REMARKS ON FLEXIBLE QUAD MESHES

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1. ABSTRACT

A *quad mesh* is a discrete surface consisting of planar quadrangles. There are some examples where this polyhedral structure consisting of rigid faces, but variable dihedral angles, is continuously flexible, e.g., Miura-ori, Voss surfaces or Kokotsakis' example which starts from a regular tiling of the plane by congruent convex quadrangles. The classification of all flexible quadrangular meshes is an open problem. However, the determination of all flexible meshes of 3×3 quadrangles, the socalled *Kokotsakis meshes*, is close to be solved. The goal of this paper is to bring insight into the geometry behind some flexible examples and to analyze their flexions. The treated problems are also related to paper folding.

KEYWORDS: Paper folding, Miura-ori, Kokotsakis mesh, quadrangular surface, Descriptive Geometry.

2. INTRODUCTION

In discrete differential geometry, but also in architecture there is an interest in polyhedral structures composed from quadrilaterals, i.e., in *quadrilateral surfaces* or - by short - *quad meshes*. When all quadrilaterals are planar, the edges form a *discrete conjugate net* (see, e.g., [1]). When each quadrilateral is seen as a rigid body and only the dihedral angles can vary, the question arises under which conditions such structures are *flexible*. In the flexible case we call the process of continuous isometric deformation *folding* and the obtained polyhedral structures *flexions* of the initial quad mesh.

A complete classification of all continuously flexible quad meshes is an open problem. In [1, p. 75] the following theorem can be found: A discrete conjugate net in general position is continuously flexible if and only if all its 3×3-complexes, the so-called *Kokotsakis meshes*, are continuously flexible. In [4] an updated list of known examples of flexible Kokotsakis meshes is presented.

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In the following the geometric properties of flexions of some well-known flexible quad meshes are analyzed.

3. MIURA-ORI

Miura-ori is a Japanese folding technique named after Prof. Koryo Miura, The University of Tokyo. It is used for solar panels because it can be unfolded into its stretched flat shape by pulling on one corner only. On the other hand it is used as kernel to stiffen sandwich structures. Let us analyze the process of folding the sheet of paper depicted in Fig. 1 with given valley and mountain folds, thus proving that it is really continuously flexible.
We start with two coplanar parallelograms with aligned upper and lower sides (Fig. 2). Then we rotate the right parallelogram against the left one about the common side through the angle $2\delta \neq 0^0, \pm 180^0$.

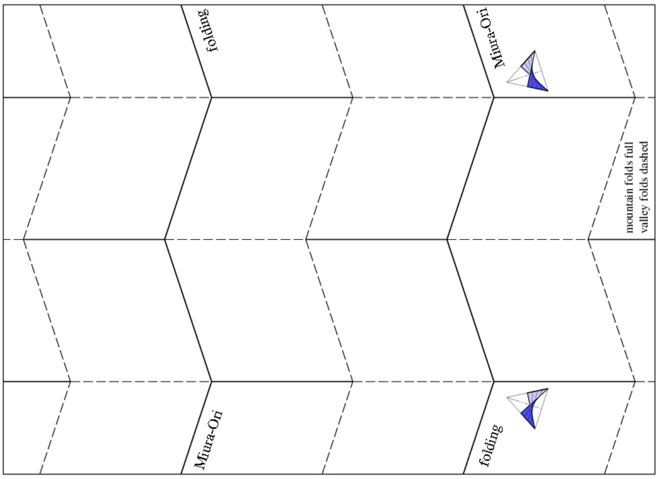

Fig. 1. The map of Miura-ori unfolded; dashs are valley folds, full lines are mountain folds

Then the lower sides span a plane ε_1 and the upper sides span a plane ε_2 parallel to ε_1. Now we extend the two parallelograms to a zig-zag strip by adding alternately parallelograms translatory congruent to the left or to the right initial parallelogram. After this the complete strip has its upper zig-zag boundary still placed in ε_1 and the lower one in ε_2 (see Fig. 2). This remains valid when we fix the plane ε_1 but vary the bending angle 2δ.

After iterated reflection in planes ε_i parallel to ε_1 or after translation orthogonal to ε_1 the complete miura-ori flexion is obtained as depicted in Fig. 3. The edges can be subdivided into two kinds of foldings. The horizontal ones are copies of the zig-zig line in ε_1 and placed in horizontal planes. Due to the iterated reflections or translations, the others are located in mutually parallel vertical planes.
The complete flexion can also be generated by iterated horizontal translations and reflections in horizontal planes from the hexagonal compound of two non-coplanar parallelograms as depicted in the top-right illustration of Fig. 2. Miura-ori admits two flat poses, one for $\delta = 0^0$, one for $2\delta = 180^0$.

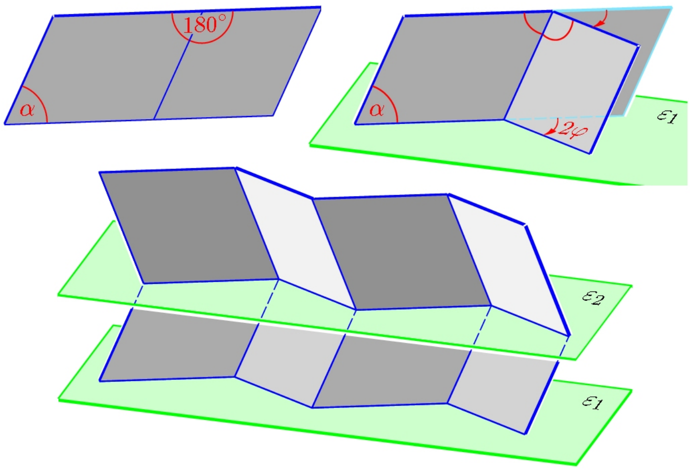

Fig. 2. Zig-zag strips of Miura-ori

Remarks: 1) In [4], the eqs. (1) and (2) together with Figure 5 show how the bending angle 2δ is related to the angle 2φ of the horizontal fold in ε_1 (see Fig. 2) and that of the vertical folds, in dependance of the interior angle α at the given parallelograms.
2) The mentioned one-parameter flexion of Miura-ori is not the only one. Trivial flexions arise, e.g., when in the stretched position we fold about the aligned horizontal folds. Or we fold adjacent horizontal strips one behind the other and treat them like one single strip.
3) There are several generations of Miura-ori, among them the impressive freeform-like versions presented in the recent paper of Tomihiro Tachi [7].

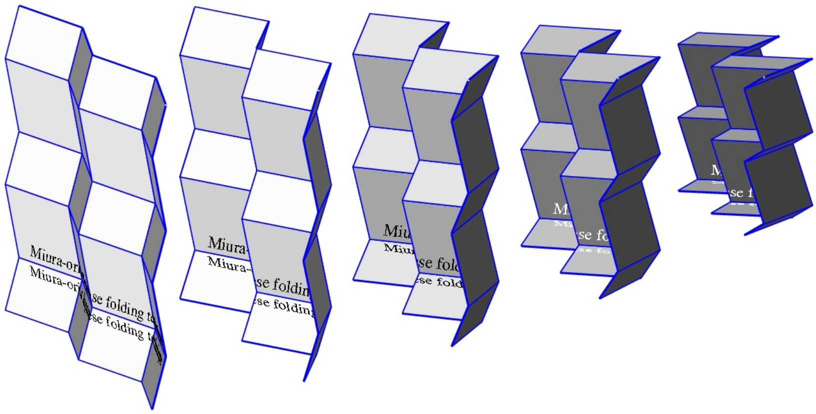

Fig. 3. Snapshots of the folding procedure

3. KOKOTSAKIS' FLEXIBLE TESSELLATION

There is one remarkable continuously flexible quad mesh which also starts from a flat initial pose. This example displayed in Fig. 4 dates back zu Antonios Kokotsakis (1899-1964) [2, p. 647]. He proved its flexibility, but did not present any geometric property of the obtained flexions:

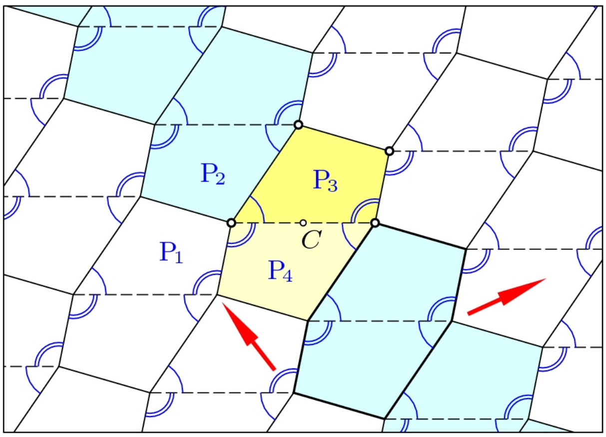

Fig. 4. Kokotsakis' flexible mesh unfolded

Take any arbitrary plane quadrangle like the yellow P_3 in Fig. 4. By iterated 180^0-rotations about the midpoints of the sides we obtain a wellknown regular tessellation of the plane. The same tessellation arises when the two adjacent quadrangles P_3 and P_4 are glued together thus forming a hexagon symmetric with respect to the center C and when this hexagon undergoes iterated translations indicated in Fig. 4 by the red arrows.
When the quadrangles are convex and seen as planar faces of a polyhedral structure with variable dihedral angles, then this structure is flexible (Kokotsakis [2], p. 647).

Proof: First we extract four pairwise congruent faces $P_1,...,P_4$ adjacent to the vertex V_1 from our tessellation (note the shaded area in Fig. 5). These faces form a four-sided pyramid with apex V_1; it is flexible, provided the fundamental quadrangle is convex. We start with any nonplanar flexion.
According to the labelling in Fig. 5, for any pair $(P_1,P_2),\dots,(P_4,P_1)$ of neighbouring faces there is a respective 180^0-rotation ρ_1, …, ρ_4 which swaps the two faces. So, e.g., $P_2 = \rho_1(P_1)$ and $P_1 = \rho_1(P_2)$. The axis of ρ_1 (see Fig. 5) is perpendicular to the common edge V_1V_2, and it is located in a plane which bisects the dihedral angle between P_1 and P_2.

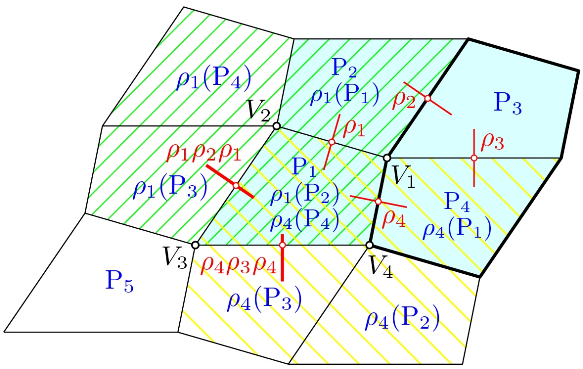

Fig. 5. The flexion is generated by iterated coaxial helical motions $\rho_1\,\rho_2$ and $\rho_1\,\rho_4$

After applying all four 180^0-rotations (= reflections in lines) $\rho_1, \ldots, \rho_4$ consecutively to the quadrangle P_1, this is mapped via P_2, P_3, P_4 onto itself, hence $\rho_4\,\rho_3\,\rho_2\,\rho_1 = \mathrm{id}$. (We indicate the composition of mappings by left multiplication.) Because of $\rho_i^{-1} = \rho_i$ we obtain

$$\rho_4\,\rho_3 = \rho_1\,\rho_2\,. \tag{1}$$

After that we extend this flexion of the pyramid stepwise by adding congruent copies of the initial pyramid without restricting the flexibility:
The rotation ρ_1 exchanges not only P_1 with P_2 but transforms the pyramid with apex V_1 onto a congruent copy with apex V_2 sharing two faces with its preimage. This is the area hatched in green in Fig. 5. Analogously, ρ_4 generates a pyramid (hatched in yellow) with apex V_4 sharing the faces P_1 and P_4 with the initial pyramid.
Finally there are two ways to generate a pyramid with apex V_3. Either, we transform ρ_2 by ρ_1 and use

- $\rho_1\,\rho_2\,\rho_1$, which exchanges $\rho_1(\mathrm{P}_2) = \mathrm{P}_1$ with $\rho_1(\mathrm{P}_3)$ and swaps V_2 and V_3.

Or we proceed with

- $\rho_4\,\rho_3\,\rho_4$, which exchanges $\rho_4(\mathrm{P}_4) = \mathrm{P}_1$ with $\rho_4(\mathrm{P}_3)$ and swaps V_4 and V_3.

Thus we obtain mappings $(\rho_1\,\rho_2\,\rho_1)\rho_1 = \rho_1\,\rho_2$ and $(\rho_4\,\rho_3\,\rho_4)\rho_4 = \rho_4\,\rho_3$ with $V_1 \mapsto V_3$ and $\mathrm{P}_1 \mapsto \mathrm{P}_5$. Both displacements are equal by (1), and we notice

$$\rho_1\,\rho_2 = \rho_4\,\rho_3:\ \ \mathrm{P}_1 \mapsto \mathrm{P}_5\,,\ \mathrm{P}_2 \mapsto \rho_1(\mathrm{P}_3)\,,\ \mathrm{P}_3 \mapsto \mathrm{P}_1\,,\ \mathrm{P}_4 \mapsto \rho_4(\mathrm{P}_3)\,. \tag{2}$$

Hence each flexion of the initial pyramid with apex V_1 is compatible with a flexion of the compound of 3×3 quadrangles like that schematically displayed in Fig. 5, and this can be extended to the complete tessellation. Hence this is continuously flexible. □

On the other hand, $\rho_1\,\rho_4 = \rho_2\,\rho_3$ maps the pyramide with apex V_4 onto that with apex V_2 and

$$\rho_1\,\rho_4:\ \mathrm{P}_1 \mapsto \rho_1(\mathrm{P}_4)\,,\ \mathrm{P}_4 \mapsto \mathrm{P}_2\,,\ \rho_4(\mathrm{P}_2) \mapsto \mathrm{P}_1\,,\ \rho_4(\mathrm{P}_3) \mapsto \rho_1(\mathrm{P}_3)\,. \tag{3}$$

The product of two reflections in lines is a *helical motion*. Its axis is the common perpendicular of the two axes of reflections. The angle of rotation and the length of translation is twice the angle and distance, respectively, between the two axes. When the pyramid with apex V_1 is not flat, then the axes of the line-reflections $\rho_1, \ldots, \rho_4$ are pairwise skew; the common perpendicular for any two of these axes is unique. Hence (1) implies that the axes of the four line-reflections have a common perpendicular n. The motions $\rho_1\,\rho_2 = \rho_4\,\rho_3$ and $\rho_1\,\rho_4 = \rho_2\,\rho_3$ are *helical motions with the common axis* n.

When P_3 and P_4 are glued together, we obtain a line-symmetric skew hexagon, one half of our initial pyramid with apex V_1. The line-reflection ρ_4 maps this hexagon onto itself. By (2), the helical motion $\rho_1\,\rho_2$ maps this hexagon onto the compound of P_1 and $\rho_4(\mathrm{P}_3)$, and furthermore P_1 onto P_5. The inverse $\rho_2\rho_1$ is the spatial analogon of the translation indicated in Fig. 4 by the red arrow pointing upwards to the right. On the other hand, $\rho_4\rho_1$ maps the compound of P_1 and $\rho_4(\mathrm{P}_3)$ onto $\rho_1(\mathrm{P}_4)$ and $\rho_1(\mathrm{P}_3)$. This is the spatial analogon of the second generating translation of the flat tessellation.

Theorem: *Any flexion of the Kokotsakis' mesh is obtained from the line-symmetric hexagon consisting of the two planar quadrangles* P_3 *and* P_4 *by applying the discrete group of coaxial helical motions generated by* $\rho_1\,\rho_2$ *and* $\rho_1\,\rho_4$. *In the flat pose these generating motions are the translations applied to a centrally symmetric hexagon thus generating the regular tessellation of the plane.*

<u>Remark:</u> The flat initial pose of the pyramid with apex V_1 admits a bifurcation between two differentiable constraint motions of the pyramid. Hence the complete mesh admits two differentiable floldings when starting from the planar tessellation. In the case of a trapezoid P_1 one type of folding results in prismatic flexions and is therefore trivial.

4. AN EXAMPLE DUE TO SAUER AND GRAF

In order to obtain the flexible *T*-flat (German: *T*-Flach) detected 1931 by Sauer and Graf [3], we start with a non-closed prism Φ with horizontal generators e_0, …, e_n as depicted in Fig. 6 in top view and side view (on the left hand side). This prism is terminated by two vertical planes ε_1 and ε_2, hence its faces are trapezoids. Let α_1, …, α_n denote the angles of inclination of the faces $e_{i-1}e_i$ for $i = 0,...,n$ under $0 < \alpha_i < 90^0$.

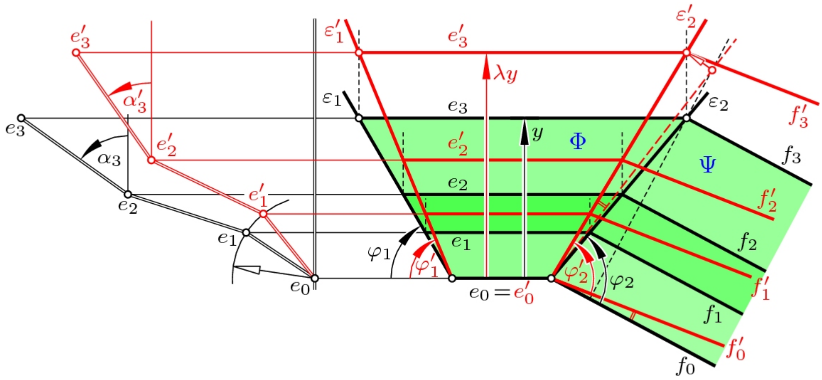

Fig. 6. *T*-flat together with a flexion (in red)

Now we apply a folding $e_i \mapsto e'_i$, $i = 0,...,n$, to Φ such that the generators remain horizontal and the top view undergoes an axial dilatation perpendicular to the generators with a factor λ close to 1. In order to transform all faces isometrically, the angles α_i of inclinations of the faces must be replaced by α'_i obeying

$$\cos\alpha'_i = \lambda \cos\alpha_i \,, \tag{4}$$

provided $0 \le \lambda\cos\alpha_i \le 1$ for all i. The vertical planes ε_j remain vertical as the top view undergoes an affine transformation. The angle φ_1 between e_0 and ε_1, e.g., is replaced by φ'_1 obeying

$$\tan\varphi'_1 = \lambda \tan\varphi_1 \,. \tag{5}$$

For the polygonal sections $\Phi \cap \varepsilon_1$ each side length is preserved. The aligned top view of these polygons undergoes a scaling with factor $\cos\varphi_1 / \cos\varphi'_1$; therefore all ratios are preserved.

Suppose, $\Phi \cap \varepsilon_2$ is a section of another prism Ψ with horizontal generators f_0, …, f_n (see Fig. 6). Then the folding of Φ implies a folding of Ψ

under which in the top view the generators again are transformed by an axial dilation with a certain factor. Iteration gives a flexible quad mesh called *T-flat* with trapezoidal faces only.

Remark: In [5] the flexibility of T-flats is proved by checking the spherical image of the included 3×3-meshes (see [5], Fig. 7).

5. CONCLUSIONS

This paper focuses on three well-known examples of continuously flexible quad meshes and analyzes their geometric background. These examples are

- Miura-ori (Fig. 1),
- Kokotsakis' flexible quad meshes [2] starting from plane tessellations by congruent convex quadrangles (Fig. 4), and
- *T*-flats due to Sauer and Graf [3] displayed in Fig. 6 with horizontal and vertical folds and trapezoidal faces only.

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