Canal surfaces containing four straight lines

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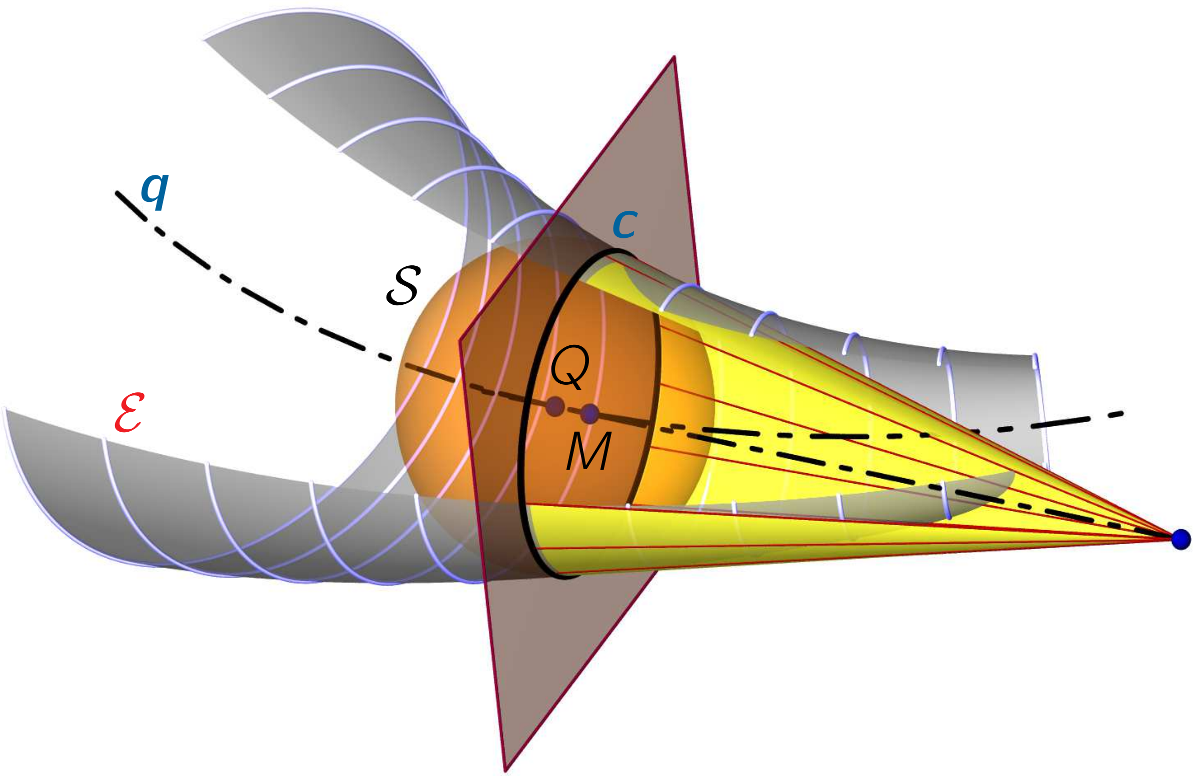
1. Canal surfaces in general

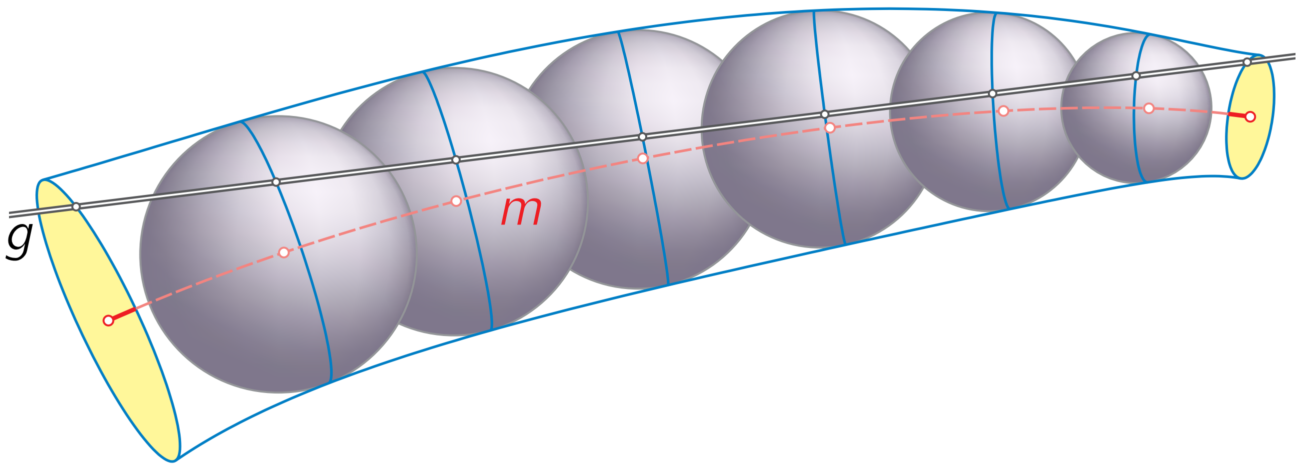


A **canal surface** (**channel surface**) $\mathcal{E}$ is the envelope of a one-parameter family of spheres $\mathcal{S}$ with (in general) variable radius.

The path of the spheres' centers Q is called **spine curve** q.

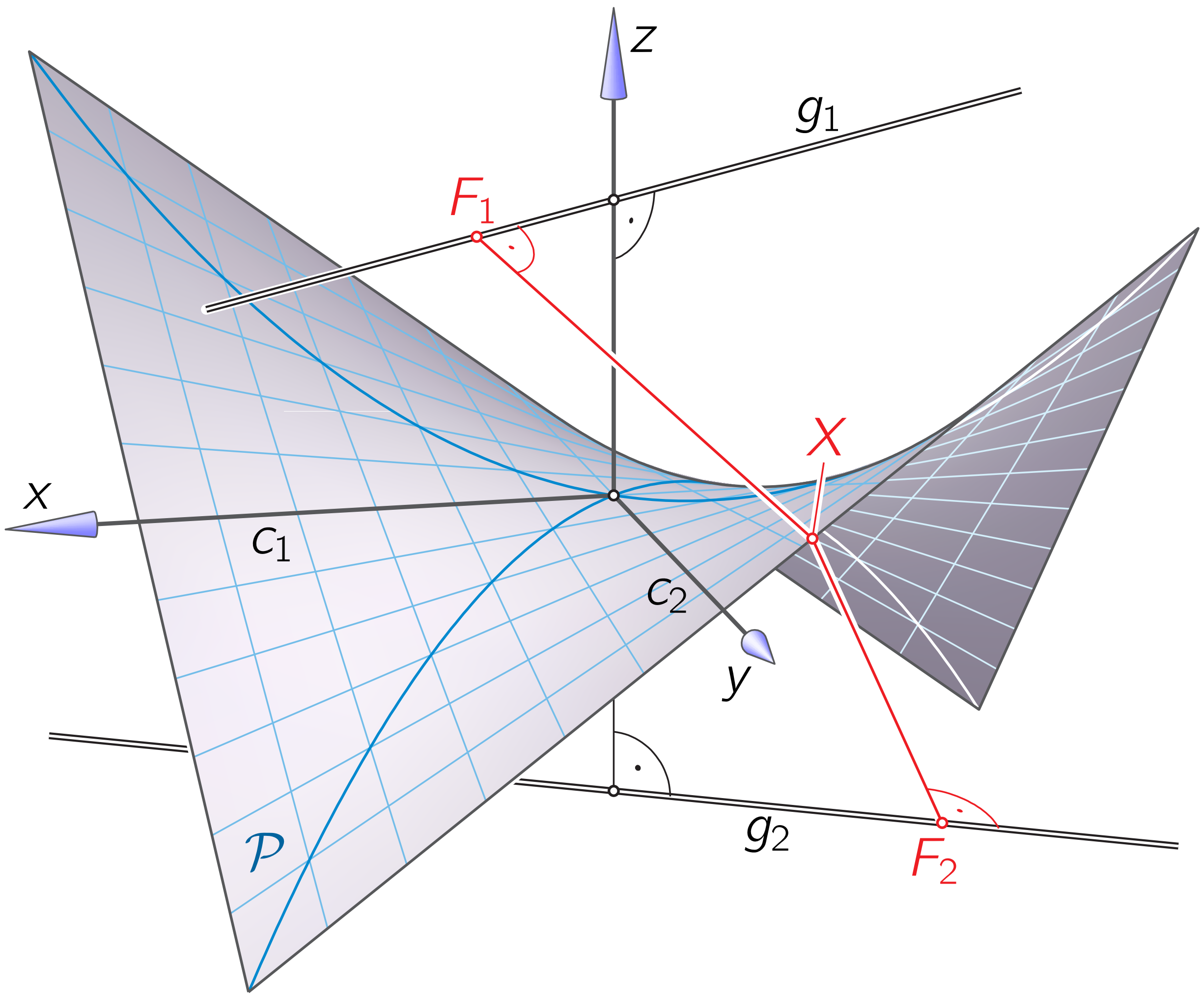
Each sphere $\mathcal{S}$ contacts $\mathcal{E}$ along a circle c called **characteristic** with center M (in general $\neq Q$). The tangent planes to $\mathcal{E}$ along c envelop a cone of revolution.

1. Canal surfaces in general



1. If all spheres of a canal surface contact a line g, then the points of contact belong to the envelope $\mathcal{E}$, i.e., $g \subset \mathcal{E}$.
2. If $Q \in q$ is the center of an enveloping sphere $\mathcal{S}$, then the pedal point of Q on g belongs to the characteristic c of $\mathcal{S}$.
3. If $\mathcal{E}$ contains **four** lines $g_1, \ldots, g_4$, then for each $Q \in q$ the four pedal points must be **coplanar** and **concircular**.

1. Canal surfaces in general



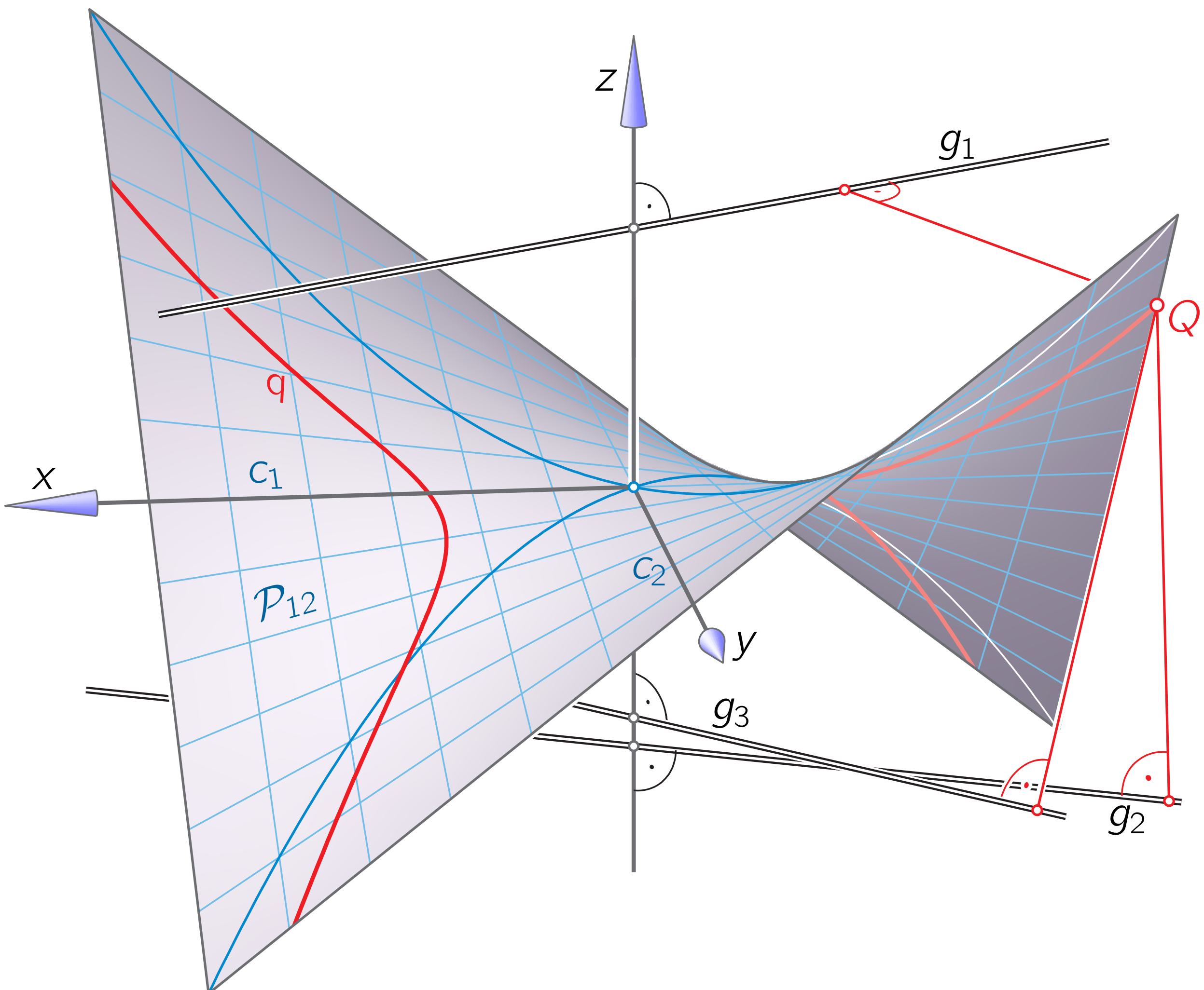
Given two skew lines g_1, g_2, the **bisector**, i.e., $\{X \mid \overline{Xg_1} = \overline{Xg_2}\}$, is an orthogonal parabolic hyperboloid $\mathcal{P}$.

$2d := \overline{g_1g_2}$, $2\varphi := \sphericalangle g_1g_2$

$$\mathcal{P}: \ z + \frac{\sin 2\varphi}{2d}\, xy = 0\,.$$

The axes of symmetry c_1, c_2 of g_1 and g_2 are the vertex generators of $\mathcal{P}$.

1. Canal surfaces in general



If all spheres contact **two** lines g_1, g_2, then the spine curve q lies on the bisector, for skew lines an orthogonal hyperbolic paraboloid $\mathcal{P}_{12}$, otherwise the two planes of symmetry.

If all spheres contact **three** lines g_1, g_2, g_3, then q is the **quartic** $\mathcal{P}_{12} \cap \mathcal{P}_{13}$, a pair of parabolas or a line,

Can a fourth line be tangent to infinitely many spheres?

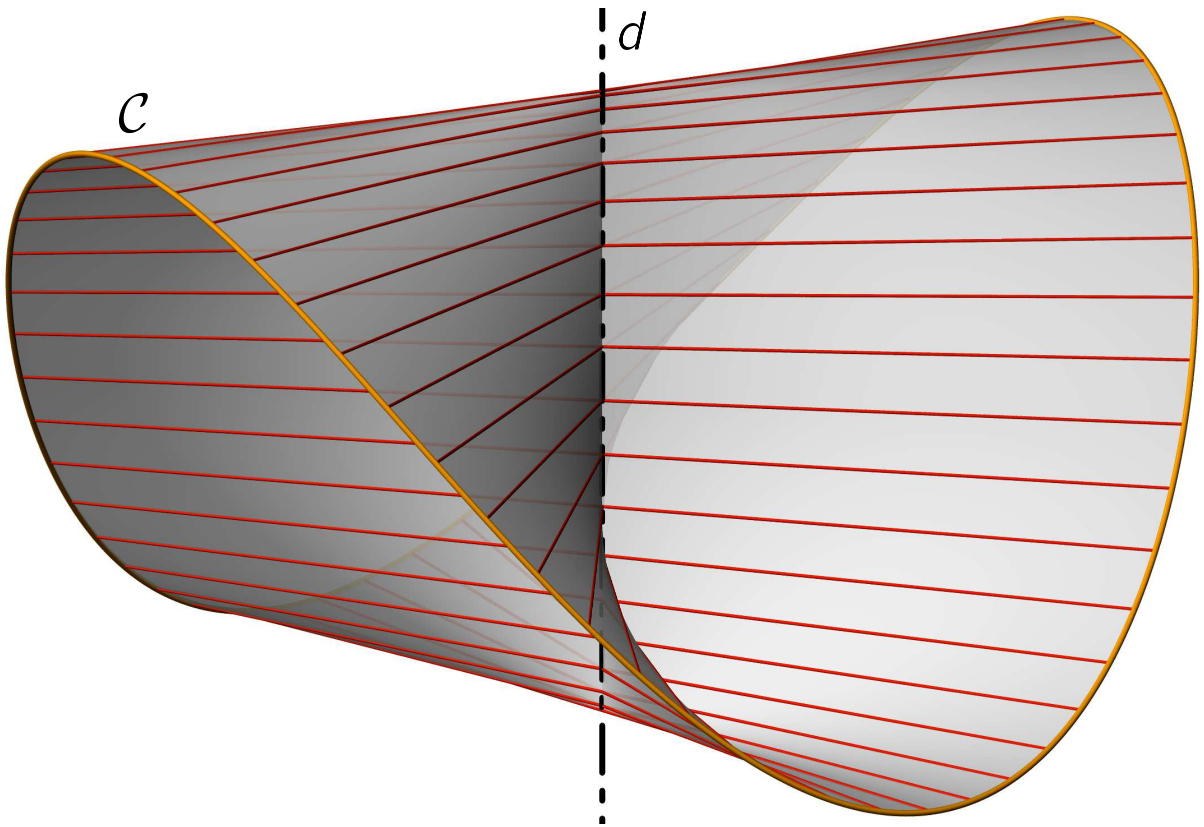
1. Canal surfaces in general

Theorem. (H.S. 1995)
There are only two cases where **four** mutually skew lines $g_1, \ldots, g_4$ have a continuum of contacting spheres: The given lines either belong to a hyperboloid of revolution or they are **concyclic**, i.e., they belong to a Plücker conoid $\mathcal{C}$ and their points of intersection with a tangent plane to $\mathcal{C}$ are located on a circle.

In the latter case, the six bisecting hyperbolic paraboloids $\mathcal{P}_{ij}$ of the pairs (g_i, g_j), $i, j \in \{1, \ldots, 4\}$, $i \neq j$, belong to a pencil. In the first case, the paraboloids share the hyperboloid's axis.

Corollary.
If all spheres of an irreducible algebraic canal surface contact a finite number of lines, then this number is less or equal to **four**.

2. Plücker's conoid and concyclic lines



Plücker's conoid (or **Cylindroid**) $\mathcal{C}$ is a ruled surface of degree three with a finite double line d.

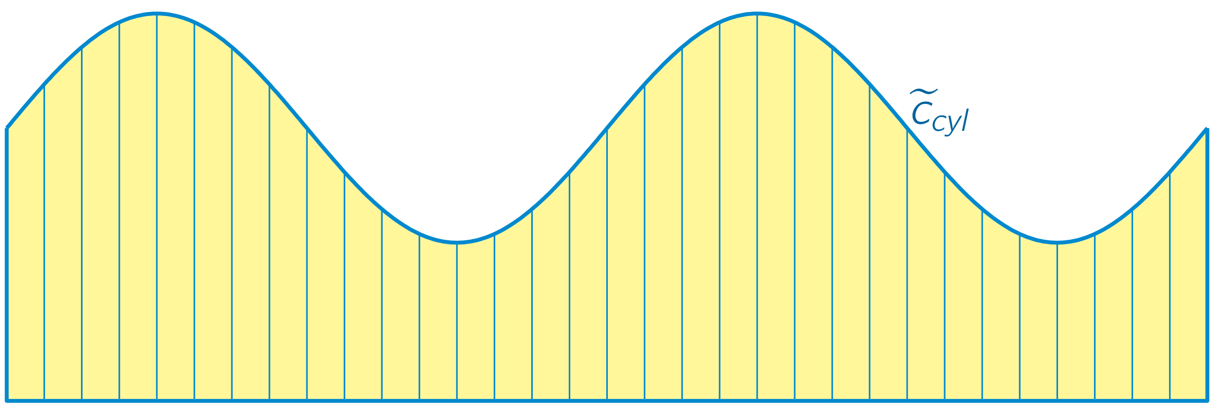
In cylinder coordinates (r, φ, z) the conoid satisfies

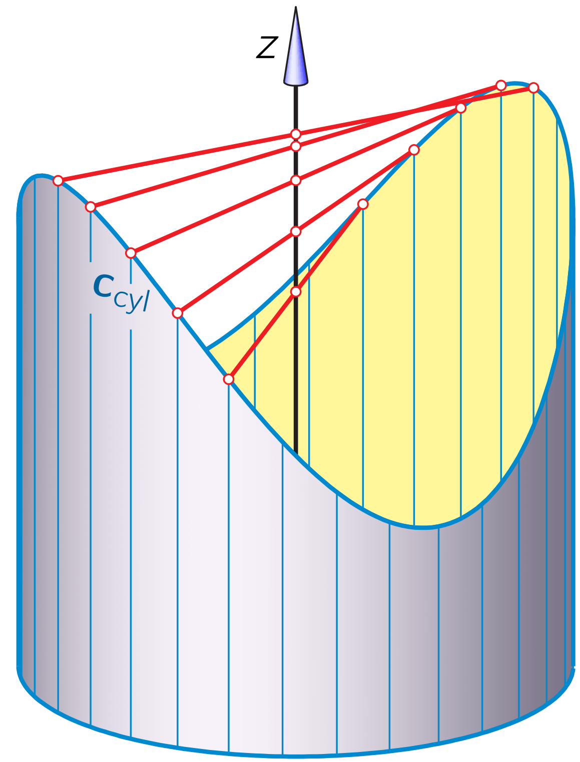
$$z = c \cdot \sin 2\varphi$$

with the width $c = \text{const.}$

Cartesian equation: $(x^2 + y^2)\, z - 2c\, xy = 0.$

2. Plücker's conoid and concyclic lines

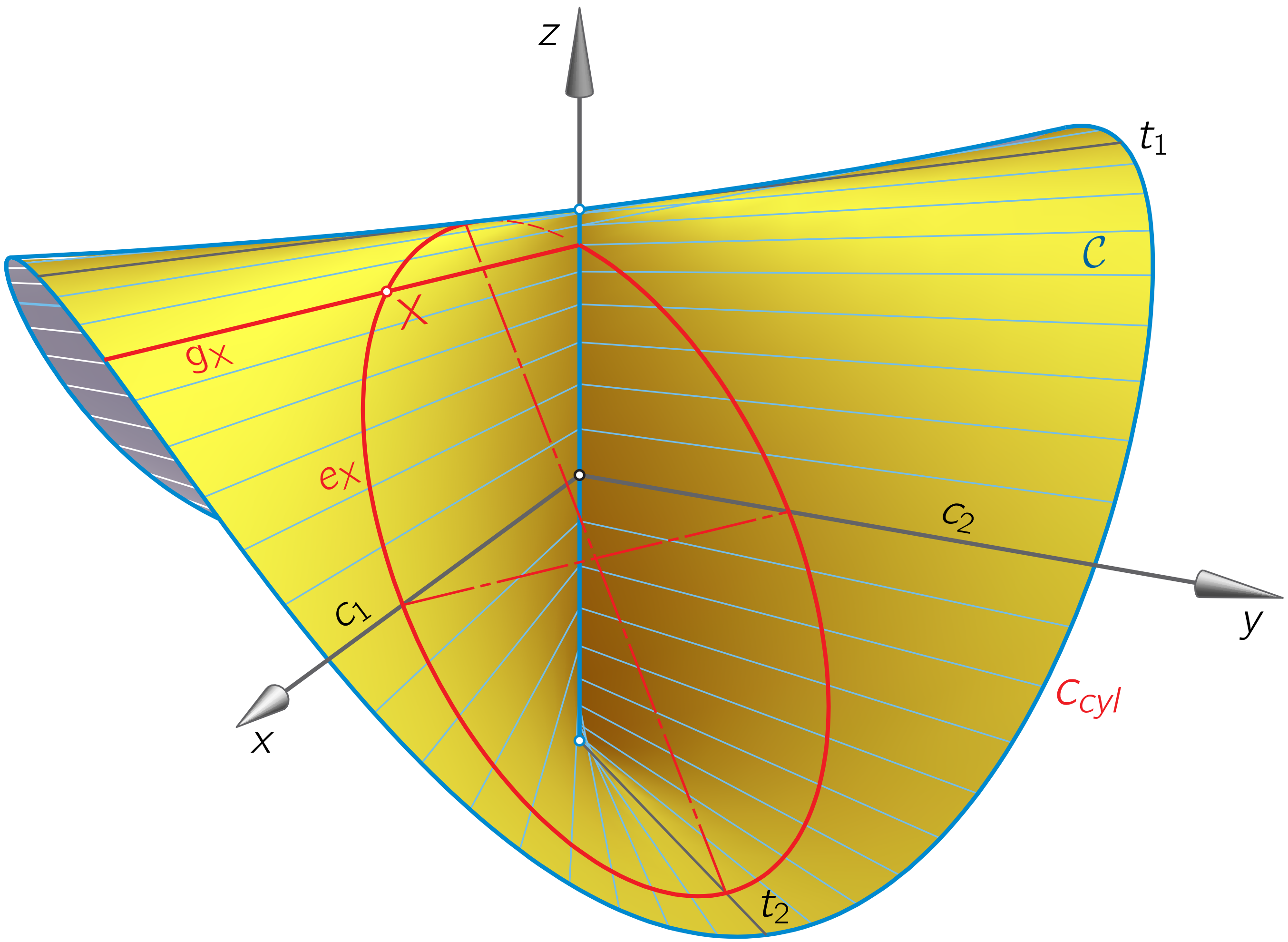




The equation $z = c \cdot \sin 2\varphi$ implies:

The curve of intersection c_{cyl} of Plücker's conoid $\mathcal{C}$ with a right cylinder about the z-axis appears in the cylinder's development as two periods of a Sine curve. The generators of $\mathcal{C}$ connect opposite points of c_{cyl}.

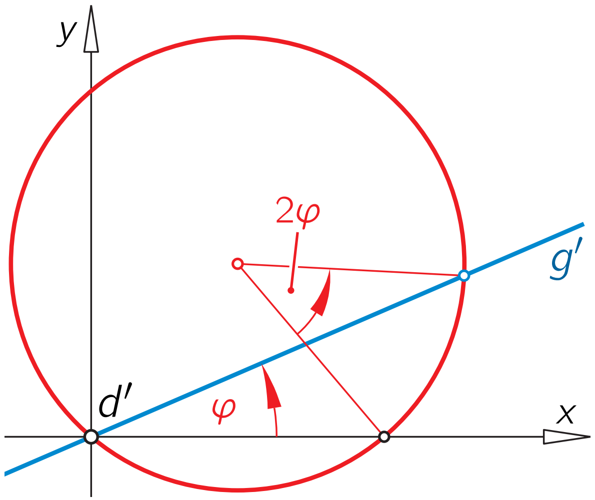
2. Plücker's conoid and concyclic lines



$\mathcal{C}$ has two torsal generators t_1, t_2 and two central generators c_1, c_2 in the $[xy]$-plane as axes of symmetry.

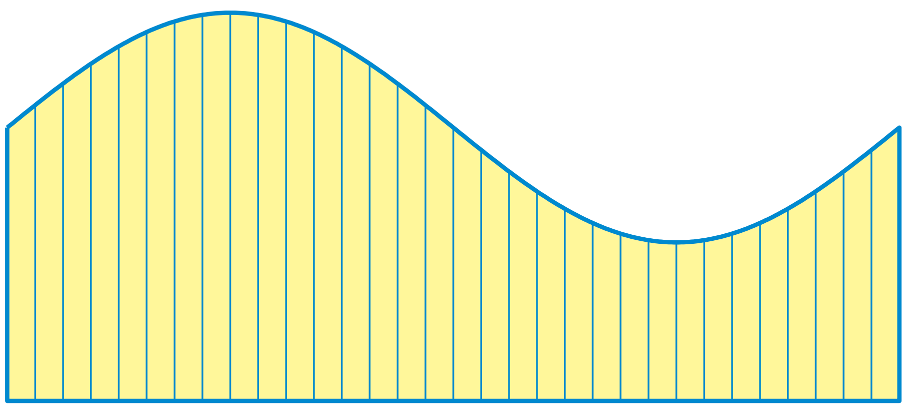
$\mathcal{C}$ contains a two-parameter set of ellipses e_X. The tangent plane τ_X at the point X intersects $\mathcal{C}$ along the generator g_X and the ellipse e_X.

2. Plücker's conoid and concyclic lines

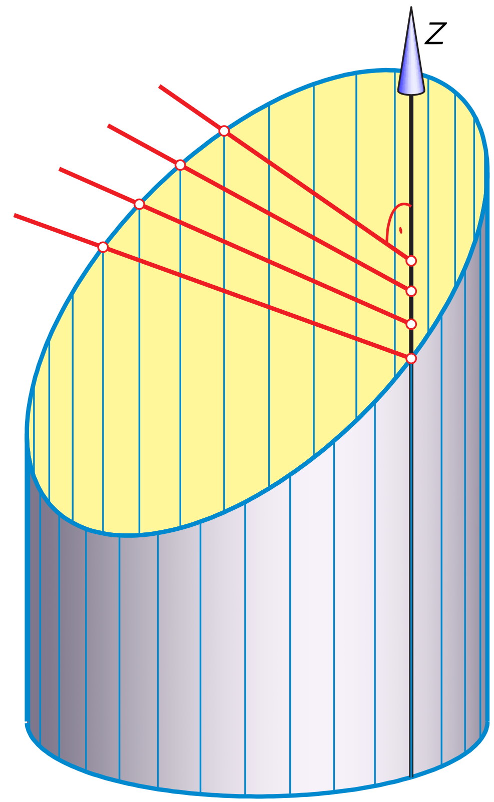


The top view of
$z = c \cdot \sin 2\varphi$ reveals:

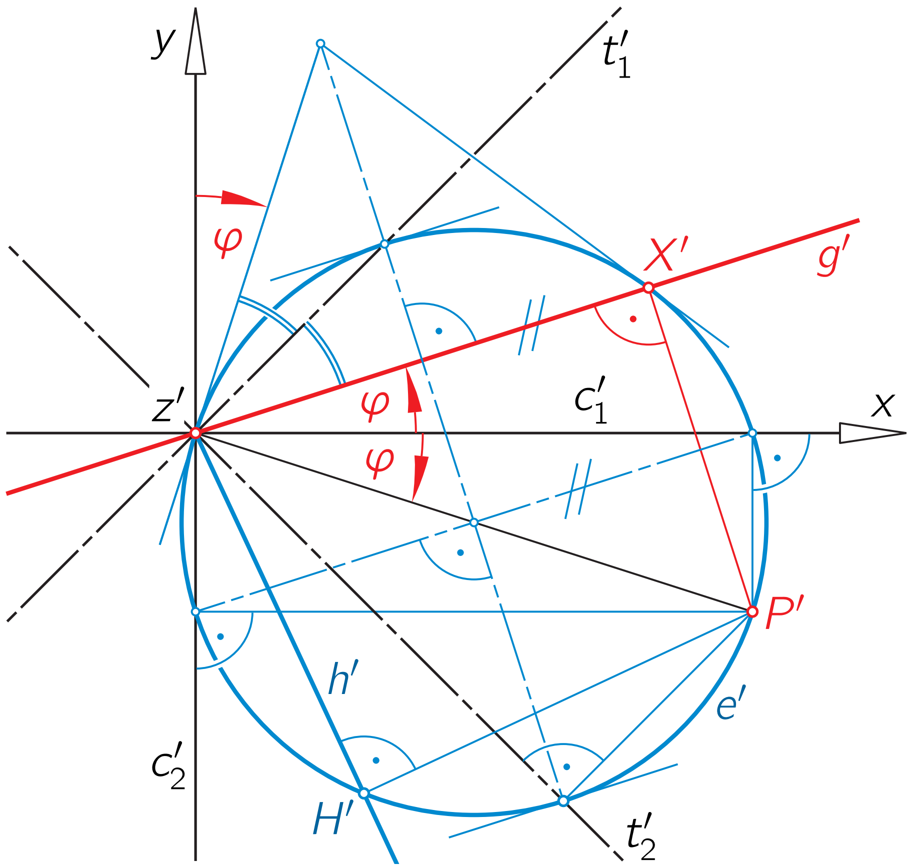
The intersection of the conoid $\mathcal{C}$ with a right cylinder through the double line d gives only one period of the Sine curve in the cylinder's development.



This results in an ellipse on Plücker's conoid.



2. Plücker's conoid and concyclic lines



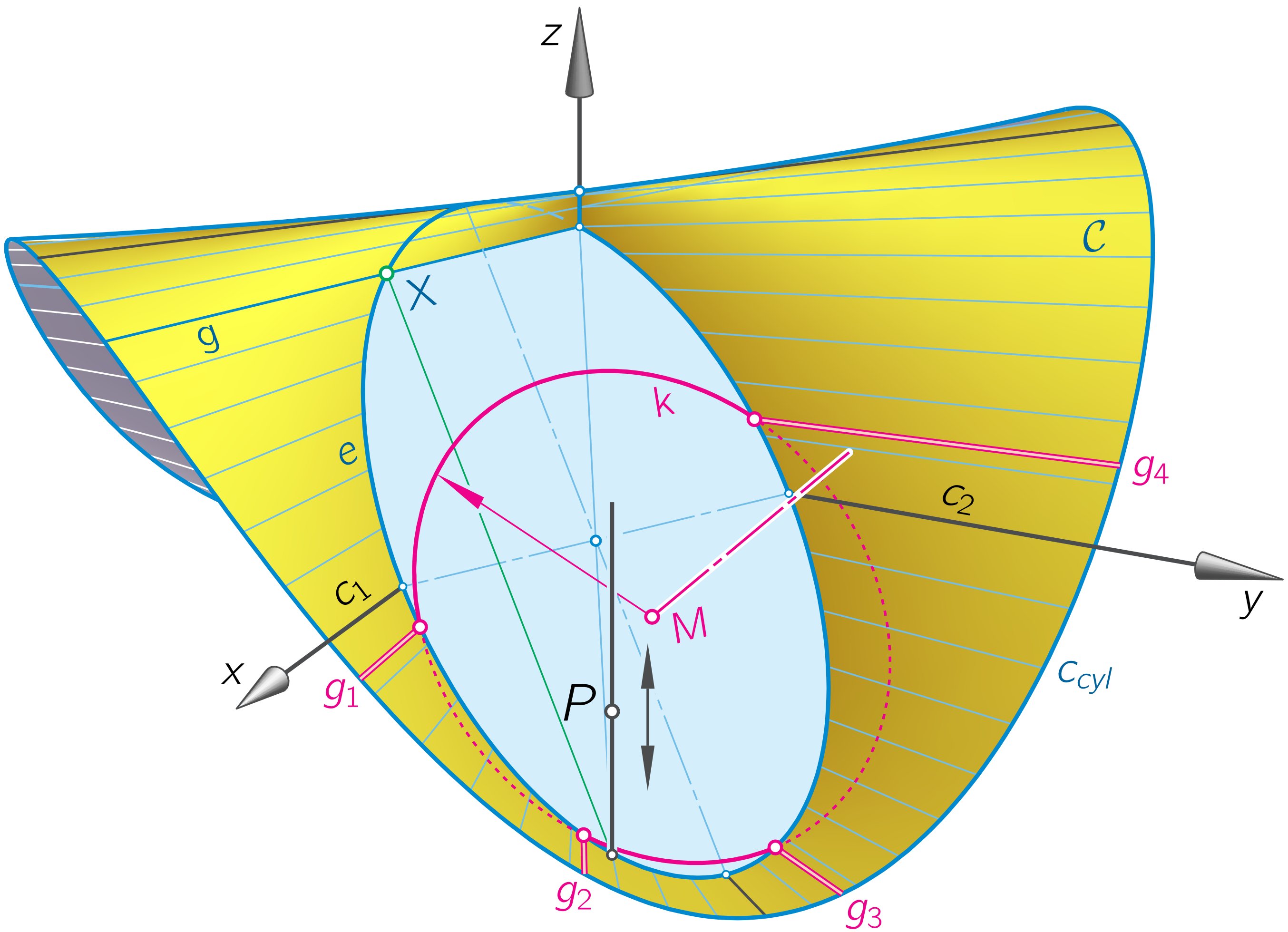
The top view reveals:

H is the pedal point of the generator $h \subset \mathcal{C}$ for all points P with the top view P'; the ellipse $e \subset \mathcal{C}$ is the **pedal curve** of P.

Theorem.
All pedal curves of Plücker's conoid $\mathcal{C}$ are planar.

Due to P. Appell (1900) Plücker's conoid is the only algebraic non-torsal ruled surface with this property.

2. Plücker's conoid and concyclic lines

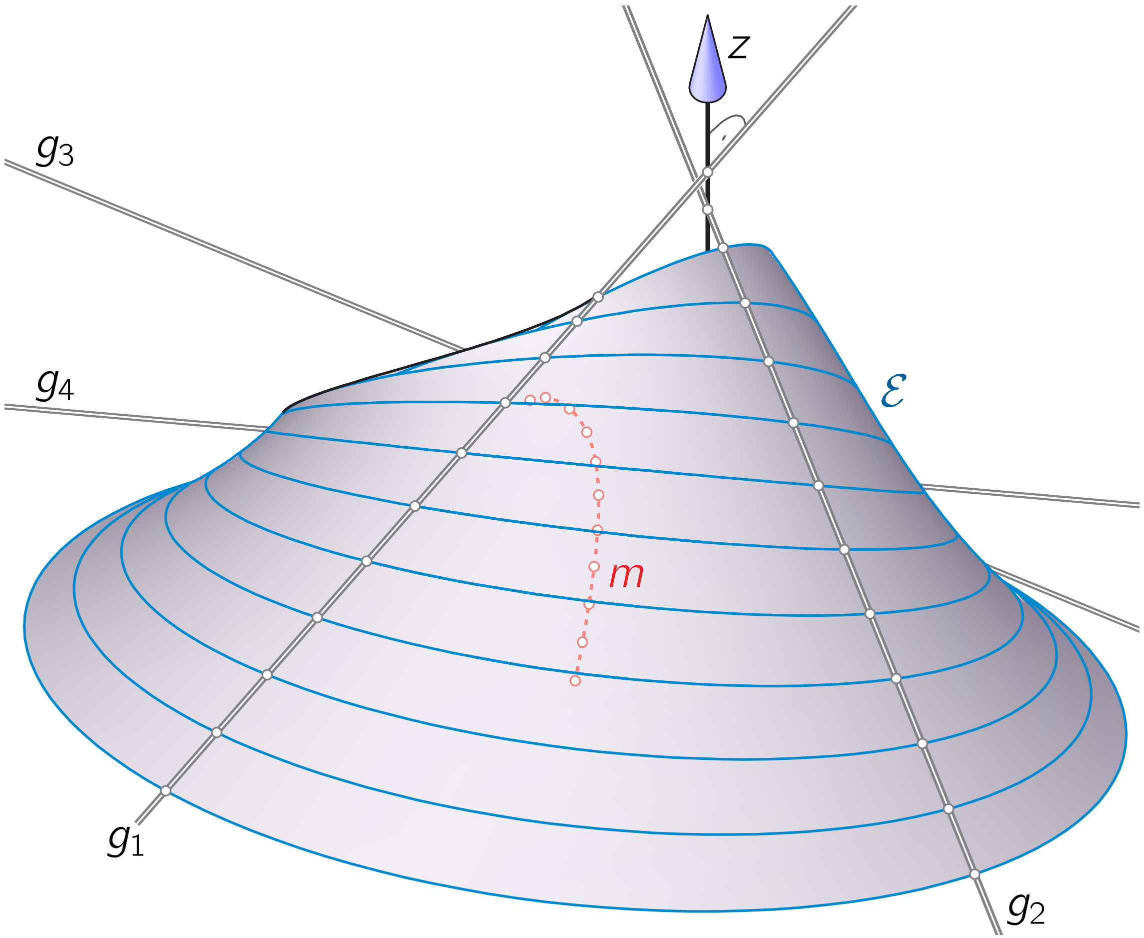


If $g_1, \dots, g_4 \subset \mathcal{C}$ are **concyclic** generators, then they intersect each tangent plane of $\mathcal{C}$ in concyclic points.

Infinitely many spheres are tangent to $g_1, \dots, g_4$.

The circle k is a characteristic $\iff$ the axis of k lies in the vertical plane through P and X orthogonal to g.

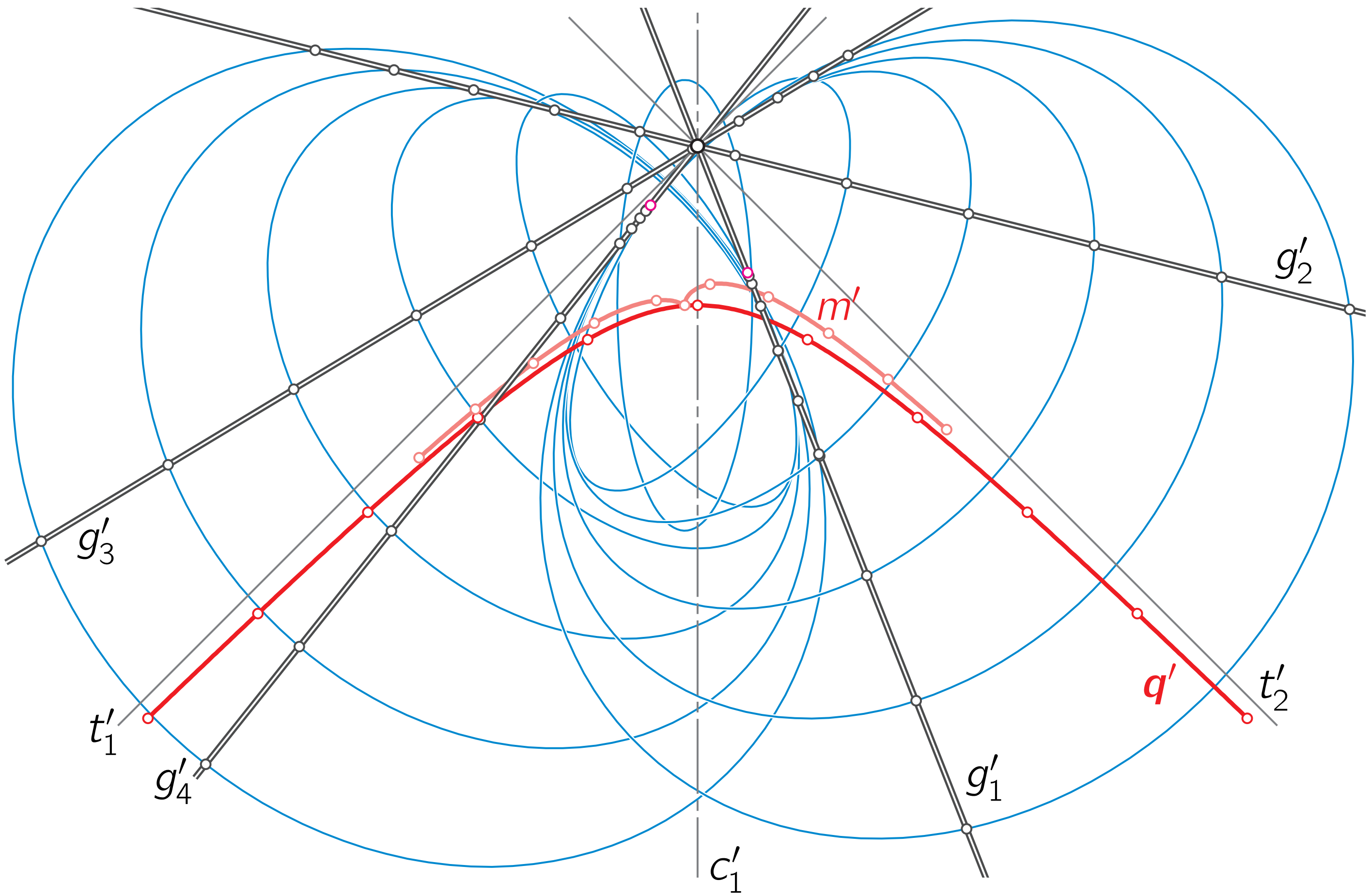
3. Generic case and singularities



A part of one component of the canal surface $\mathcal{E}$ through mutually skew concyclic lines $g_1, \ldots, g_4$ along with the locus m of characteristics' centers. The second component is symmetric w.r.t. the z-axis.

$\mathcal{E}$ has always singularities. The black curve is a cuspidal edge. The points of the lines are biplanar or uniplanar points of the complete surface.

3. Generic case and singularities

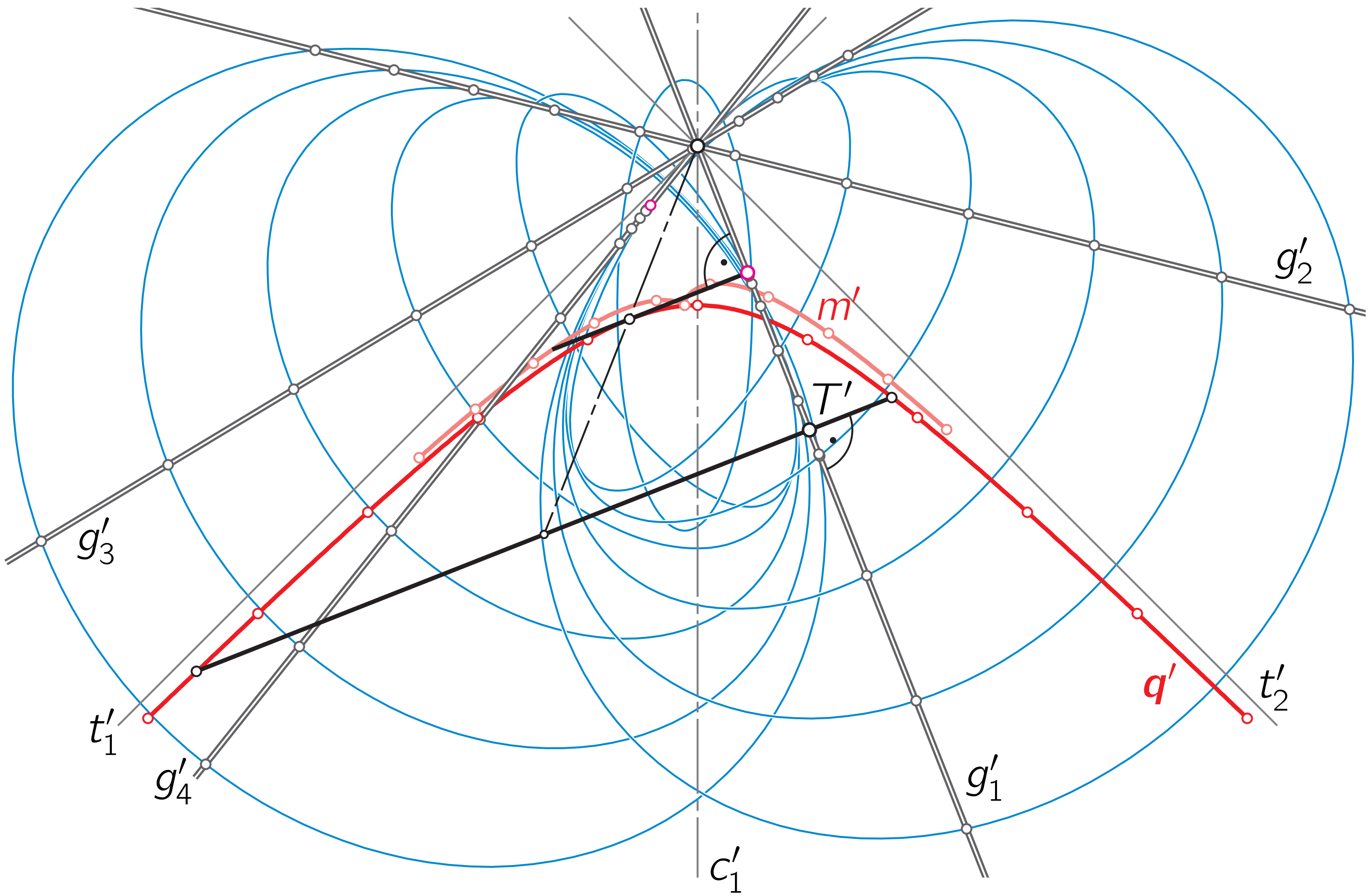


Top view of circles of the previous canal surface $\mathcal{E}$ through skew concyclic lines $g_1, \ldots, g_4$.

The **hyperbola** $\boldsymbol{q'}$ is the top view of the spine curve and m' that of the curve of blau circles' centers.

Not all points of g_1 are contact points with a sphere.

3. Generic case and singularities



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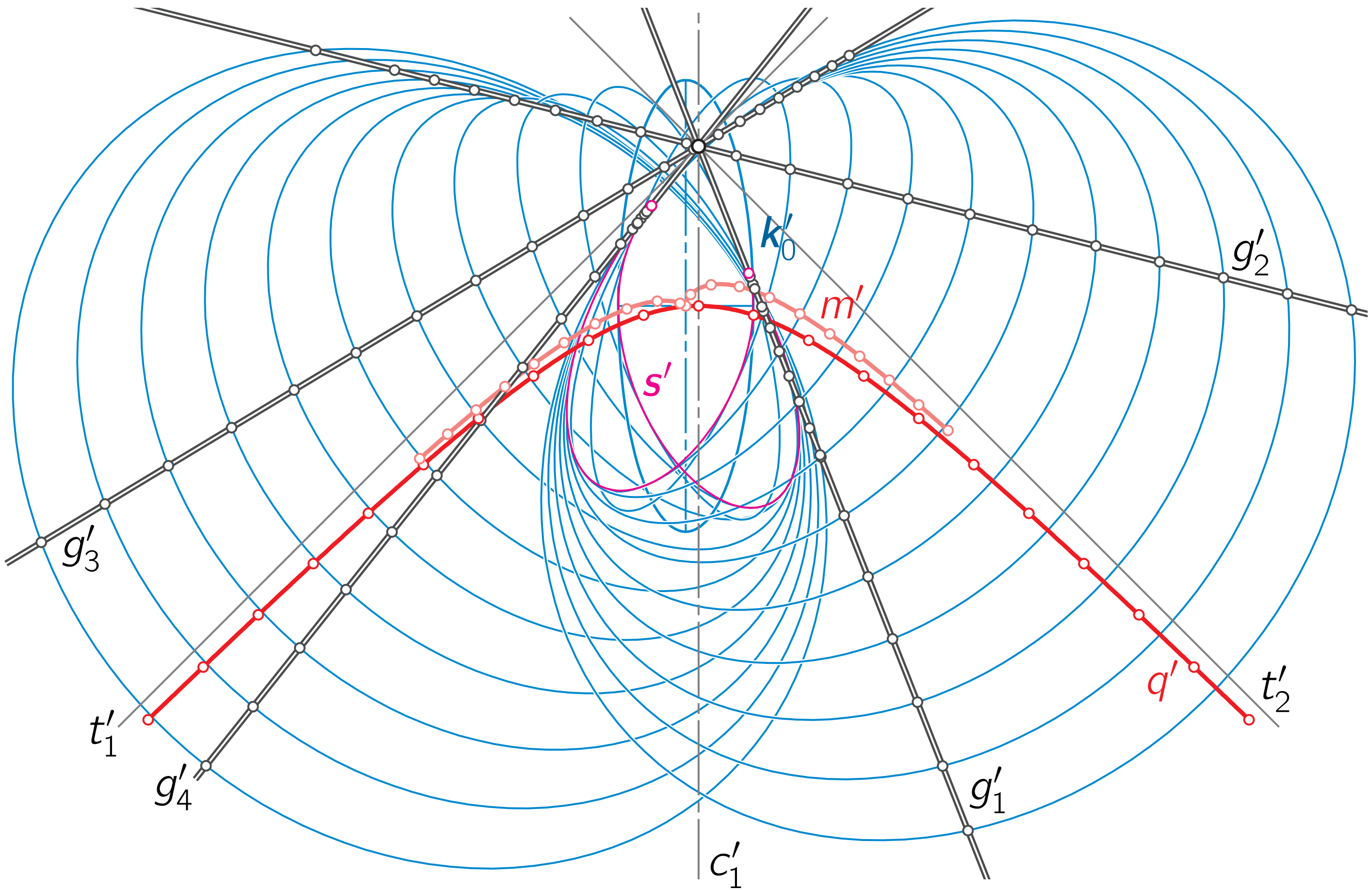
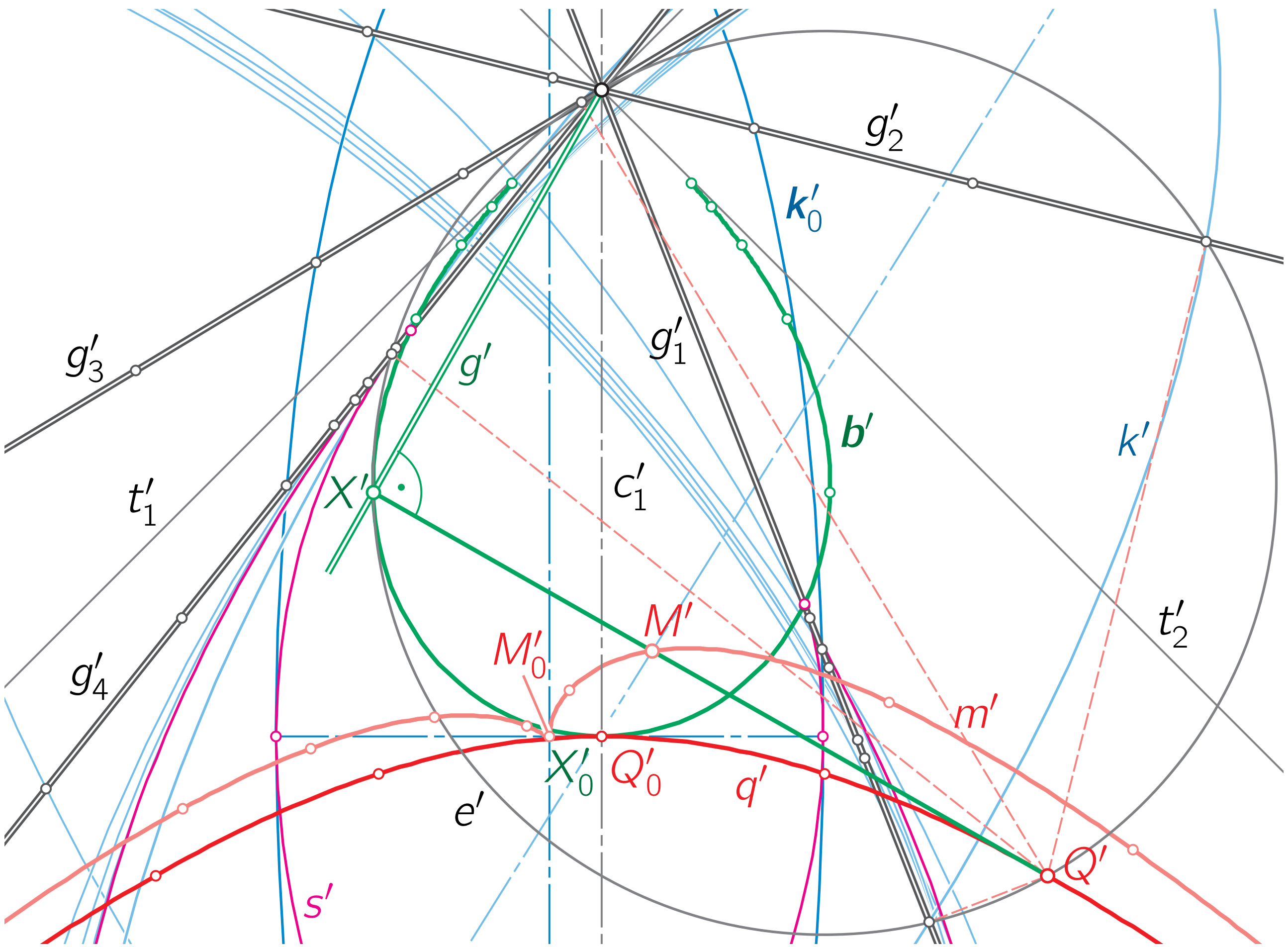


Top view of circles of the canal surface $\mathcal{E}$ through skew concyclic lines $g_1, \dots, g_4$.

The curve s is the cuspidal edge. It passes through the uniplanar points of g_1 and g_4.

k_0 is the smallest circle.

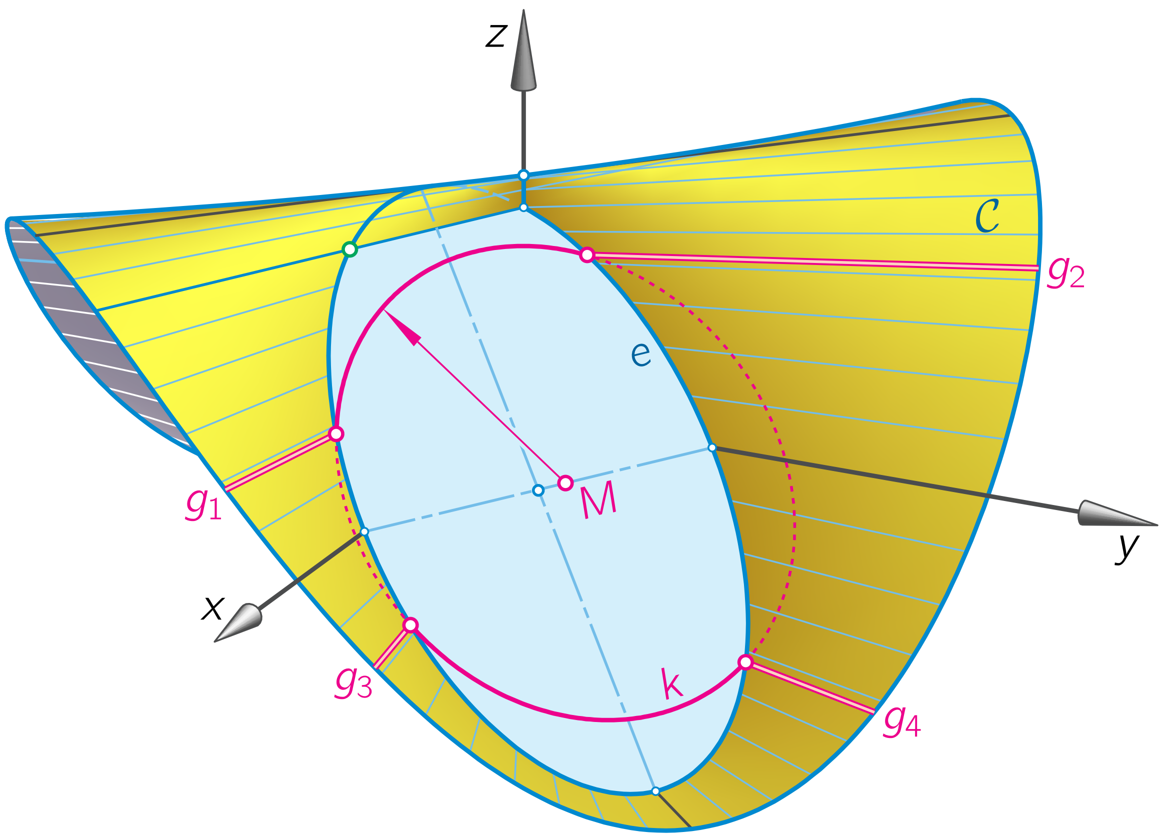
3. Generic case and singularities



The characteristic's planes contact the Plücker conoid $\mathcal{C}$ along an asymptotic curve b with a lemnniscate of Bernoulli $\boldsymbol{b}'$ as top view.

Hence, b is the edge of regression of the developable formed by the characteristics' planes. Tangents of b intersect the characteristics at singular points.

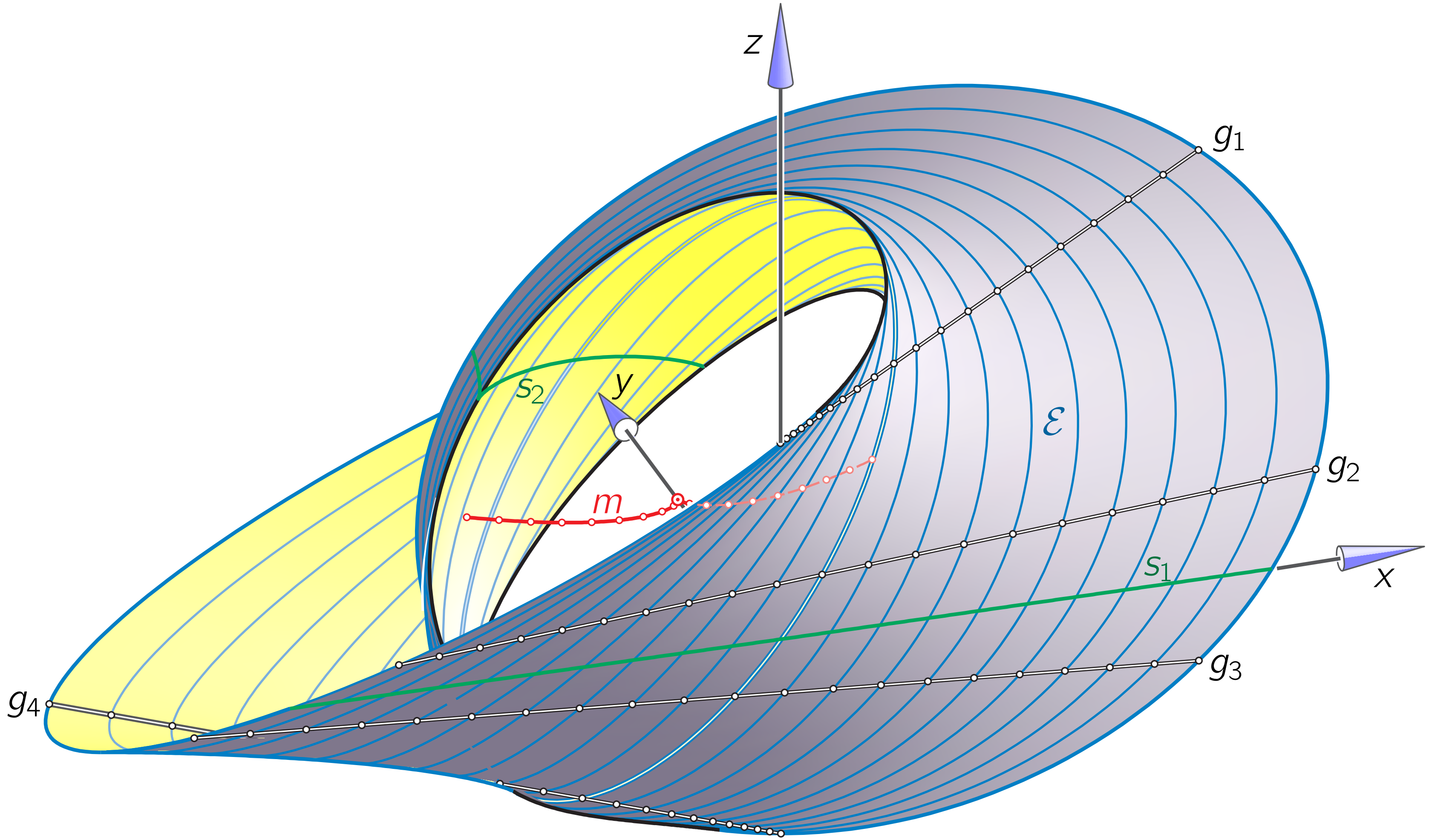
4. Symmetric canal surfaces through four lines



There are two symmetric cases: Either the center M of k lies on the secondary axis of e (left) on the principal axis.

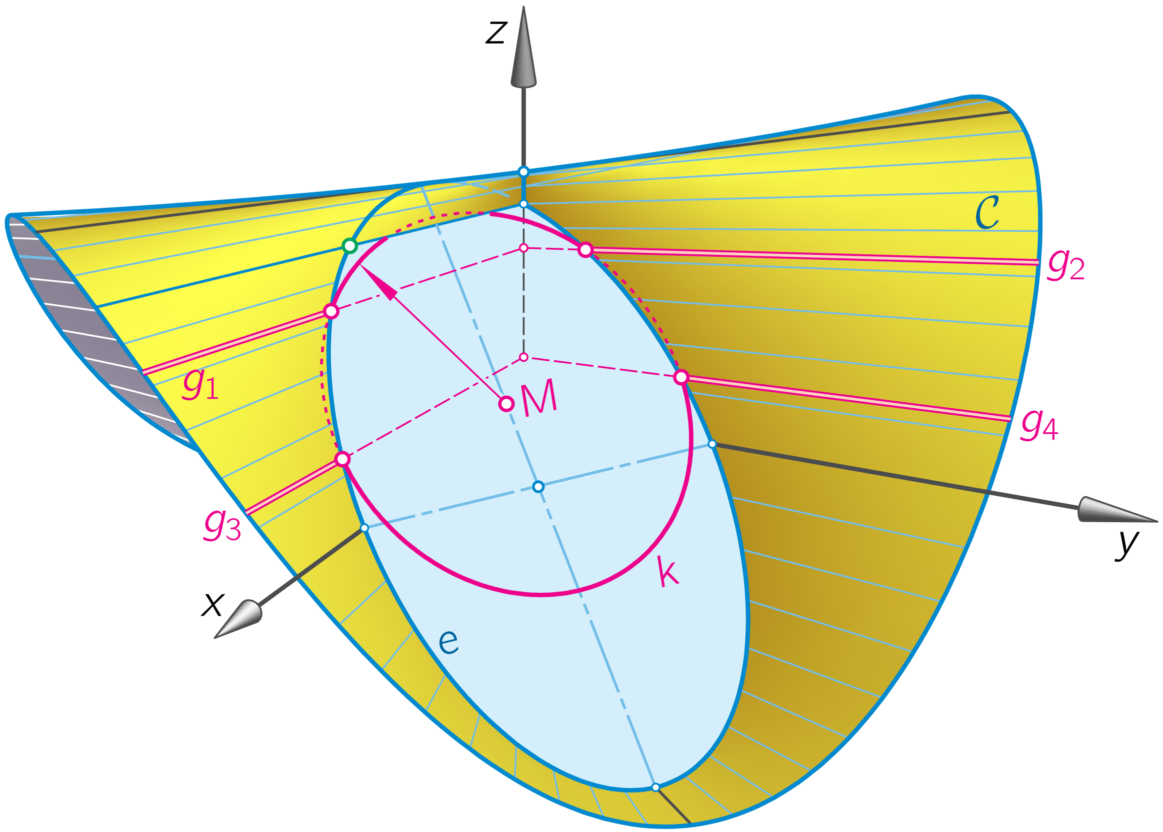
In the first case the concyclic lines are **skew**.
In the latter case the pairs (g_1, g_2) and (g_3, g_4) are **intersecting**, and $\mathcal{C}$ is symmetric w.r.t. the z-axis and the plane $x = y$.

4. Symmetric canal surfaces through four lines



Skew symmetric case with edges of regression (black) and the smallest circle (double line).

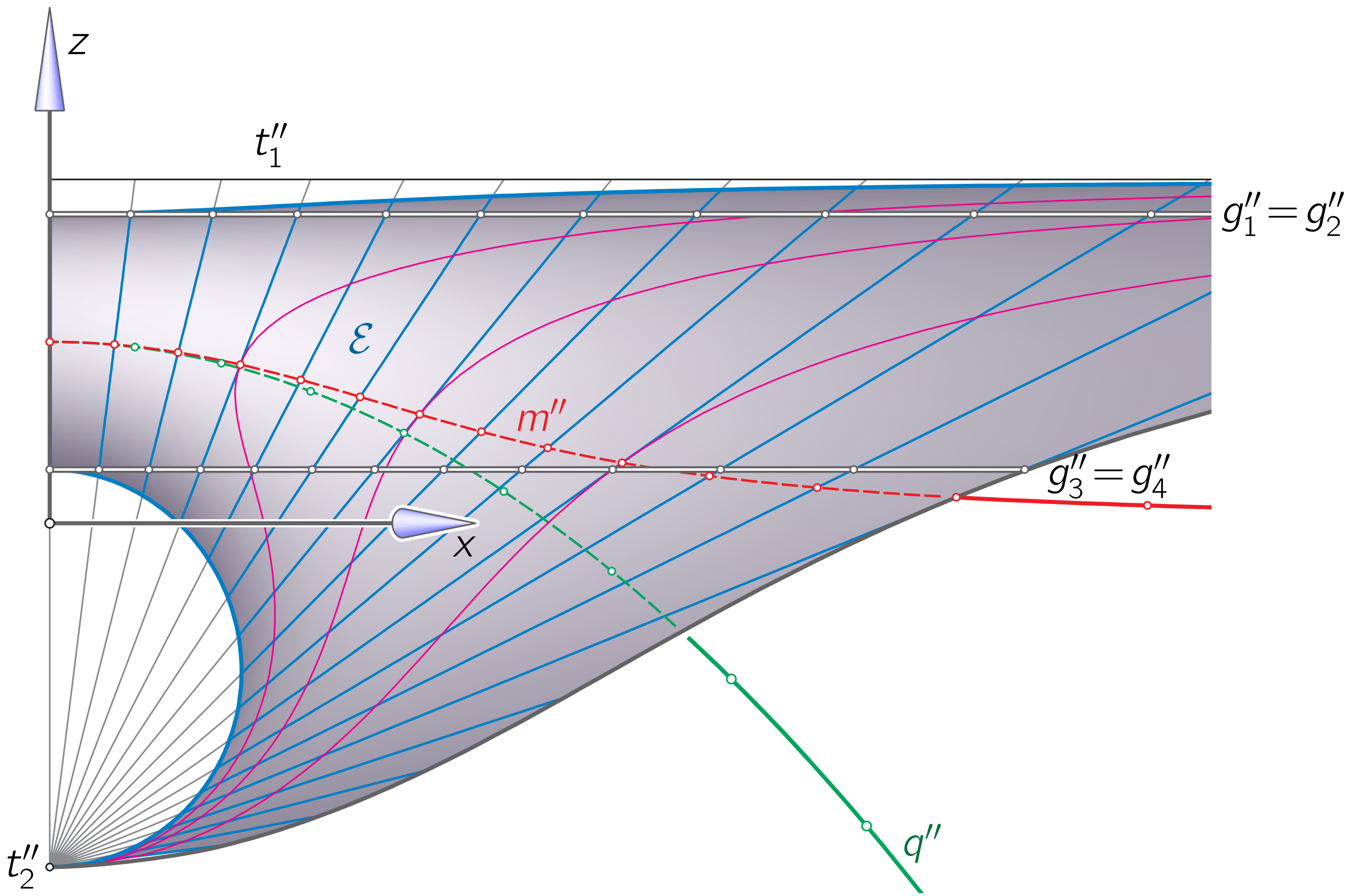
4. Symmetric canal surfaces through four lines



This is the **symmetric intersecting case**. The lines (g_1, g_2) and (g_3, g_4) are intersecting.

The canal surface $\mathcal{E}$ is symmetric w.r.t. the z-axis and the plane $x = y$ through one torsal generator of $\mathcal{C}$.

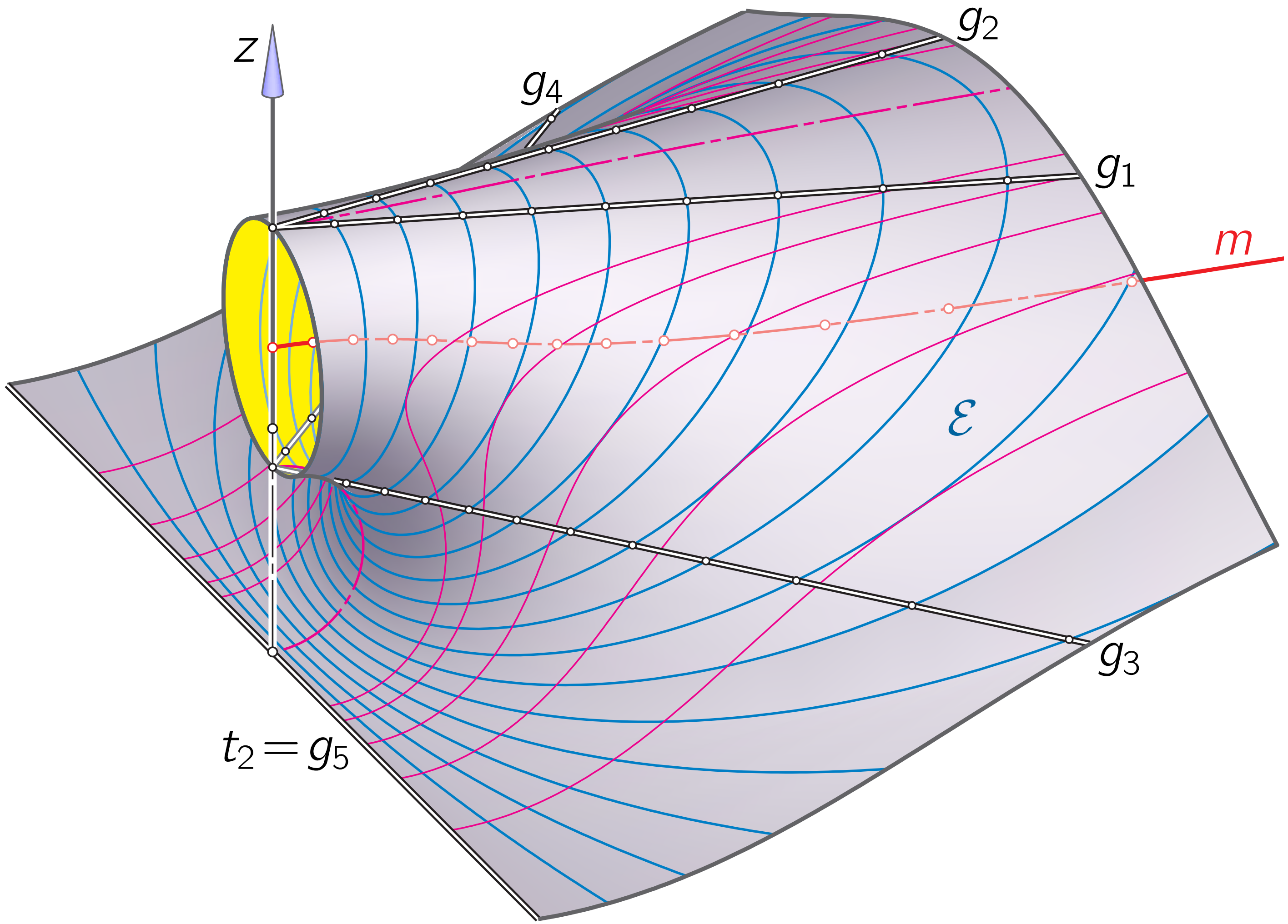
4. Symmetric canal surfaces through four lines



Front view of the right half of one component of the canal surface $\mathcal{E}$ in the intersecting case with the parabola q as spine curve.

The border line is the intersection with a plane orthogonal to t_2. Other contour lines are depicted in magenta.

4. Symmetric canal surfaces through four lines

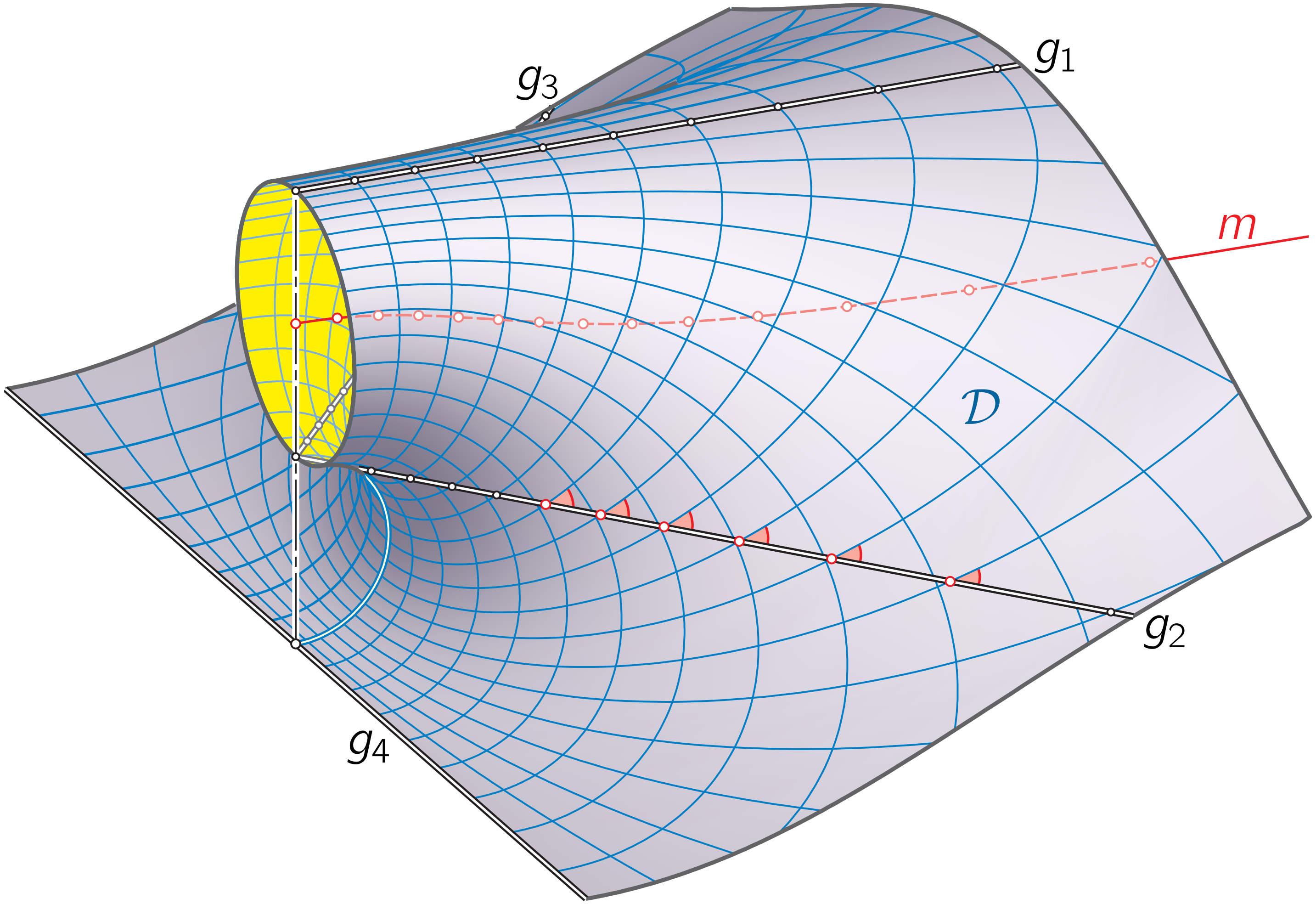


One component of the canal surface $\mathcal{E}$ through $g_1, \dots, g_4$. The second component looks similar.

The fifth line g_5 is the limit of a characteristic.

If more than four lines contact all spheres, then infinitely many $\Rightarrow$ $\mathcal{E}$ is a right cone or cylinder or a one-sheeted hyperboloid.

4. Symmetric canal surfaces through four lines

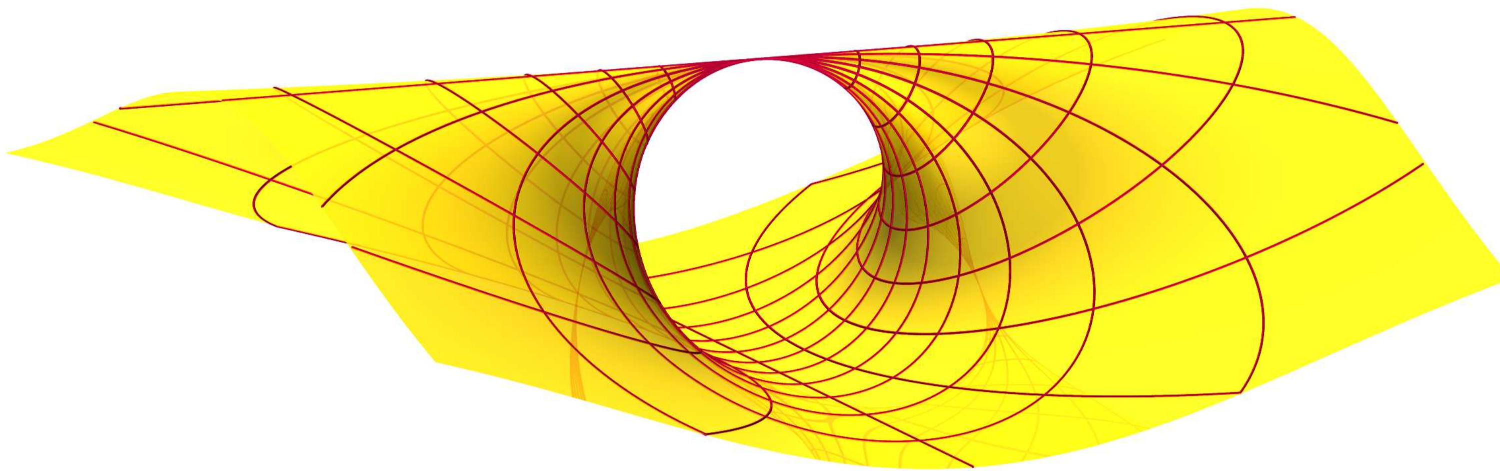


The limit $g_1 = g_3$ is a parabolic **Dupin ring cyclide** $\mathcal{D}$.

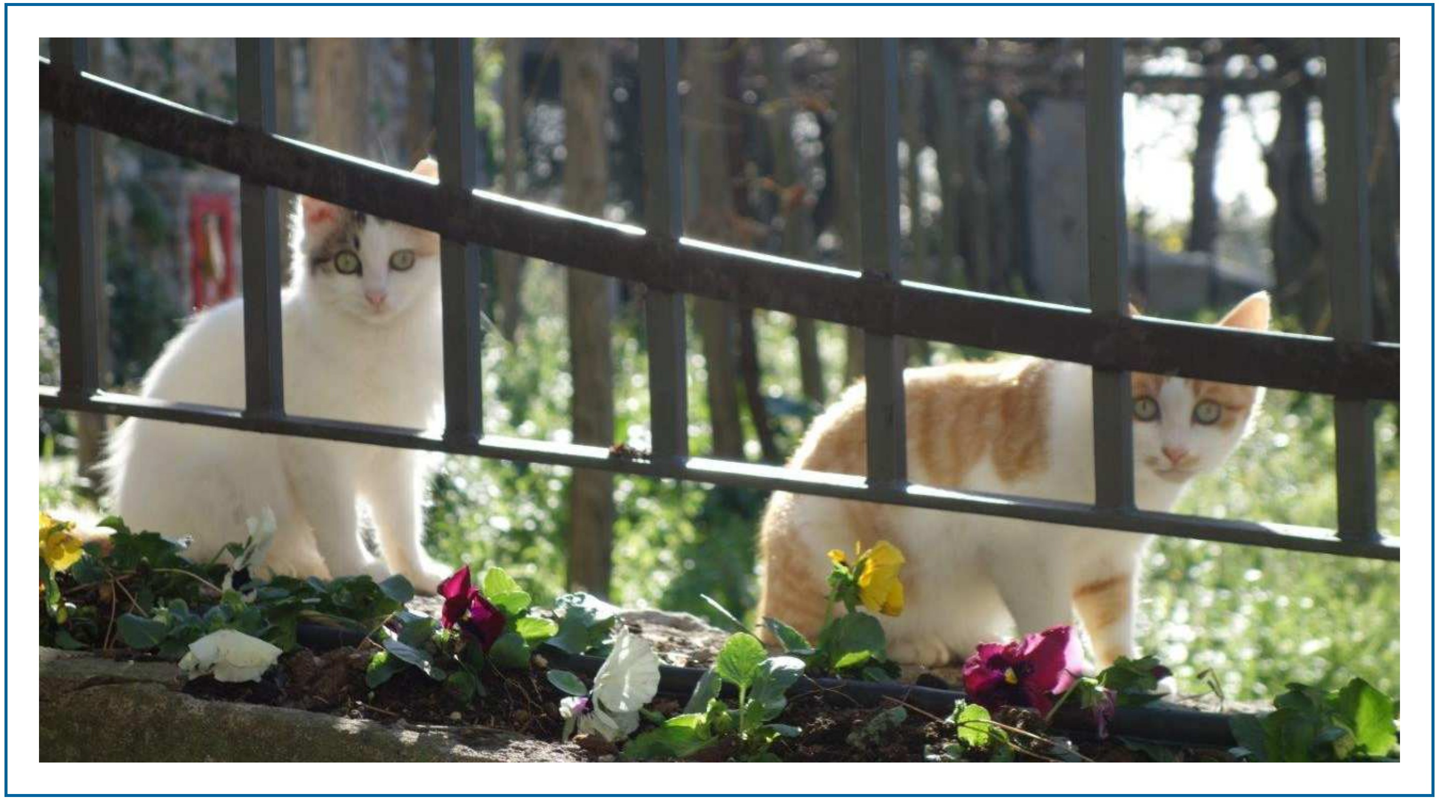
Like all Dupin cyclides, it can be generated in two ways as a canal surface.

The spine curves are focal parabolas.

4. Symmetric canal surfaces through four lines



All four lines coincide when k is hyperosculating circle of e at a principal vertex. In this limit of the symmetric intersecting case we obtain a **parabolic Dupin needle cyclide**.



Thank you for your attention!

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