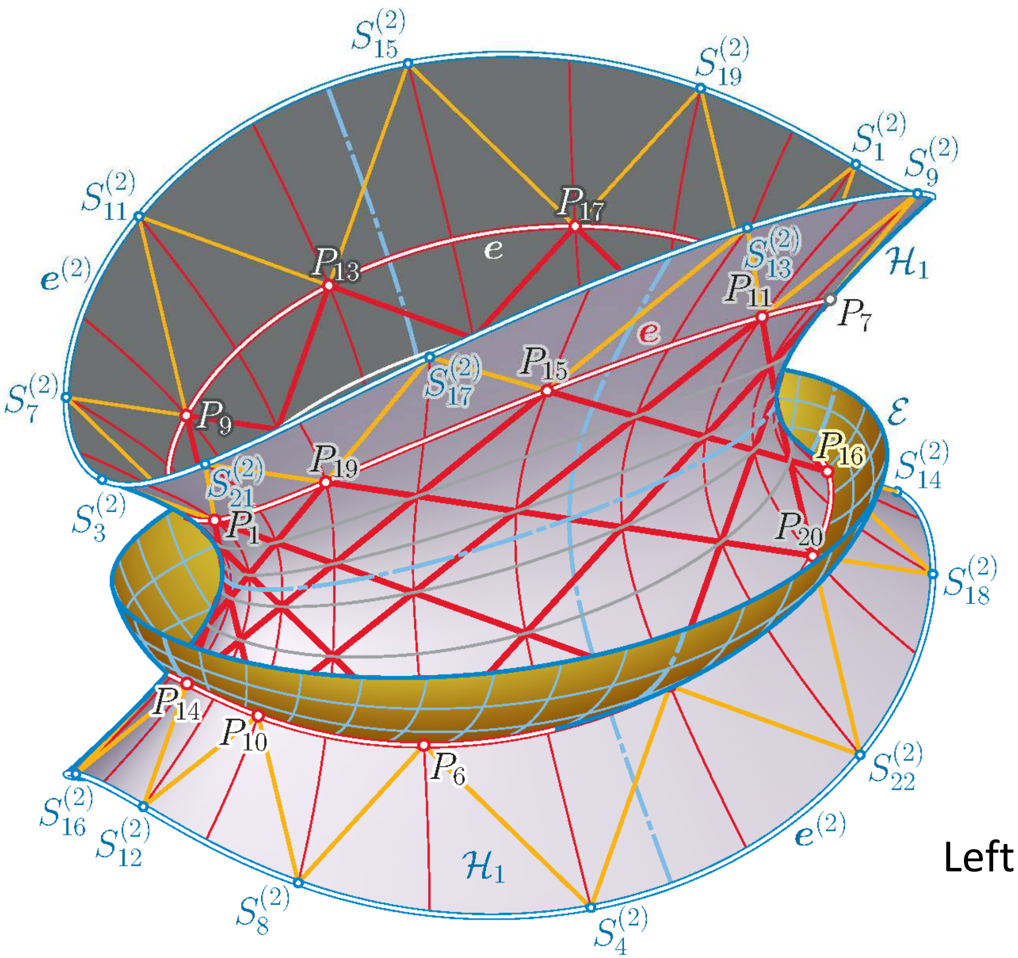
3. Diagonals of focal billiards



If the generators (red) of a Henrici hyperboloid end on a confocal ellipsoid $\mathcal{E}$, then they remain on $\mathcal{E}$ during the flexion.

At one flat limit the generators (green) contact the focal ellipse *c*' and end on principal section e' of $\mathcal{E}$.

3. Diagonals of focal billiards



Theorem. For even j, the j-th diagonals P_iP_{i+j+1} of a focal billiard are generators of one-sheeted hyperboloid, which belongs to the pencil spanned by $\mathcal{E}$ and $\mathcal{H}_1$.

Proof: We extend the sides of the focal billiard to the associated spatial Poncelet grid on $\mathcal{H}_1$ and apply the affine scaling $e^{(j)} \to e$.

Left: Case with N = 22 and j = 2.

3. Diagonals of focal billiards

Analytic proof:

There is a canonical parametrization of focal billiards in terms of **Jacobian elliptic functions** sn *u*, cn *u*, dn *u* to the modulus d/a :

$$\mathbf{e}_{1,2}(u) = \left(-\frac{a_e a_{h_1}}{a'_c} \operatorname{sn} u,\ \frac{b_e b_{h_1}}{b'_c} \operatorname{cn} u,\ \pm \frac{c_e c_{h_1}}{b'_c} \operatorname{dn} u \right)$$

Then, the last theorem is equivalent to the identity

$$\operatorname{dn}^2 \frac{u_j - u}{2} \operatorname{sn} u \operatorname{sn} u_j + \operatorname{cn} u \operatorname{cn} u_j + \operatorname{sn}^2 \frac{u_j - u}{2} \operatorname{dn} u \operatorname{dn} u_j = \operatorname{cn}^2 \frac{u_j - u}{2} .$$

Schönbrunn Castle, Vienna

Thank you for your attention !

References

Akopyan, R. Schwartz, S. Tabachnikov: *Billiards in ellipses revisited.* Eur. J. Math. 2020. doi:10.1007/s40879-020-00426-9

G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics*. Springer Spectrum, Berlin, Heidelberg 2016

C.G.J. Jacobi: *Ueber die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie.* Crelle's Journal **3**/4 (1828), 376-389

B. Odehnal, H. Stachel, G. Glaeser: *The Universe of Quadrics.* Springer-Verlag GmbH Germany, Berlin, Heidelberg 2020

J.V. Poncelet: *Traité des Propriétés Projectives des Figures.* Bachelier, Paris 1822

D. Reznik, R. Garcia, J. Koiller: *Fifty New Invariants of N-Periodics in the Elliptic Billiard*. Arnold Math. J. **7**/2 (2021), doi: 10.1007/s40598-021-00174-y

References

H. Stachel: *The Geometry of Billiards in Ellipses and their Poncelet Grids.* J. Geom. **112**:40 (2021), doi: {10.1007/s00022-021-00606-2

H. Stachel: *On the Motion of Billiards in Ellipses*. Europ. J. Math. **8**/2 (1922), doi: 10.1007/s40879-021-00524-2

H. Stachel: *Isometric Billiards in Ellipses and Focal Billiards in Ellipsoids.* J. Geometry Graphics **25**/1, 97-118 (2021)

S. Tabachnikov: *Geometry and Billiards*. American Mathematical Society, Providence/Rhode Island 2005

H. Wieners und P. Teutleins Sammlungen mathematischer Modelle. 2. Ausgabe, B.G. Teubner, Leipzig, Berlin 1912.