Geometric analysis of Bennett's spherical 8-bar linkage and its spatial counterpart

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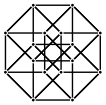
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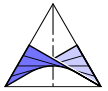
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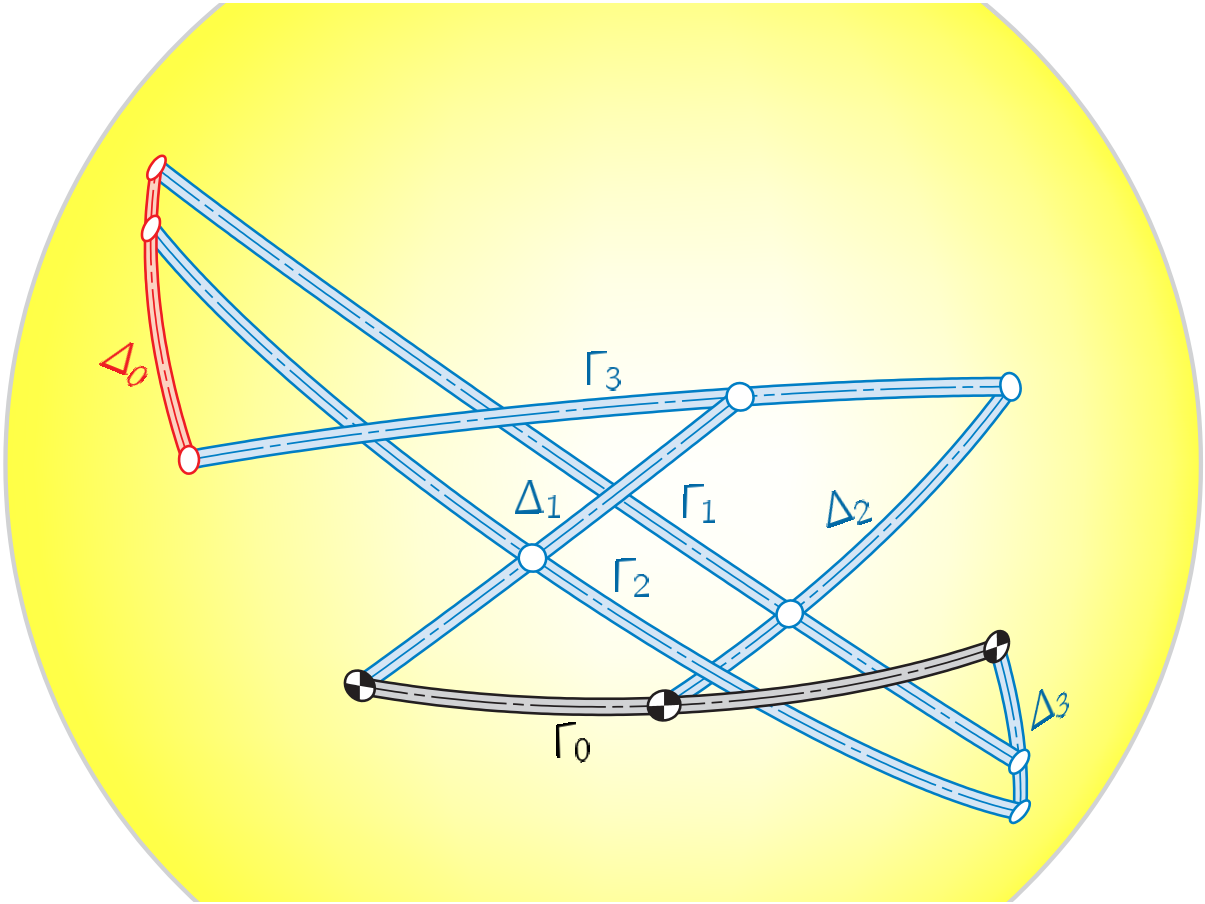
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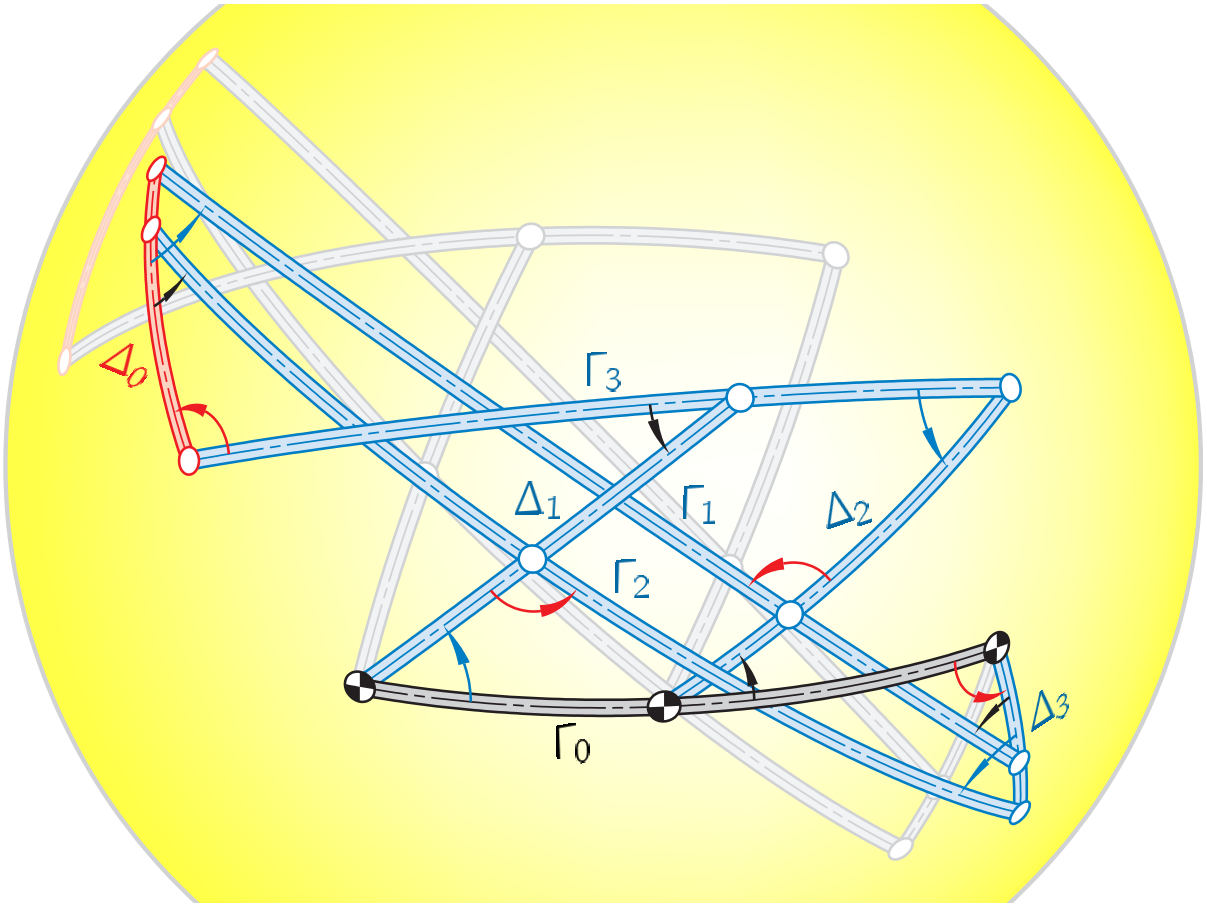
1. Introduction



In 1912, **G.T. Bennett** the flexibility of this spherical linkage with 8 bars $\Gamma_0, \ldots, \Gamma_3, \Delta_0, \ldots, \Delta_3$ and 12 joints with the aid of trigonometric relations.

It is overconstrained and has the degree of freedom d.o.f. $= 1$
The Chebyshev – Grübler – Kutzbach mobility formula yields for the generic case $3 \cdot 7 - 12 \cdot 2 = \mathbf{-3}$.

1. Introduction

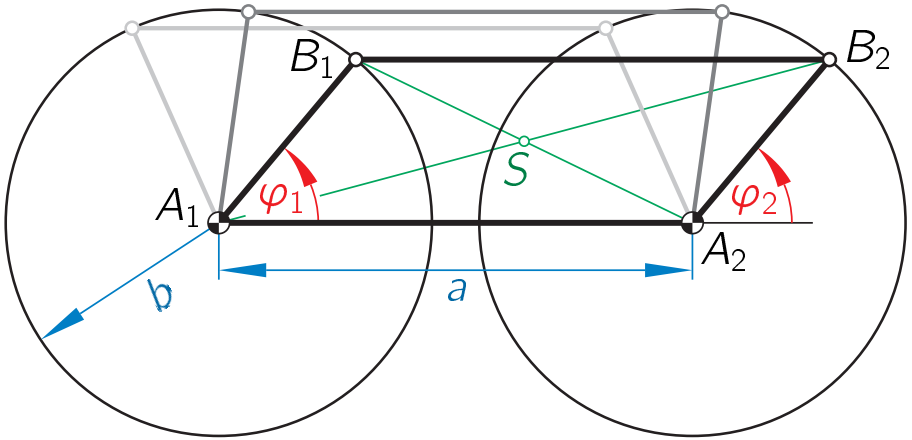


Bennett's spherical 8-bar linkage is composed from six isograms, i.e., spherical quadrangles with opposite sides of equal lengths ⇒ congruent opposite angles.

This linkage is the spherical image of **R. Bricard's** flexible octahedron of type III (1897). Bennett used the flexibility of his linkage to prove that of the type-III octahedra.

1. Introduction

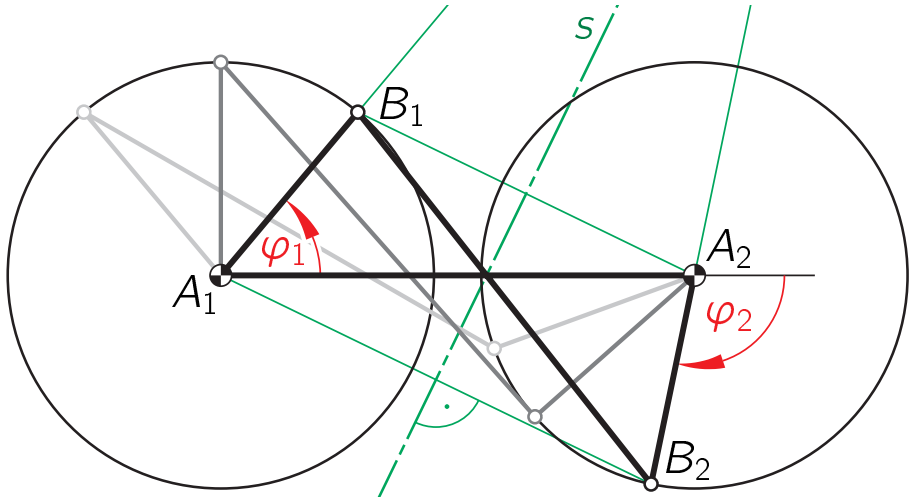
We start with a planar version.



parallelogram linkage

$$\varphi_2 = \varphi_1$$

The parallelogram has a center S of symmetry.

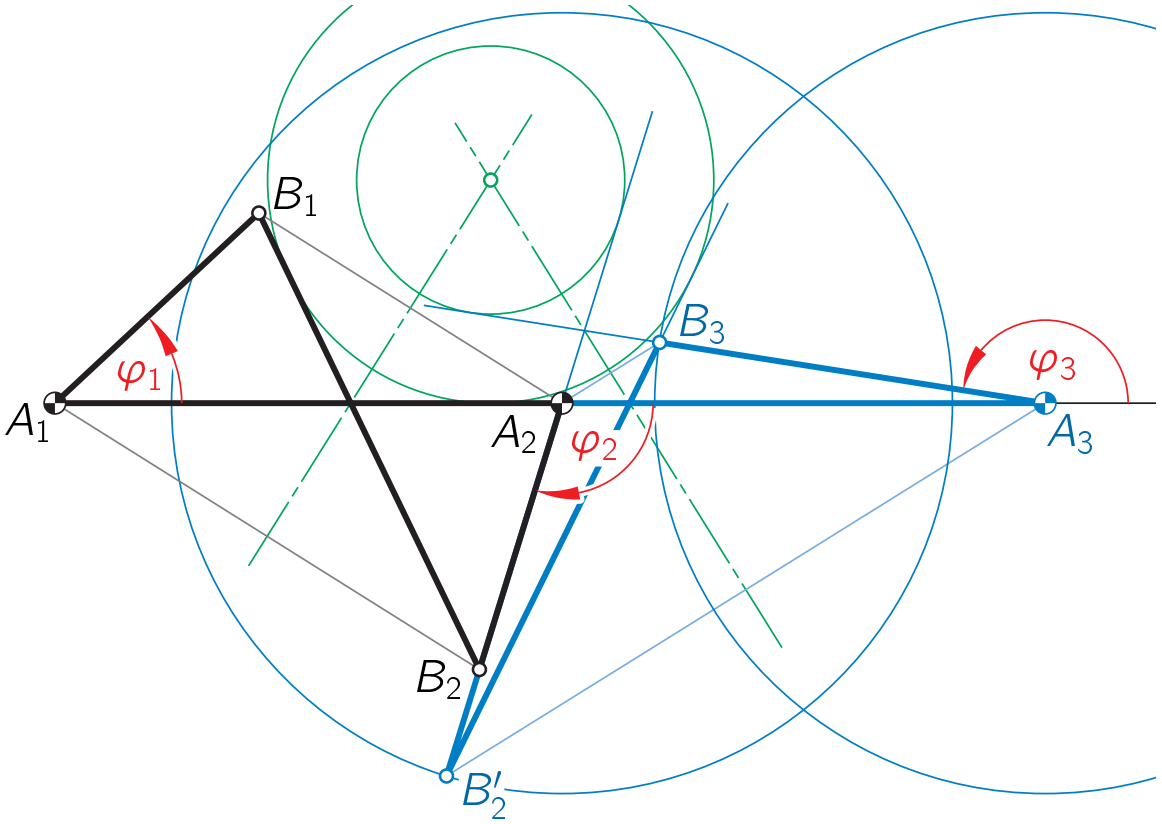


antiparallelogram linkage

$$\tan\frac{\varphi_2}{2} = \frac{b+a}{b-a}\tan\frac{\varphi_1}{2}$$

The antiparallelogram has an axis s of symmetry.

1. Introduction



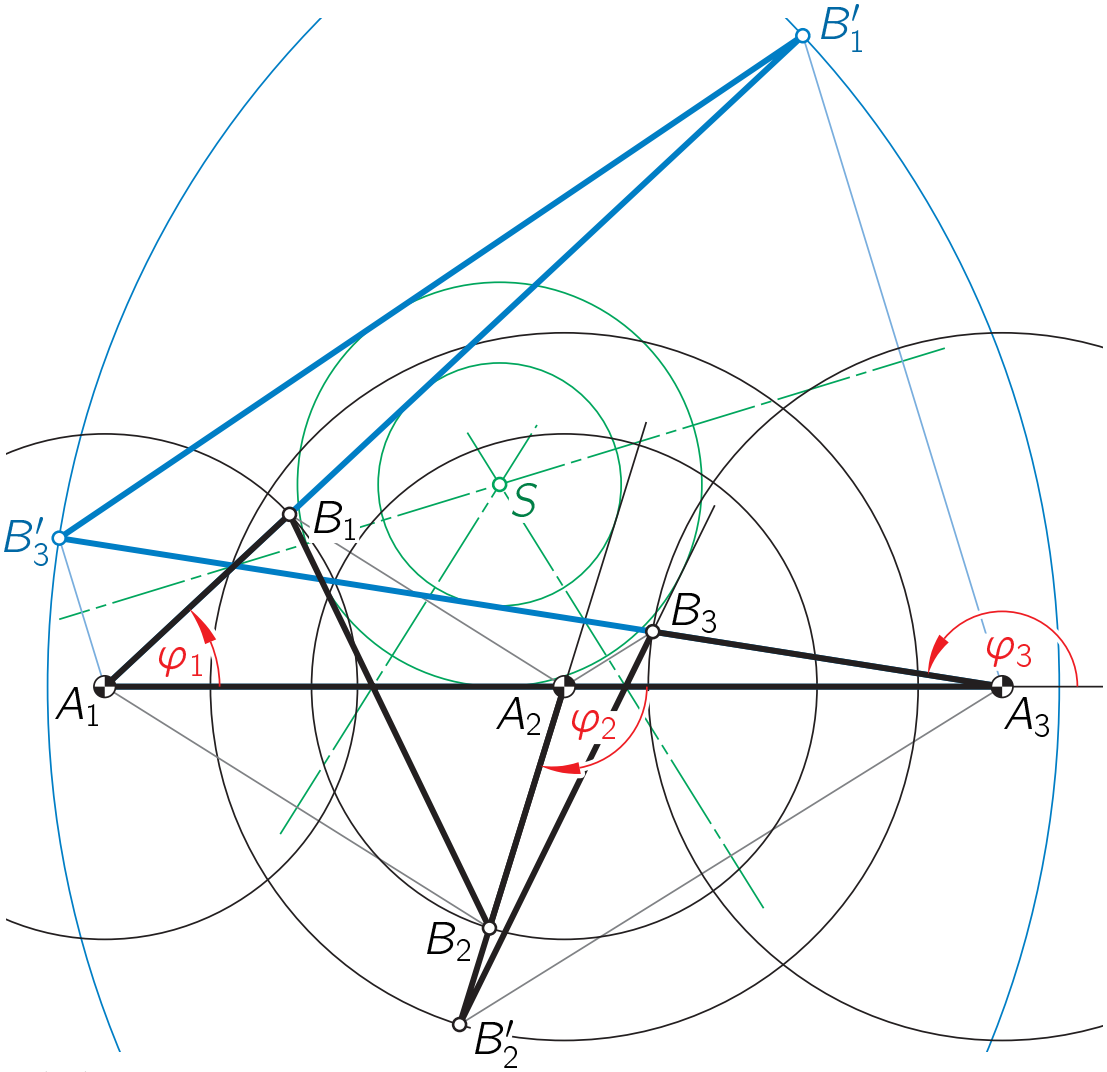
Composition of two antiparallelogram linkages

$$\tan\frac{\varphi_3}{2} = c_{32}c_{21}\tan\frac{\varphi_1}{2}\,.$$

A third antiparallelogram linkage with base A_1A_3 is compatible with the movement of the other two.

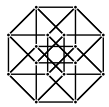
$$\tan\frac{\varphi_2}{2} = c_{21}\tan\frac{\varphi_1}{2}, \quad \tan\frac{\varphi_3}{2} = c_{32}\tan\frac{\varphi_2}{2} \quad \Longrightarrow$$

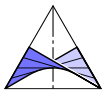
1. Introduction



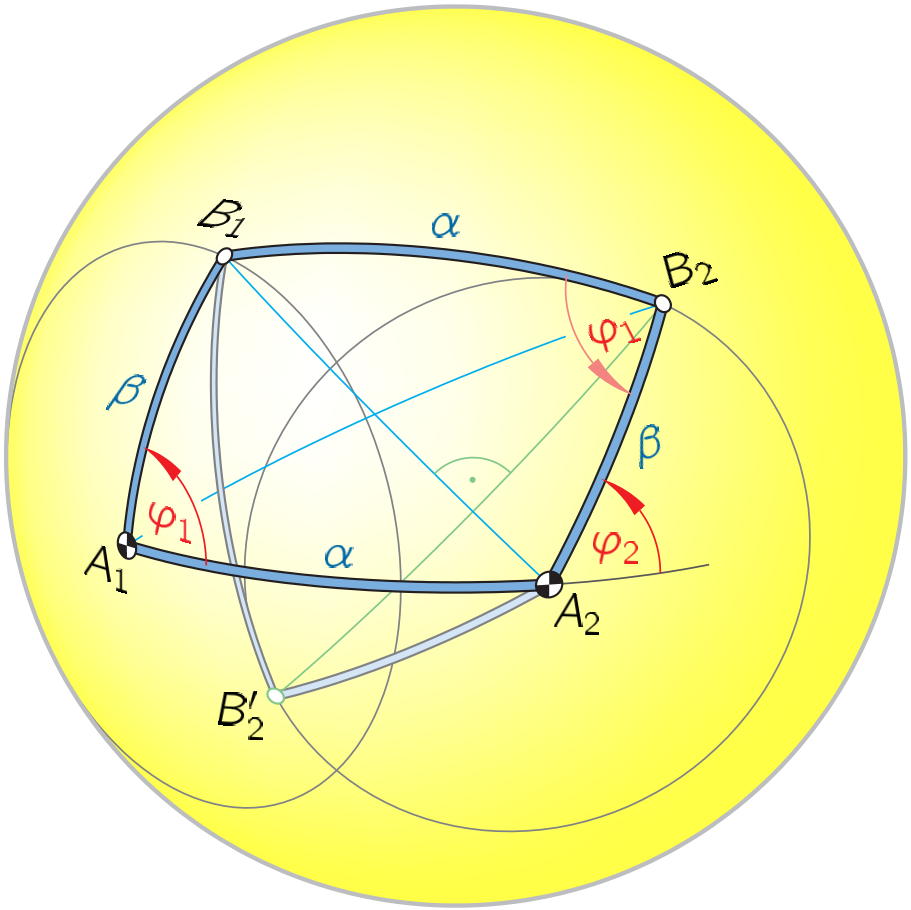
This planar linkage is composed from three antiparallelogram linkages and has d.o.f. $= 1$.

Also on the sides $A_1B'_1$, $A_2B'_2$ and $A_3B'_3$ we find pairs of antiparallelograms with aligned bases and arms. Therefore, the procedure of extension can be iterated (G. Darboux, 1879, and E. Baker, 1995). It results in a flexible planar 8-bar linkage.





2. The spherical 8-bar linkage

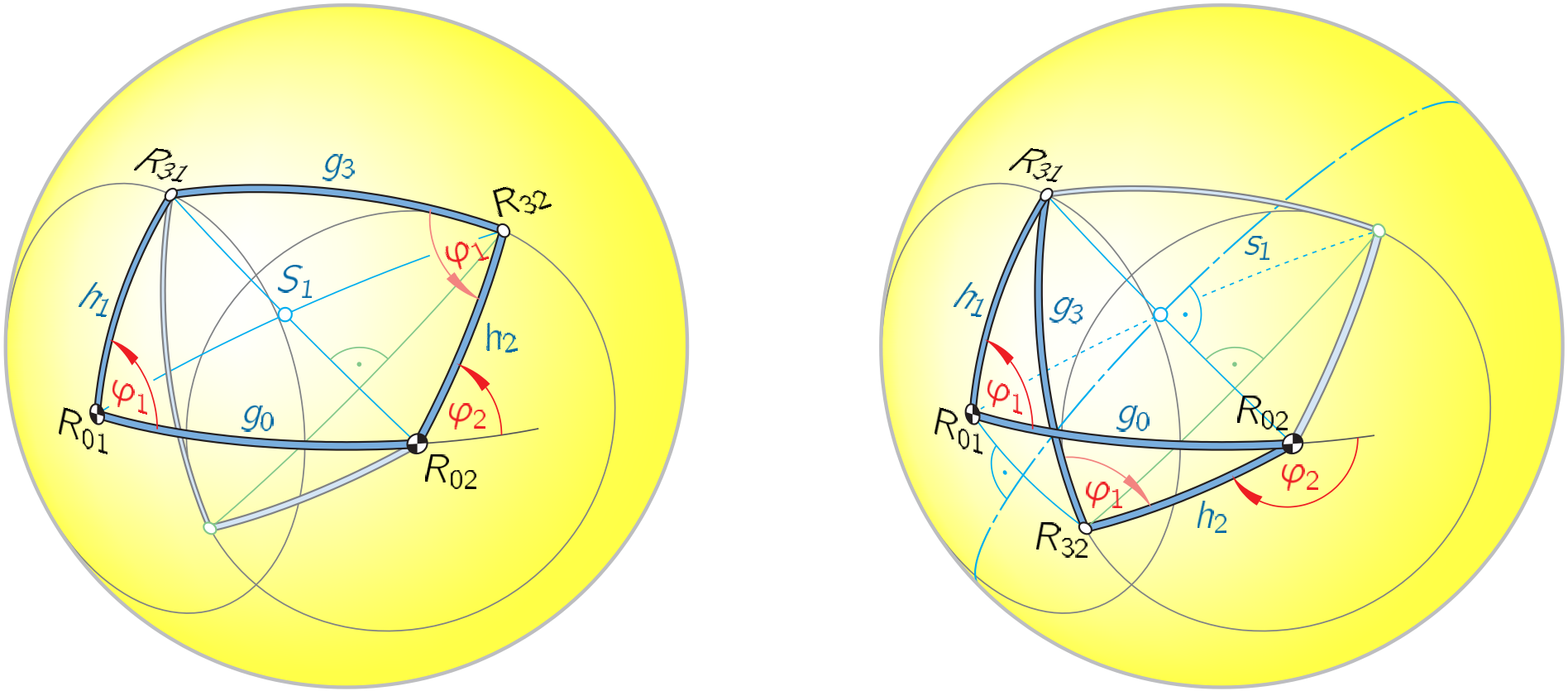


The coupler motion of a spherical **isogram** splits into two rational motions. There exist two aligned poses with bifurcations between the two motions.

$$\tan\frac{\varphi_2}{2} = \frac{\sin(\alpha-\beta)}{\sin\alpha \pm \sin\beta}\,\tan\frac{\varphi_1}{2}\,,$$

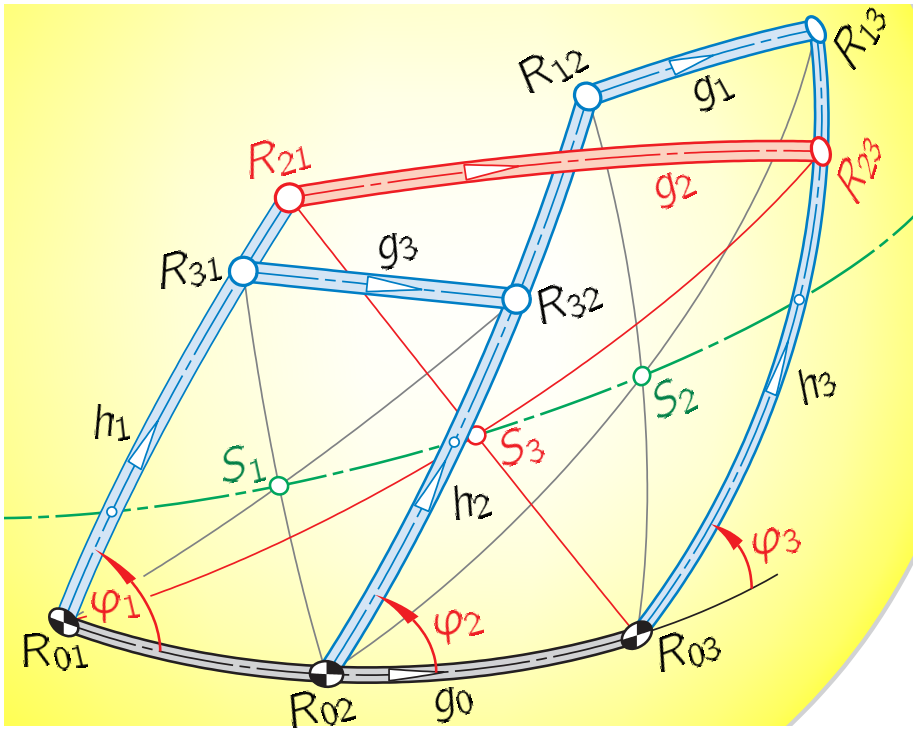
depending on whether the quadrangle is crossed or not.

2. The spherical 8-bar linkage



A non-crossed spherical isogram (left) has a center S_1, a crossed spherical isogram (right) an axis s_1 of symmetry. Both symmetries act equal on great circles.

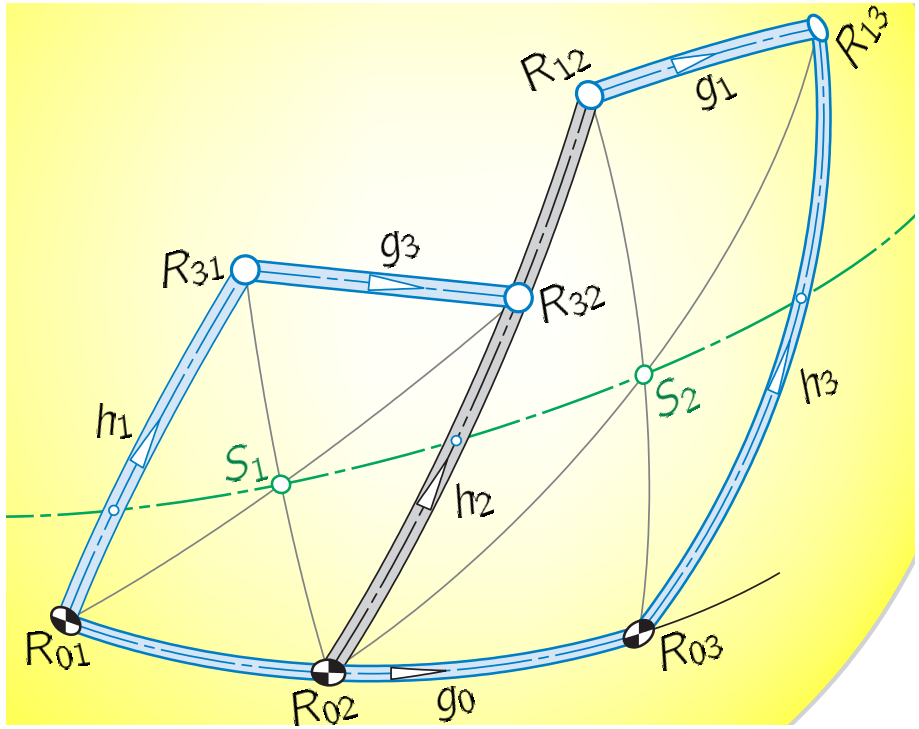
2. The spherical 8-bar linkage



Like in the plane, given two isograms with aligned bases $R_{01}R_{02}$, $R_{02}R_{03}$ on g_0 and arms $R_{02}R_{32}$, $R_{02}R_{12}$ on h_2, the transmission by two spherical isograms from h_1 via h_2 to h_3 is compatible with a third isogram with coupler $R_{21}R_{23}$ on g_2 connecting h_1 with h_3.

The three centers of symmetry S_1, S_2 and S_3 are aligned, since the composition of the three halfturns is involutive as it sends h_1 to itself but reverses the orientation.

2. The spherical 8-bar linkage



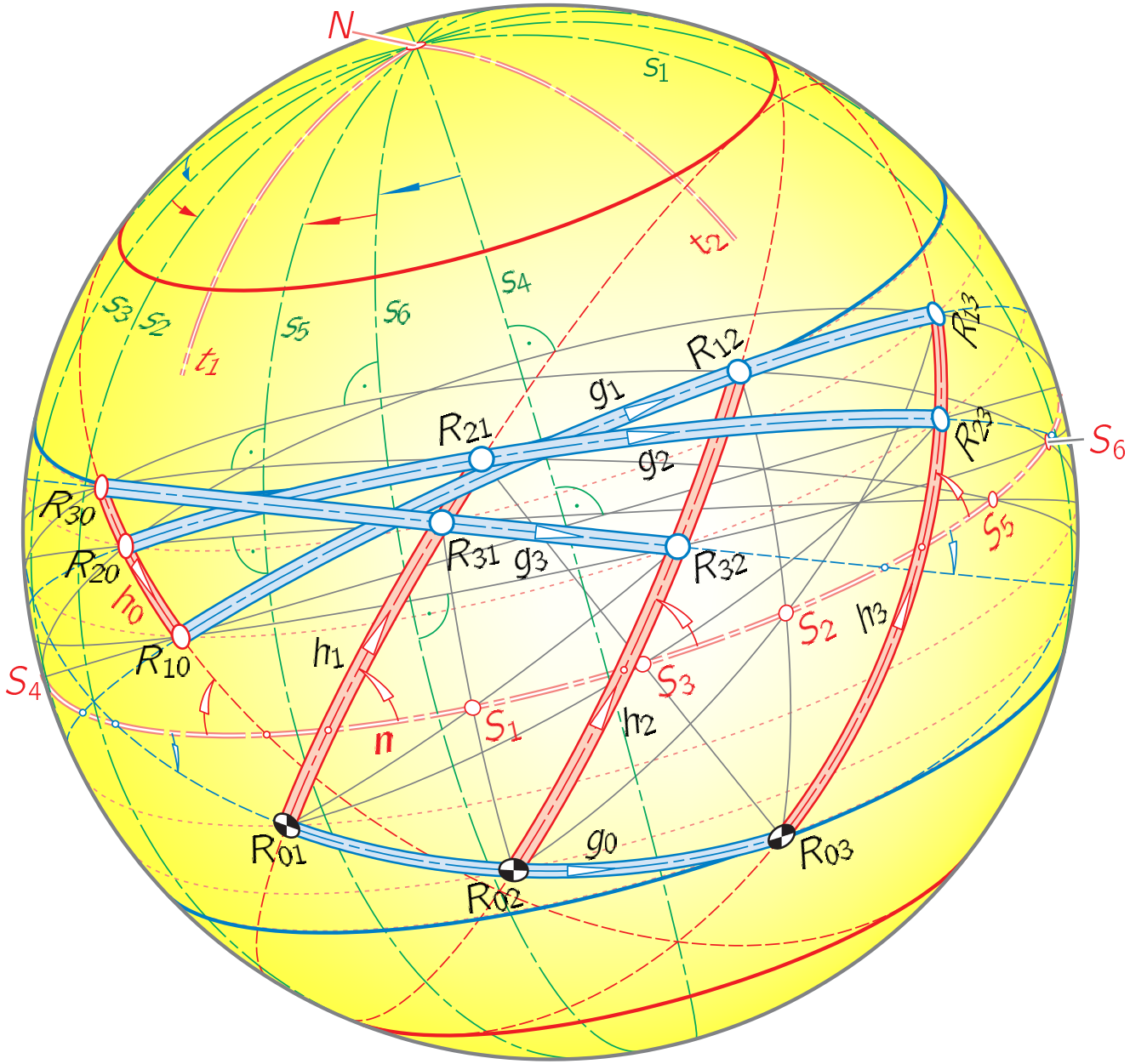
Another look to the scene:

There are also isograms with aligned bases $R_{32}R_{02}$, $R_{02}R_{12}$ on h_2 and aligned arms $R_{02}R_{01}$, $R_{02}R_{03}$ on g_0.

Again, the simultaneous movement of these two spherical isograms is compatible with a third isogram with a coupler connecting g_3 with g_1.

The new center of symmetry is again aligned with S_1 and S_2.

Finally, there are two more pairs of spherical isograms with bases h_1 and h_3 that can be extended respectively by an additional isogram. The three additional couplers are aligned on h_0, and all centers $S_1, \ldots, S_6$ are located on a great circle n.



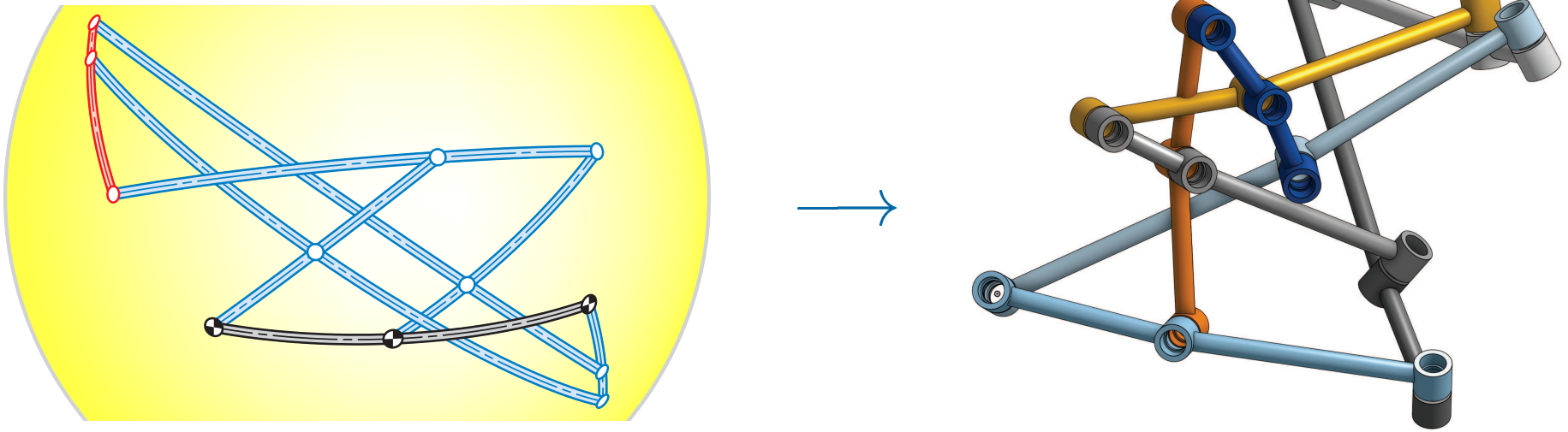
We obtain the union of 6 spherical isograms, a combinatorically symmetric **flexible linkage** with 8 bars $g_0, \ldots, h_3$ and 12 joints.

Each non-aligned pose shows several symmetries due to the 6 aligned centers of symmetry. The great circles h_i for $i = 0, \ldots, 3$ intersect the circle n through $S_1, \ldots, S_6$ under congruent angles. The same holds for all g_i .

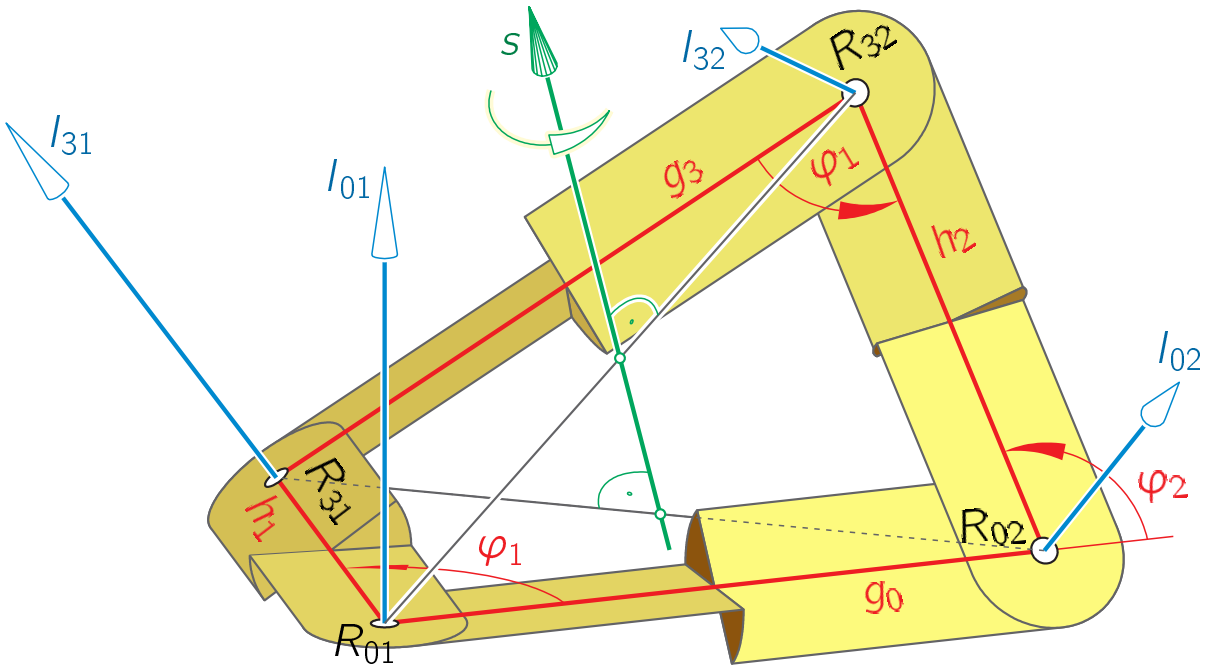
3. The spatial analogue

E. Study's principle of transference offers a possibility to transfer statements from spherical geometry to spatial geometry. It is based on the transition from quaternions to dual quaternions, i.e., from points on the sphere to oriented lines in space.

It allows to transfer Bennett's spherical linkage to a spatial linkage with eight links and twelve hinges as the union of six Bennett's isograms, first studied by E. Baker (1995) and H.-P. Schroecker et al. (2013).



3. The spatial analogue

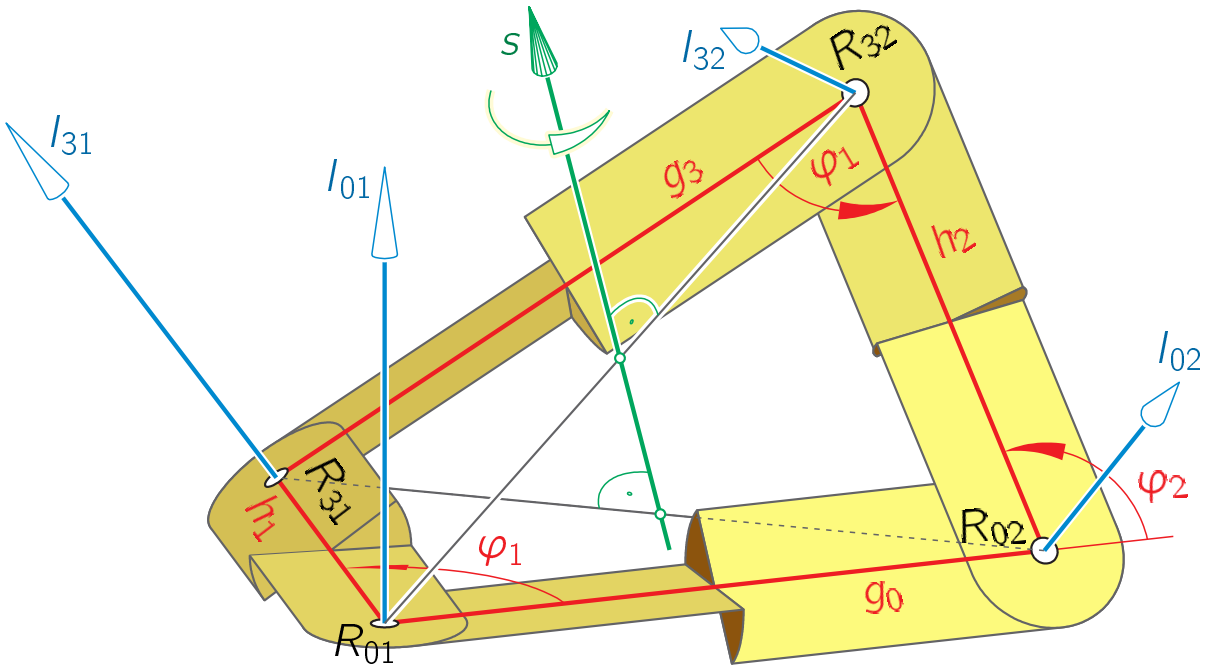


The **Bennett isogram** (1914) arises from a creased parallelogram when at each vertex the perpendicular serves as a hinge between the adjaced sides.

In each spatial pose, there exists a line s such that the reflection in s exchanges opposite vertices, hinges and angles. The line s connects the diagonals' midpoints and is the common perpendicular of the common perpendiculars between opposite sides.

The Bennett isogram is the only 4R loop which is continuously flexible.

3. The spatial analogue



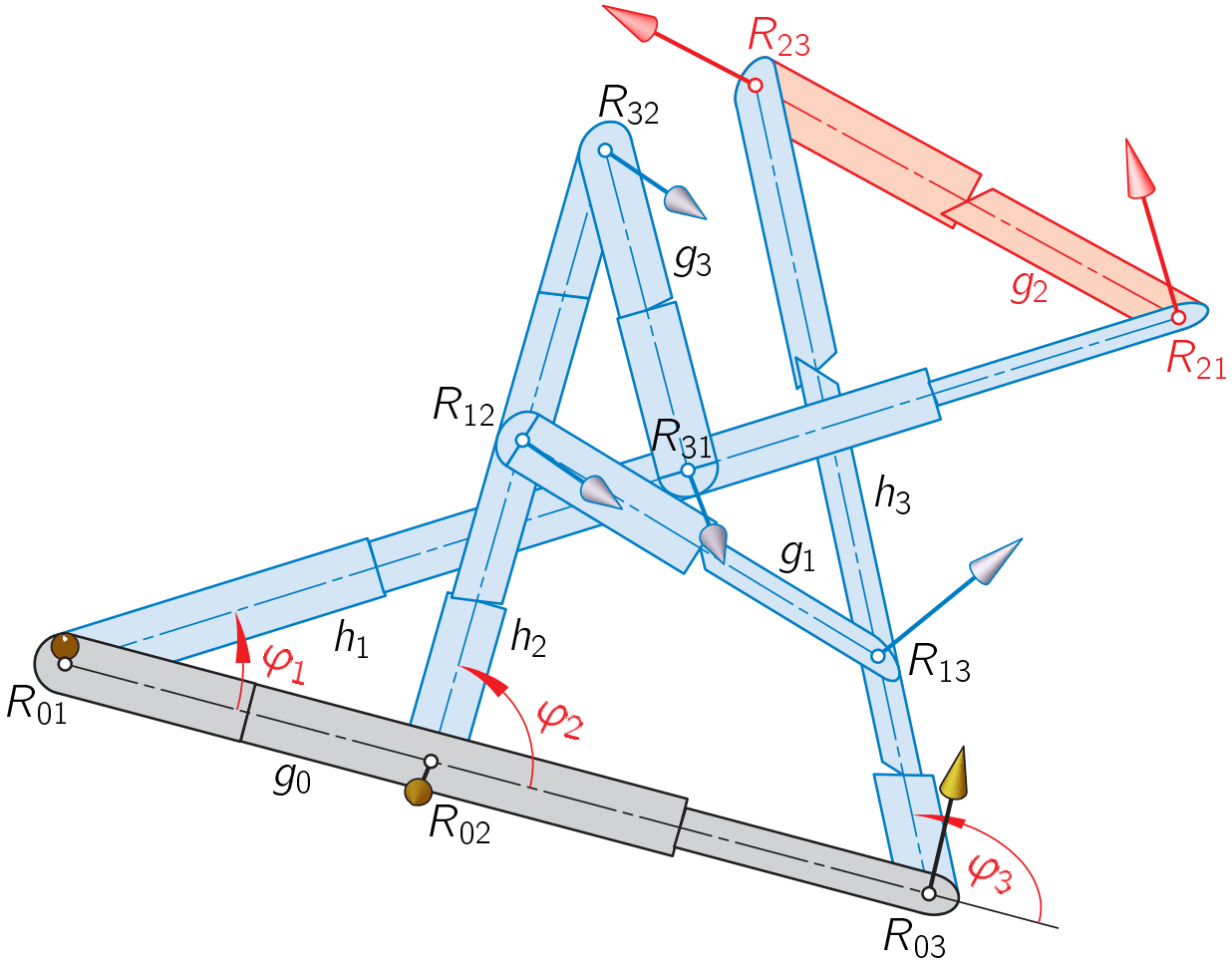
The spherical image of a Bennett isogram is a spherical isogram $\Rightarrow$ for the transmission $\varphi_1 \mapsto \varphi_2$ holds again

$$\tan\frac{\varphi_2}{2} = \frac{\sin(\alpha-\beta)}{\sin\alpha - \sin\beta}\,\tan\frac{\varphi_1}{2}$$

Basic relation of a skew isogram:

$$\overline{R_{01}R_{02}} : \overline{R_{01}R_{02}} = \sin\alpha : \sin\beta \quad \text{for} \quad \alpha := \sphericalangle\, l_{01}l_{02}, \quad \beta := \sphericalangle\, l_{02}l_{03}$$

3. The spatial analogue

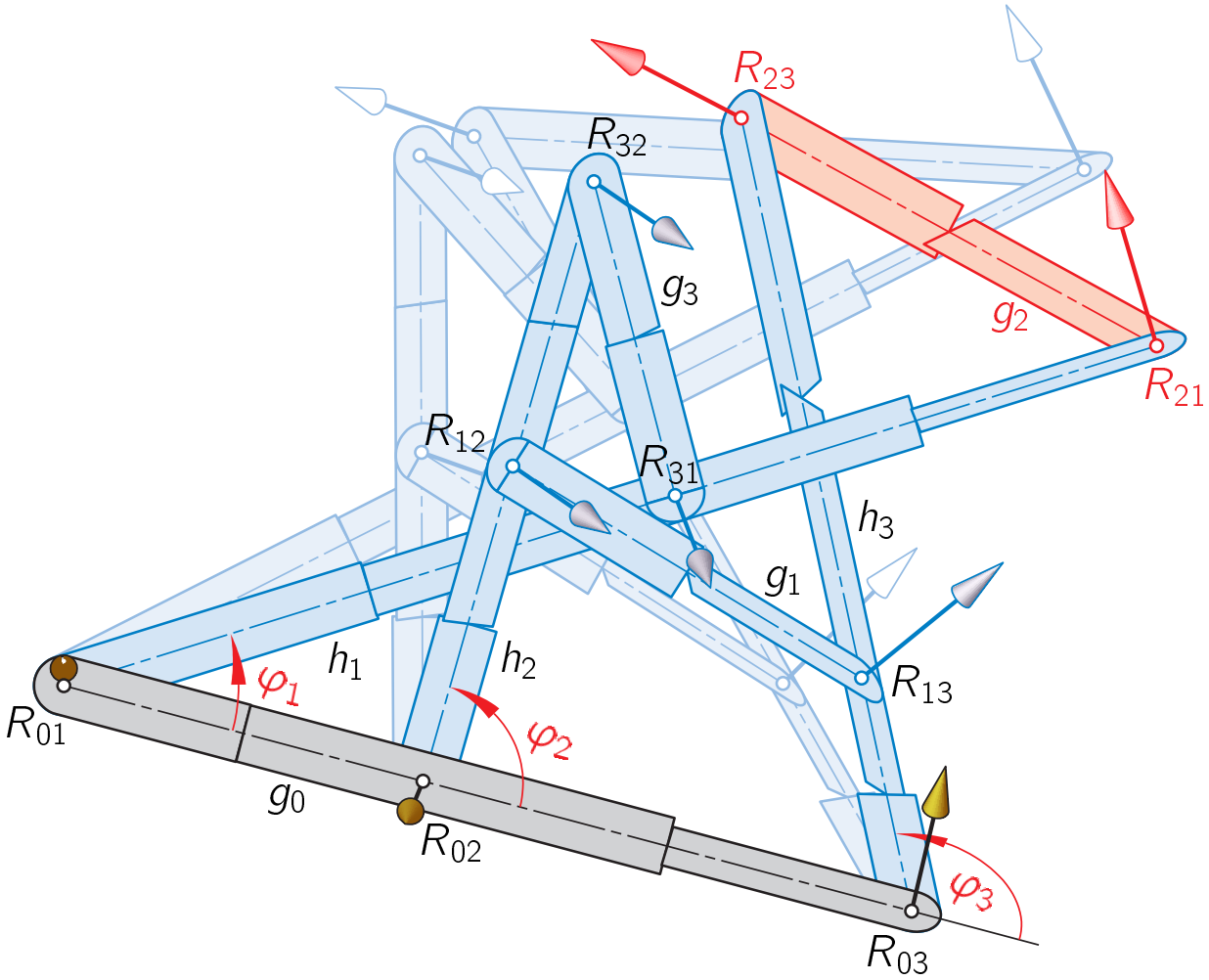


Like in the plane and on the sphere, the composition of two Bennett isograms with aligned bases $R_{01}R_{02}$, $R_{02}R_{03}$ on g_0 and arms $R_{02}R_{32}$, $R_{02}R_{12}$ on h_2 obeys

$$\tan\frac{\varphi_3}{2} = c_{32}c_{21}\tan\frac{\varphi_1}{2}\,.$$

A third Bennett isogram is compatible with the movement of the other two.

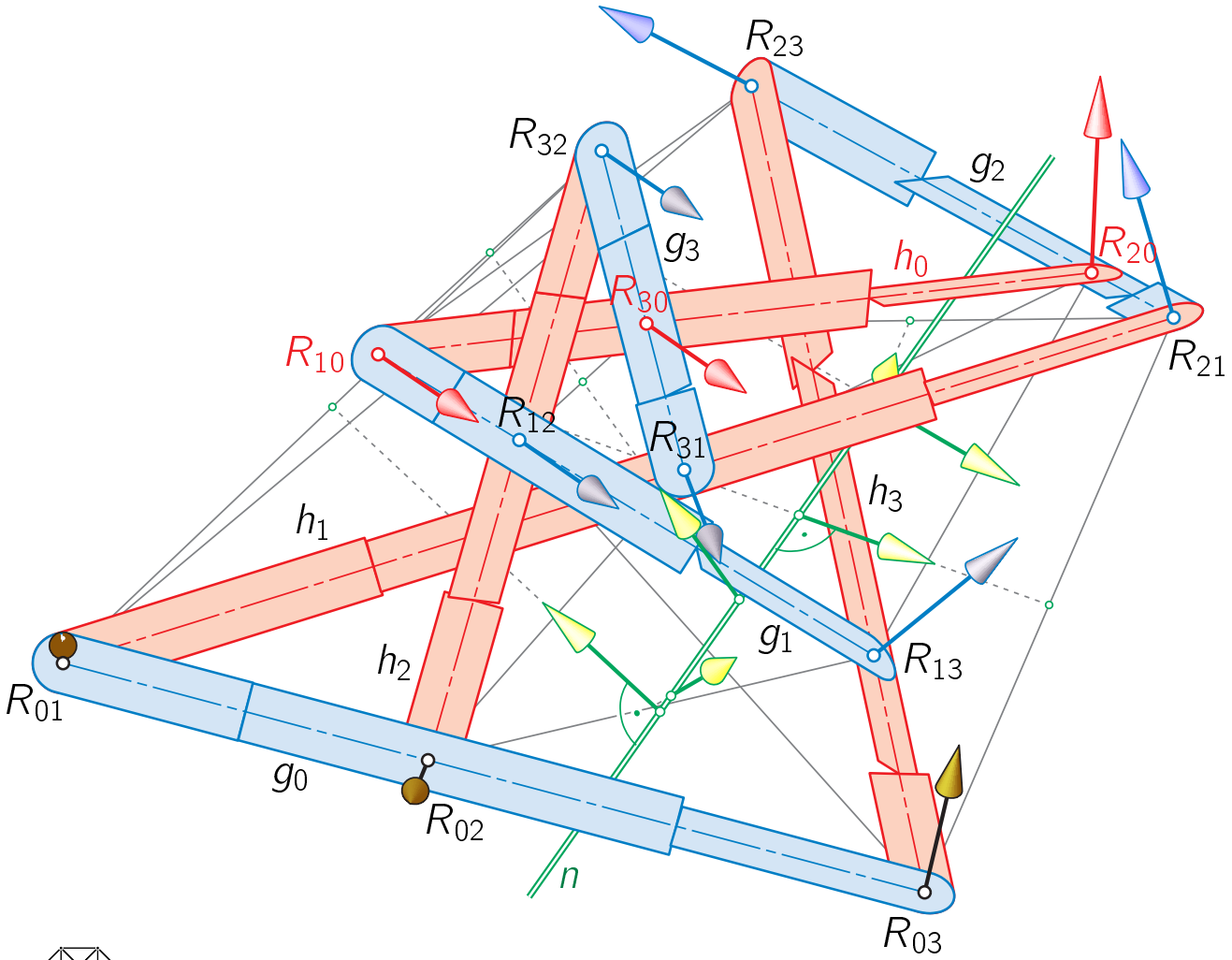
3. The spatial analogue



We obtain a flexible triple of Bennett isograms with aligned bases on g_0, arms on h_1, h_2, h_3, and couplers on g_3, g_1 and g_2.

The configuration contains three further pairs of isograms with aligned bases and arm. This yields the extension to a combinatorically symmetric spatial linkage as the union of 6 Bennett isograms.

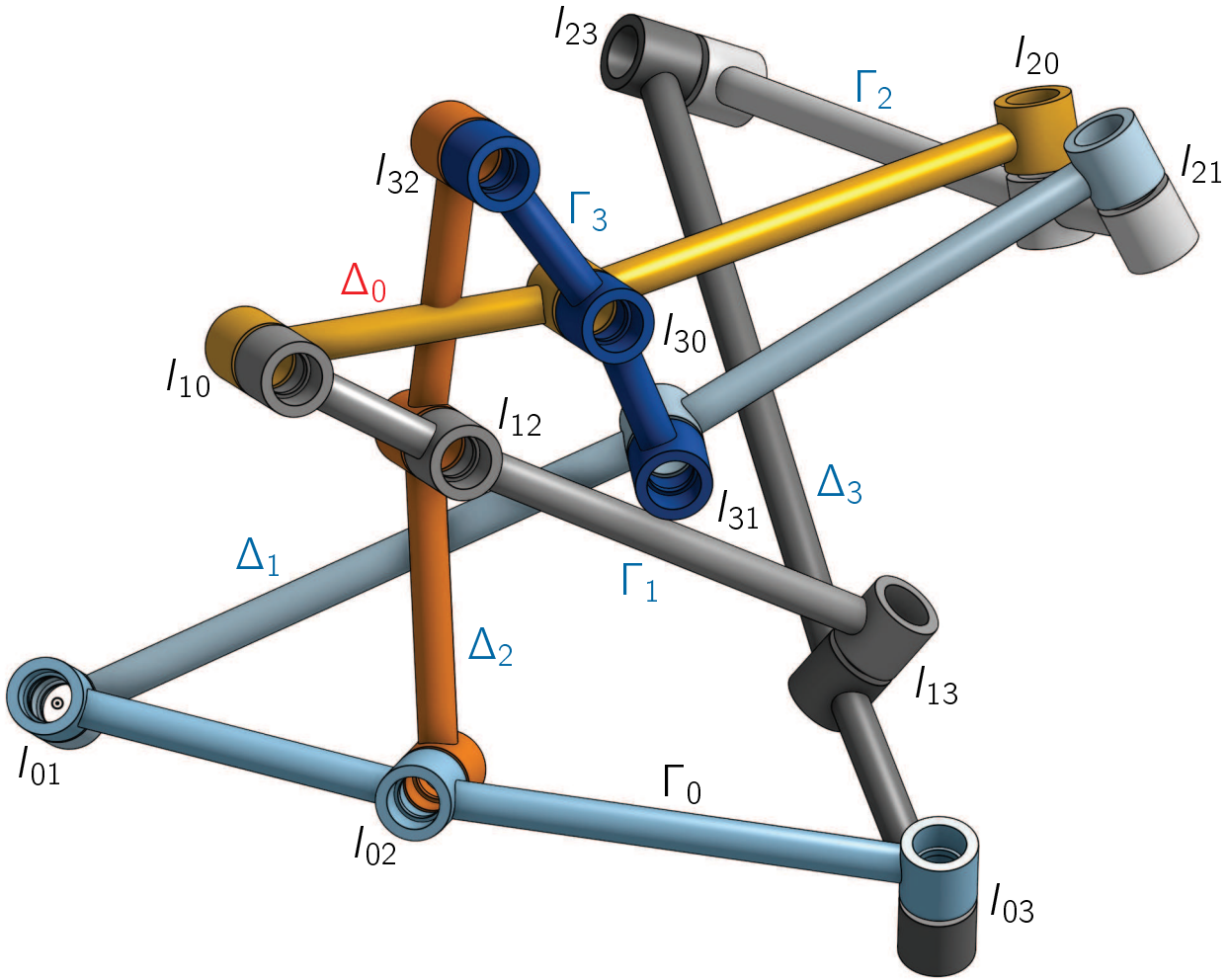
3. The spatial analogue



Each Bennett isogram has an axis of symmetry. The six axes have a common perpendicular n and are symmetrically placed along this axis.

At two poses the linkage collapses. Then the 8 lines $g_0, \ldots, h_3$ coincide.

3. The spatial analogue



Flexible linkage, composed from 6 Bennett isograms, with 8 links $\Gamma_0, \ldots, \Gamma_4$, $\Delta_0, \ldots, \Delta_4$ and 12 hinges $l_{01}, \ldots, l_{32}$; d.o.f. $= 1$.

Chebyshev – Grübler – Kutzbach mobility formula: $6 \cdot 7 - 12 \cdot 5 = \mathbf{-18}$.

left: 3D-model which avoids self-intersections, designed by D. Huczala, Seoul



Thank you for your attention!

References

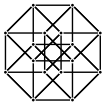
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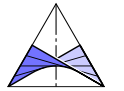
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