Two examples of curved foldings

Hellmuth Stachel

TECHNISCHE UNIVERSITÄT WIEN

stachel@dmg.tuwien.ac.at — http://www.geometrie.tuwien.ac.at/stachel

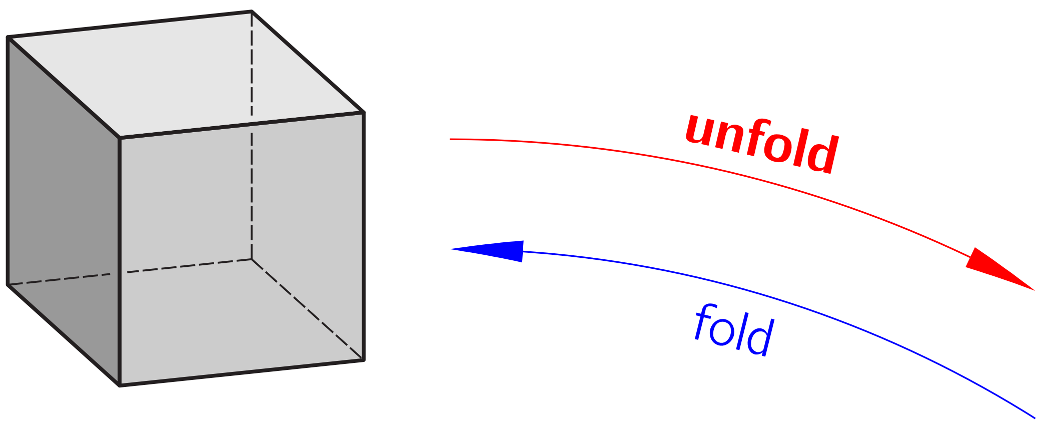
17th International Conference on Geometry and Graphics, **ICGG 2016**
August 4–8, Beijing Institute of Technology

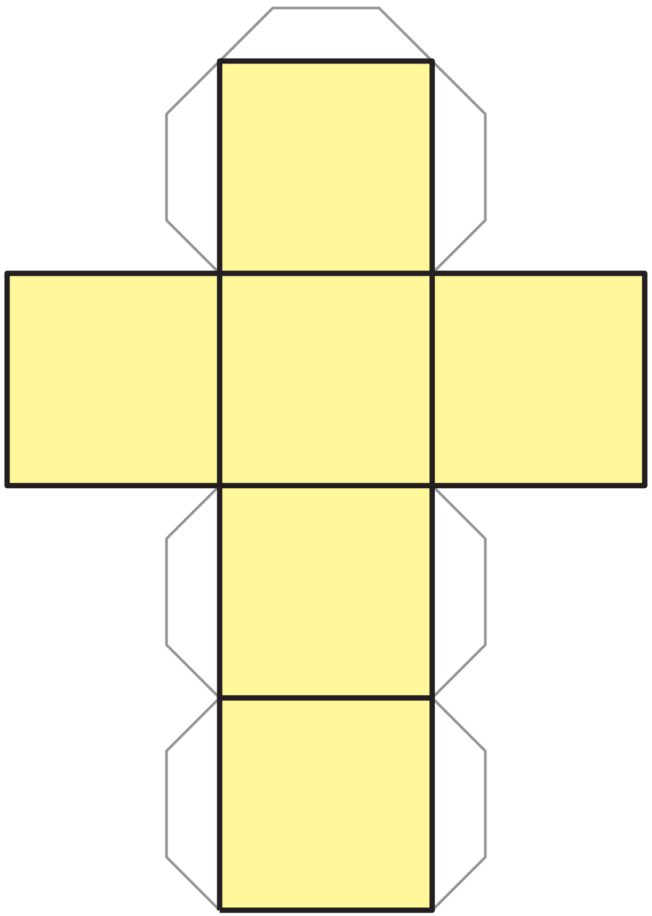
Table of contents

Funding source:

Partly supported by the **Austrian Academy of Sciences** ÖAW

1. Unfolding and folding polyhedra

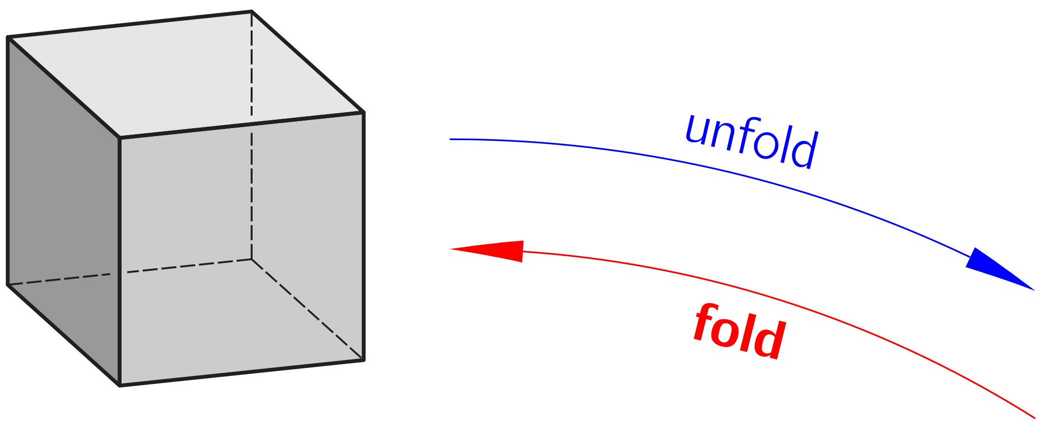


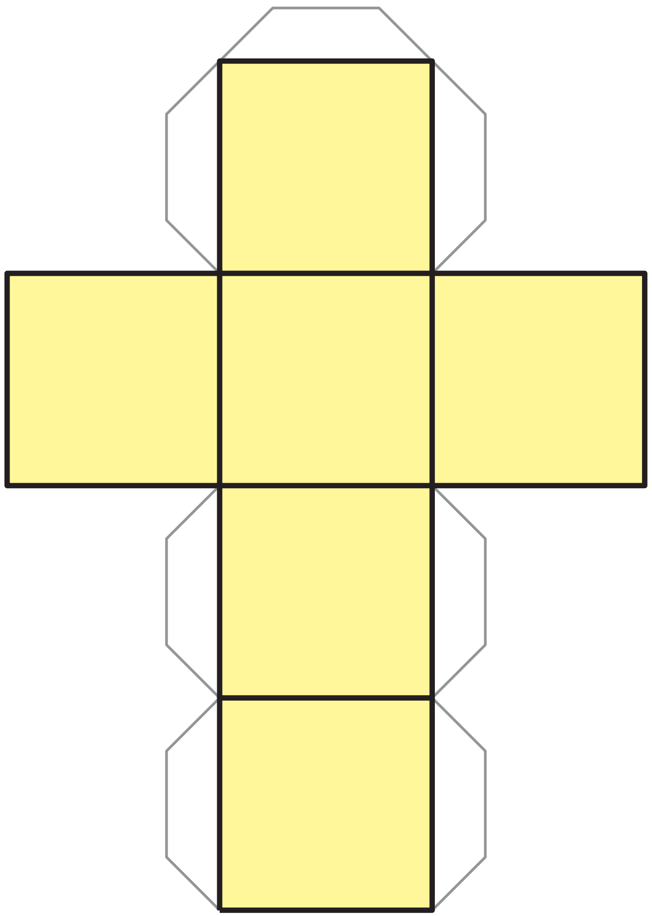


In Descriptive Geometry standard procedures are provided for the construction of the **unfolding** (development, net) of polyhedra or developable surfaces.

The result is unique, apart from the placement of the different components, and it shows the intrinsic metric of the spatial structure.

1. Unfolding and folding polyhedra





The inverse problem, i.e., the determination of a folded structure from a given unfolding is more complex. In the smooth case we obtain a continuum of bent poses.

In the polyhedral case the computation leads to a system of algebraic equations. Also here the corresponding spatial object needs not be unique.

1. Unfolding and folding polyhedra

Only if the polyhedron bounds a **convex** solid then the result is unique, due to Aleksandr Danilovich **Alexandrov** (1941).

In this case, for each vertex the sum of intrinsic angles for all adjacent surfaces is $< 360^\circ$ (= convex intrinsic metric).

Theorem: [Uniqueness Theorem]
For any convex intrinsic metric there is a unique convex polyhedron.

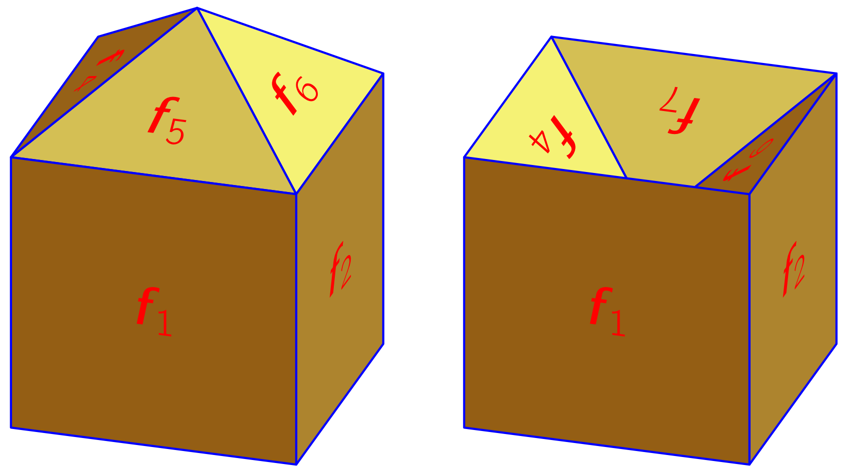
A.I. **Bobenko** and I. **Izmestiev** (2006) developed an algorithm for constructing the convex polyhedron with given intrinsic metric.

1. Unfolding and folding polyhedra

If convexity is not required the unfolding of a polyhedron needs not define its spatial shape uniquely !

Definition 1: A polyhedron is called **globally rigid** if its intrinsic metric defines its spatial form uniquely — up to movements in space.

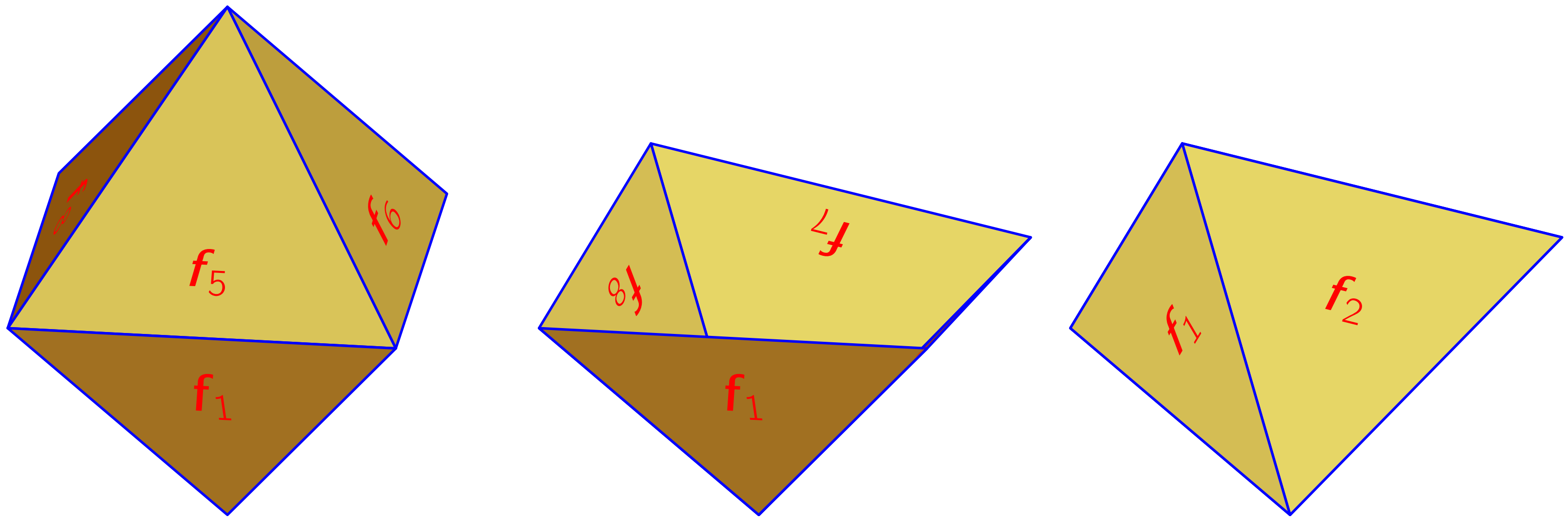
e.g., a tetrahedron



A **flipping** (or snapping) polyhedron admits two sufficiently close realizations – by applying a slight force.

1. Unfolding and folding polyhedra

Definition 2: A polyhedron is called **(continuously) flexible** if there is a *continuous family* of mutually incongruent polyhedra sharing the intrinsic metric. Each member of this family is called a **flexion**.



Even a regular octahedron is flexible — after being re-assembled. The regular pose on the left hand side is called **locally rigid**.

2. Curved folding, Example 1

A common way of producing small boxes is to push up appropriate planar cardbord forms Φ_0 with prepared creases. Below the case of creases along circular arcs c_0.

planar version with circular creases

corresponding box with planar creases

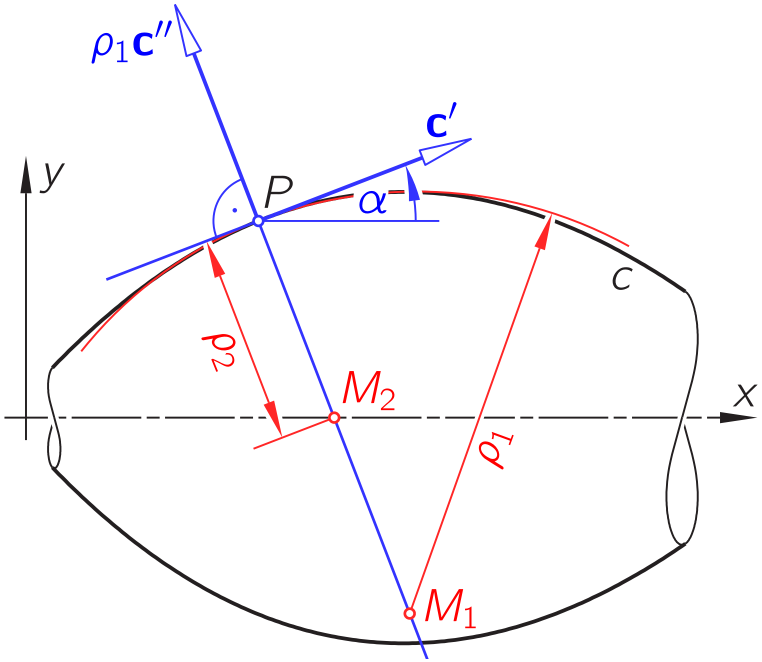
2. Curved folding, Example 1

As proved by **W. Wunderlich** (1958), the spatial creases c are again planar and known as meridians of surfaces of revolution with constant Gaussian curvature.

planar version with circular creases

corresponding box with planar creases

2. Curved folding, Example 1



On surfaces of revolution the meridians and parallel circles are the principal curvature lines. Therefore, the signed principal curvatures are

$$\kappa_1 = -\frac{y''}{\cos\alpha}, \quad \kappa_2 = \frac{\cos\alpha}{y}.$$

The Gaussian curvature is defined as $K = \kappa_1 \kappa_2$. Hence,

$$K = \text{const.} \iff$$
$$y'' + Ky = 0, \; x' = \sqrt{1 - y'^2}.$$

provided that $\cos\alpha \neq 0$.

2. Curved folding, Example 1

The general solution of $y'' + Ky = 0$ with constant $K \neq 0$ is

for $K > 0$:

$$y = a \cos s\sqrt{K} + b \sin s\sqrt{K} \,,$$

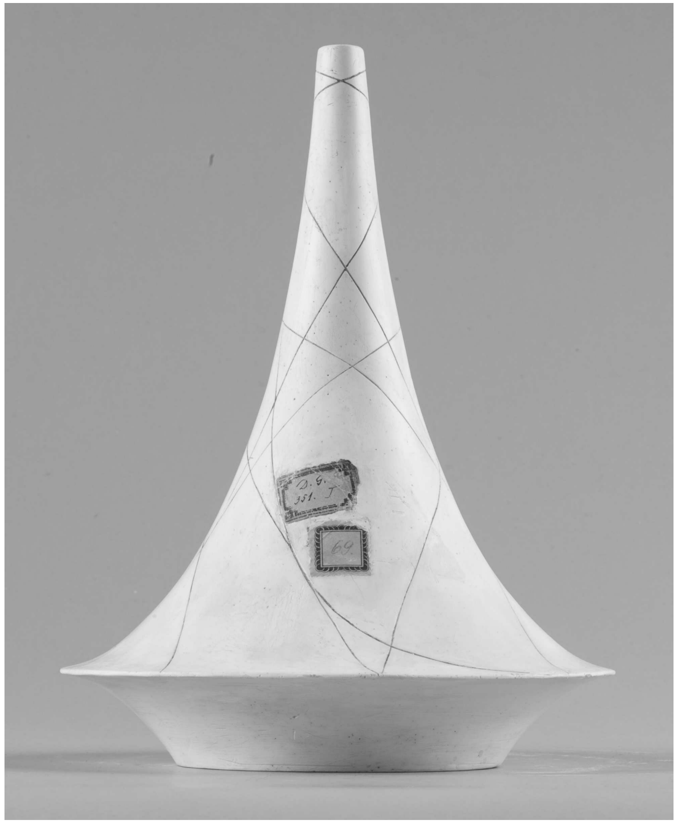
for $K < 0$:

$$y = a \cosh s\sqrt{-K} + b \sinh s\sqrt{-K}$$

with constants $a, b \in \mathbb{R}$, and

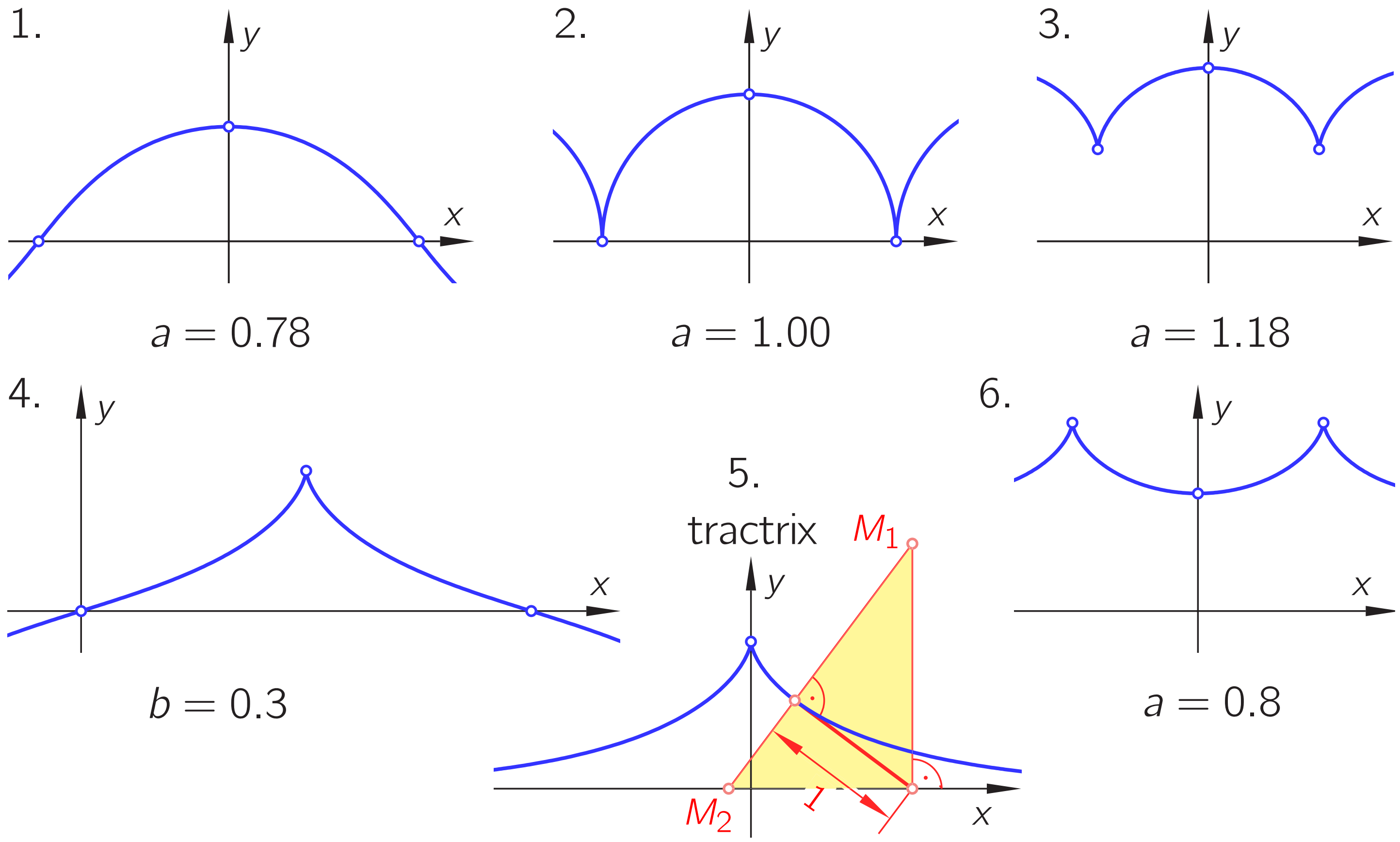
$$x = \int \sqrt{1 - y'^2}\, ds.$$

we can restrict to six cases, up to similarities (Gauß, Minding).



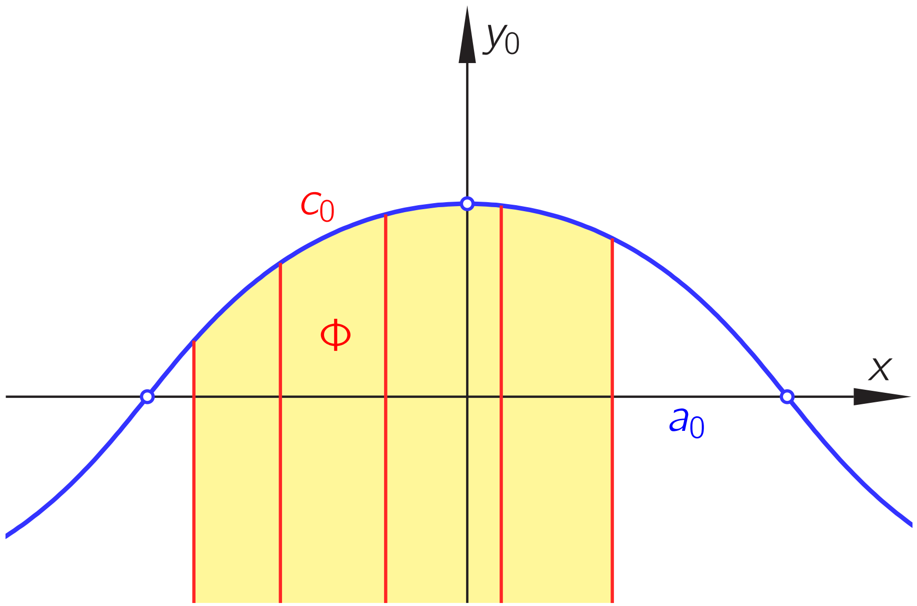
Pseudosphere (tractroid)

2. Curved folding, Example 1



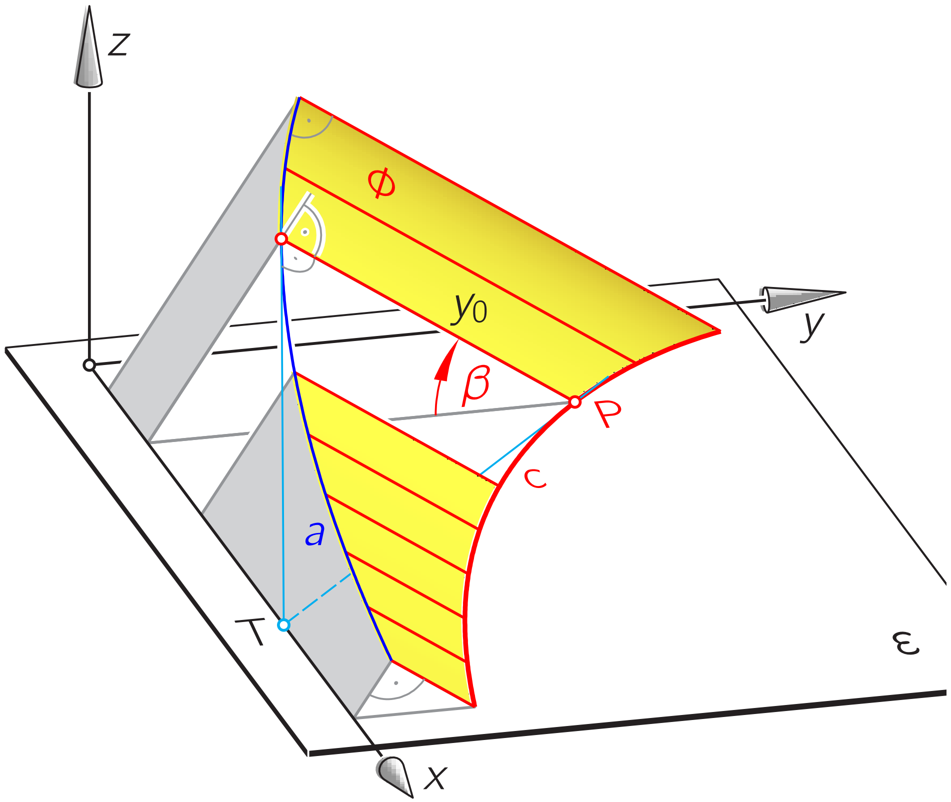
There are six types of meridians to distinguish at the surfaces of revolution with constant Gaussian curvature $K \neq 0$.

2. Curved folding, Example 1



We use such a meridian as the boundary c_0 of a flat cylindrical patch Φ_0 with generators orthogonal to the x-axis. Then we bend it such that the border c remains planar.

2. Curved folding, Example 1



Let c_0 satisfy $y_0'' + K y_0 = 0$ and bound a cylindrical patch Φ_0 with generators orthogonal to the x-axis a_0.

Theorem: If in a cylindrically bent pose Φ of Φ_0 the boundary c lies in a plane ε, then it satisfies the same differential equation as c_0.

Proof: $y_0(s) = y(s)\cos\beta$ with $\beta < \pi/2$ being the (constant) angle of inclination of the cylinder.

The axis of c is the meet of ε and the plane of the orthogonal section a, which is the bent counterpart of the original axis a_0 of $c_0 \Longrightarrow y'' + Ky = 0$.

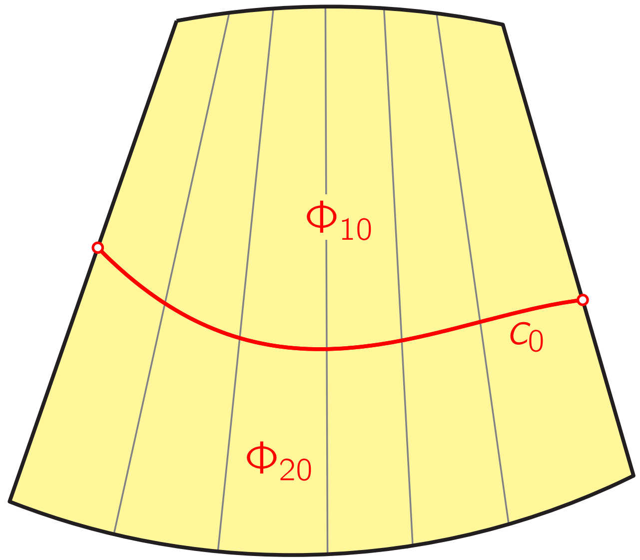
2. Curved folding, Example 1

Theorem:

Let Φ_0 be a planar 'ruled surface' with a transversal curve (crease) c_0, which separates Φ_0 into two patches Φ_{10} and Φ_{20}.

Suppose the generators of the ruling remain straight at the bent pose Φ_1, Φ_2 with a curved edge c between. Then c must be a planar curve.

If all generators of Φ_1 and Φ_2 are extended to infinity, we obtain two torses, which are symmetric with respect to the plane of c.



E.g., take a cone of revolution with a parabolic section c and reflect the part opposite to the apex in the plane of c. In Origami this is called reflection operation.

2. Curved folding, Example 1

Sketch of the *Proof:*

Let $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of c. In terms of the angle $\gamma_1(s)$ between the osculating plane of c and the tangent plane of the torse Φ_1, the geodesic curvature of c w.r.t. Φ_1 is

$$\kappa_g = \kappa \cos \gamma_1 .$$

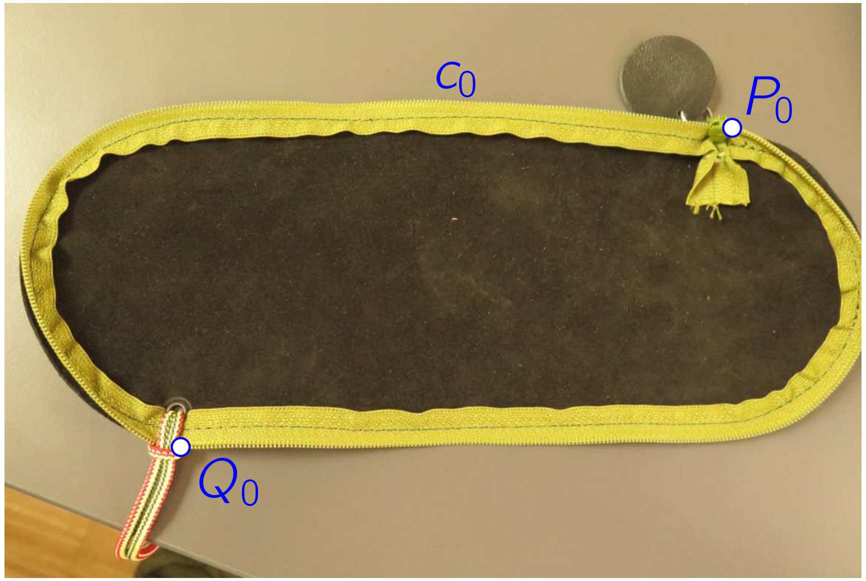
The geodesic curvature κ_g must be the same w.r.t. $\Phi_2 \implies \gamma_2 = -\gamma_1$.

The angle α between the tangent of c and the generator of Φ_1 satisfies

$$\cos \alpha : \sin \alpha = (\tau - \gamma_1') : -\kappa \sin \gamma_1 .$$

The angle α must be the same w.r.t. $\Phi_2 \implies (\tau - \gamma_1') = -(\tau + \gamma_1')$, hence $\tau = 0$.

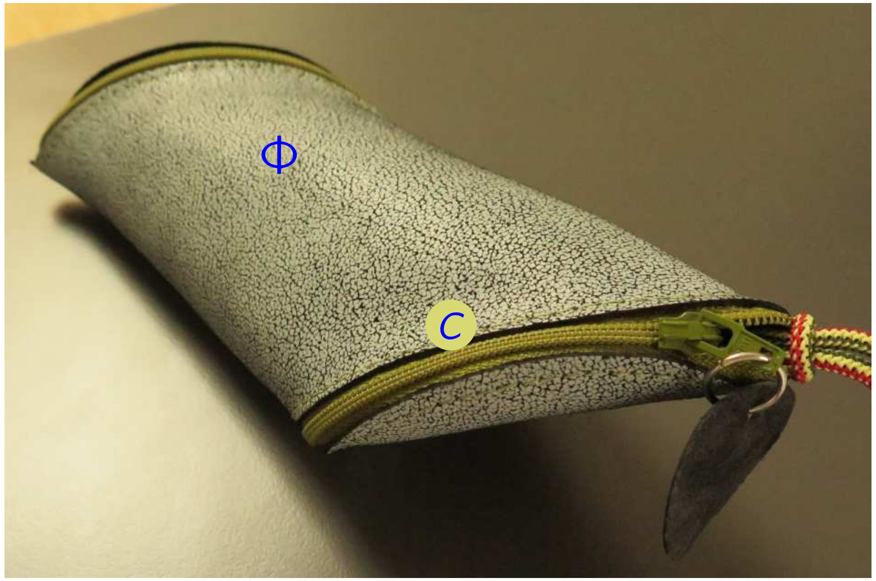
3. Curved folding, Example 2



Unfolding and corresponding spatial form (photos: **G. Glaeser**)

The spatial form Φ is obtained by gluing together the semicircles with the straight segments. How to model the resulting convex body ?

3. Curved folding, Example 2

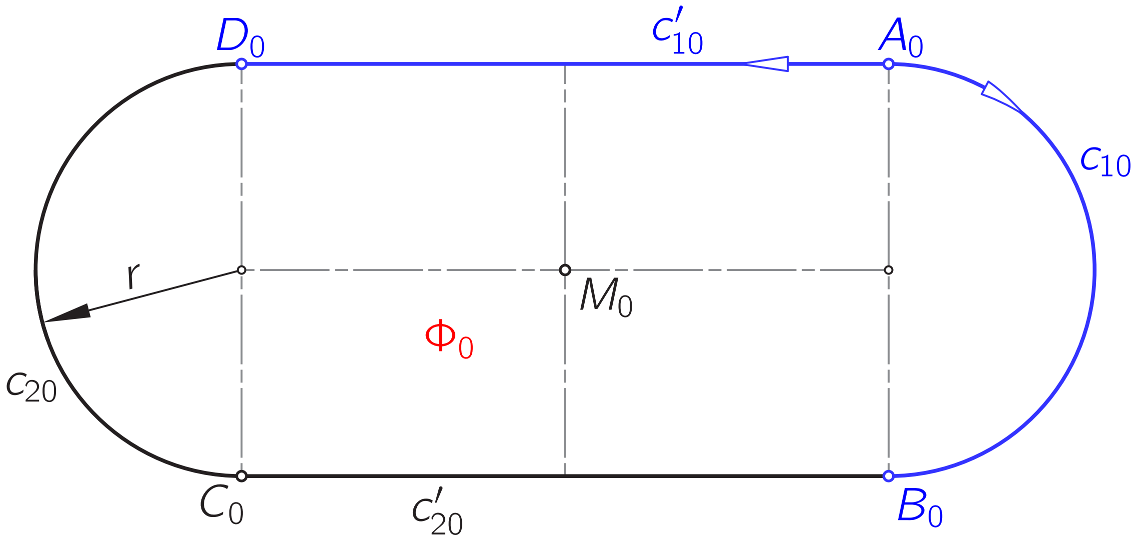


Unfolding and corresponding spatial form (photos: **G. Glaeser**)

The crucial point is here that the ruling is **unknown.**

M. Kilian, S. Flöry, Z. Chen, N.J. Mitra, A. Sheffer, H. Pottmann: *Curved Folding.* ACM Trans. Graphics **27**/3 (2008), Proc. SIGGRAPH 2008.

3. Curved folding, Example 2



A **physical model** shows:

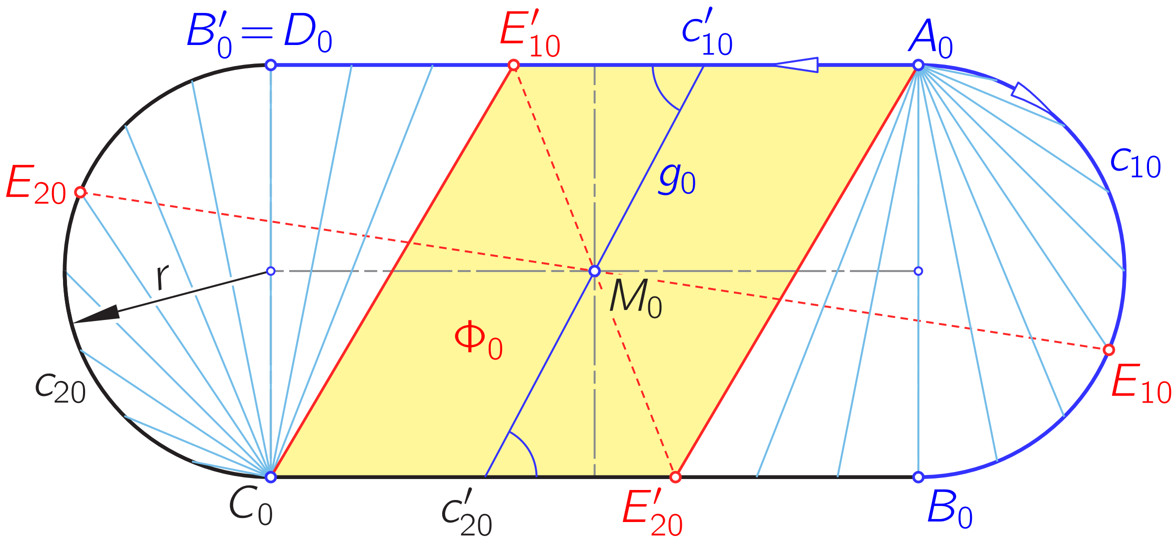
- The spatial body with its developable boundary Φ is convex and uniquely defined.

- The helix-like curve $c = c_1 \cup c_2$ is a proper edge of Φ; the resulting solid is the convex hull of c.

- The semicircular disks are bent to cones with apices A and C. Hence, Φ is a C^1-compound of two cones and a torse between.

- The body has an axis a of symmetry which connects the midpoint M with the remaining transition point $B = D$ on c.

3. Curved folding, Example 2



Consequences:

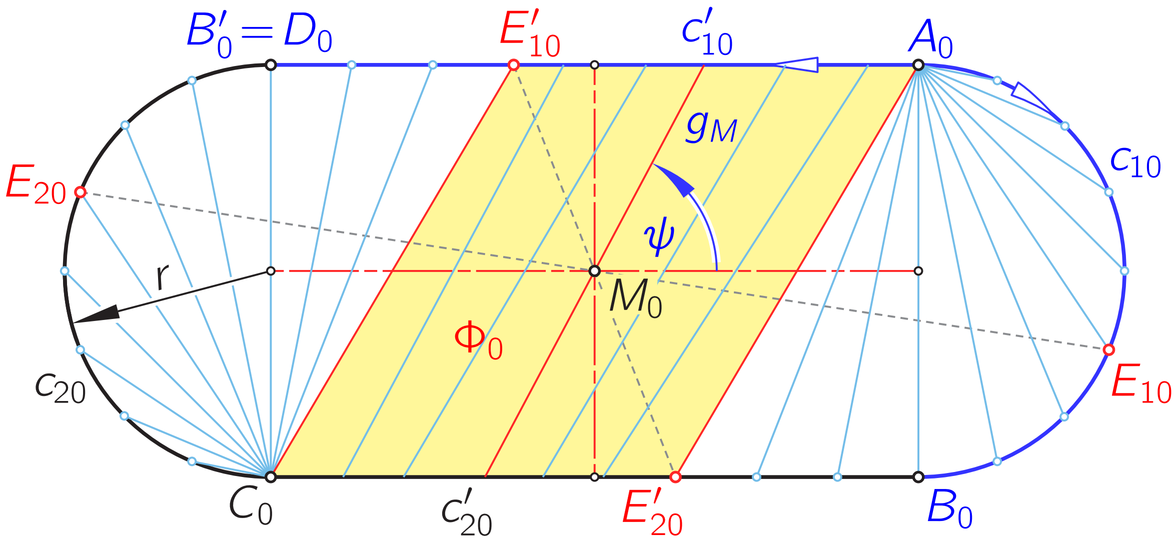
- Because of the straight segments of c_{10}, the developable surface on the left hand side of c_1 belongs to the rectifying torse of c_1.

- At A and C the surface Φ can be approximated by a right cone with apex angle 60°.

- The tangent t_A to c_1 at A is a generator, the osculating plane of c_1 a tangent plane of this cone; the rectifying plane passes through the cone's axis.

- When g_0 meets both straight sides of c_0, then g meets c_1 and c_2 at points with parallel tangents $\Longrightarrow$ coinciding tangent indicatrices.

3. Curved folding, Example 2



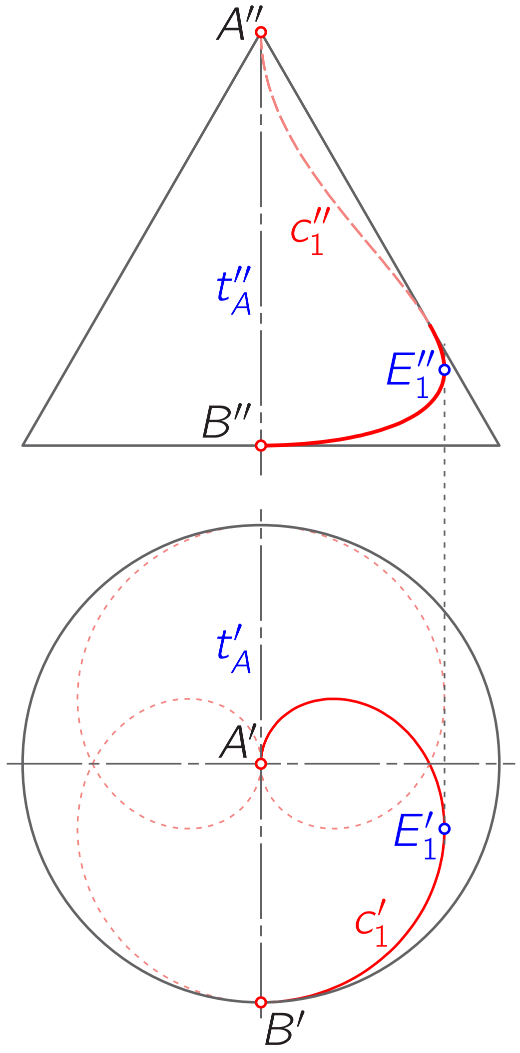
- The tangent at the point $E_2 \in c_2$ of transition between the cone with apex A and the torse must be parallel to t_A.

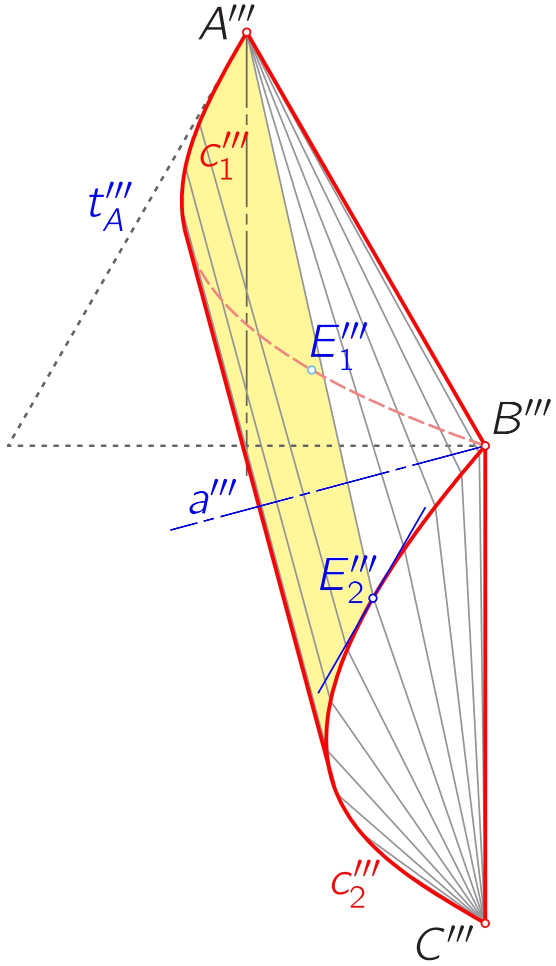
- The tangent at the analogue point $E_1 \in c_1$ is parallel to the final tangent t_C of c_2.

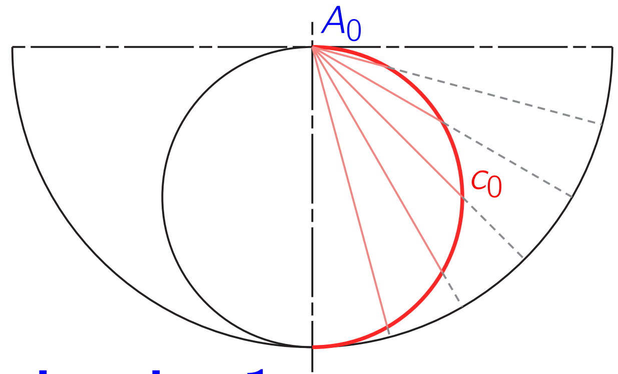
- The subcurves $AE_1 \subset c_1$ and $E_2C \subset c_2$ have conciding tangent indicatrices.

At a **first approximation** the cone with apex A is specified as right cone with apex angle 60°; c_1 is a geodesic circle on this cone.

3. Curved folding, Example 2







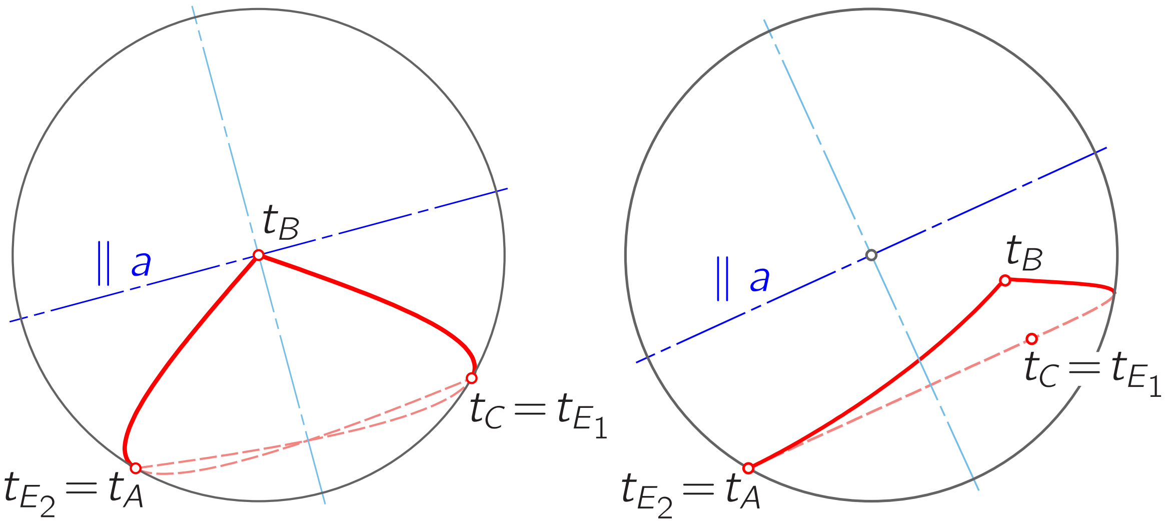
Approximation 1:

c_1 is an algebraic curve.

t_A is parallel to the tangent at $E_2 \in c_2$. Analogously, t_C is parallel to the tangent at $E_1 \in c_1$. This defines the axis a of symmetry.

We notice a contradiction since the osculating plane of c_1 at B is not orthogonal to BC.

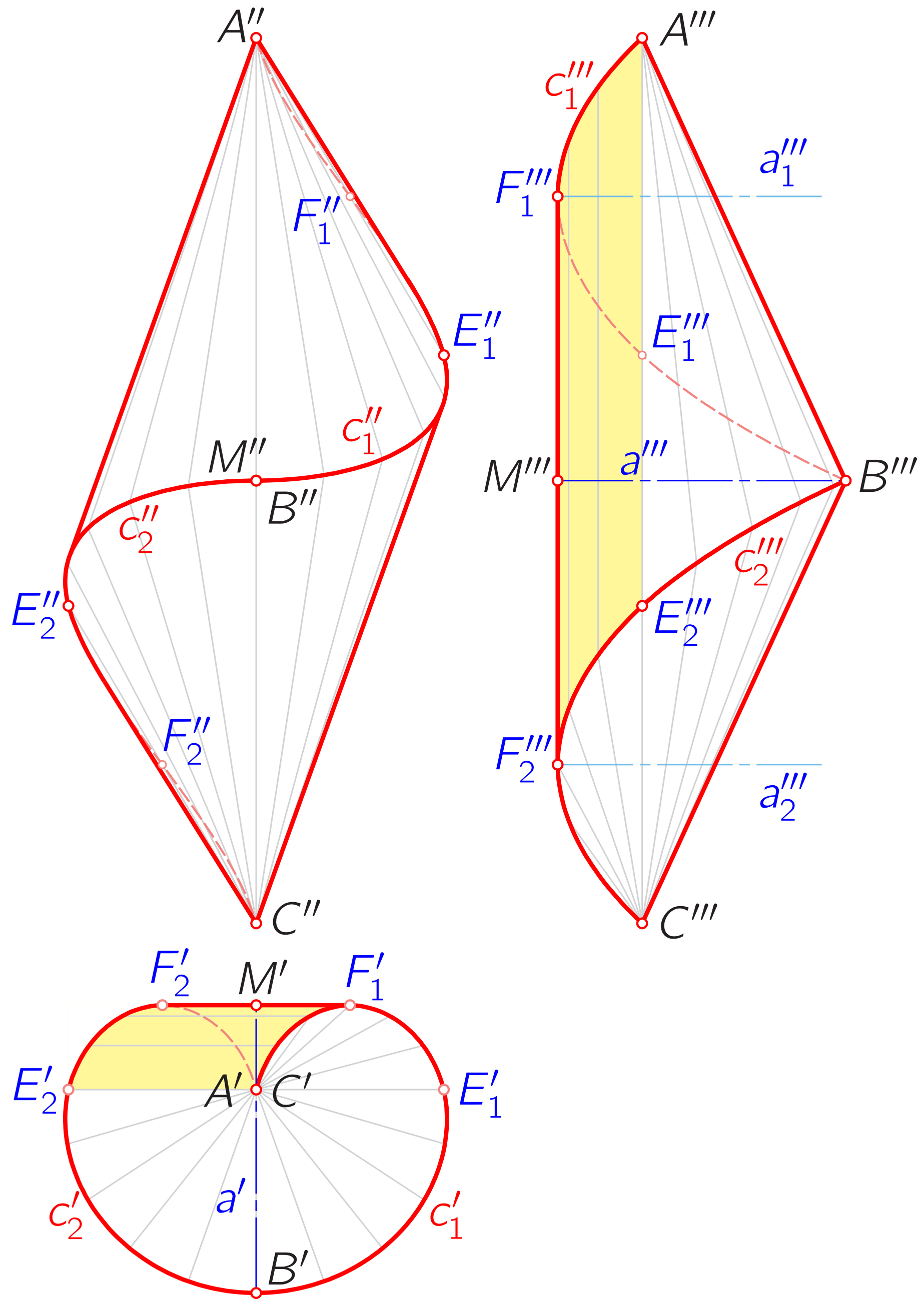
3. Curved folding, Example 2



Left: Tangent indicatrices of c_1 and c_2 for the first approximation; no coinciding subcurves!

Approximation 2 is defined by **alined** side views of the tangent indicatrices (right) $\Longrightarrow$

- the subcurve $AE_1 \subset c_1$ is a curve of constant slope.
- the central torse is a cylinder,
- a translation maps AE_1 onto the subcurve $E_2C \subset c_2$.



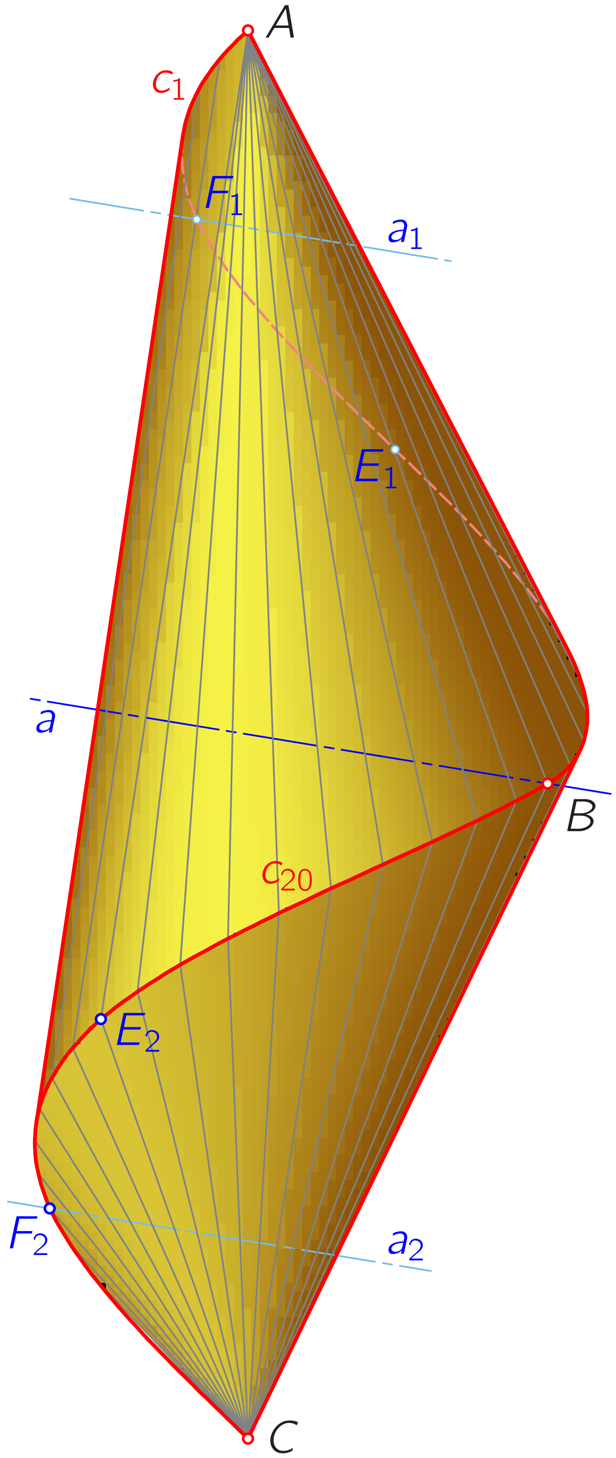
Approximation 2:

The product of the translation $A \mapsto E_2$ and the half-rotation about a maps the subcurve AE_1 onto itself, but in reverse order.

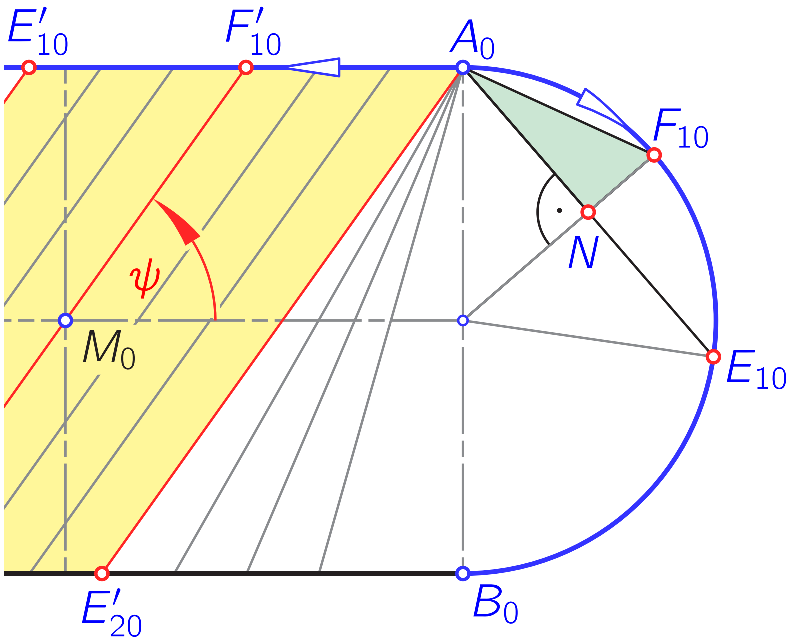
Therefore this portion AE_1 has an axis a_1 of symmetry passing through the midpoint F_1.

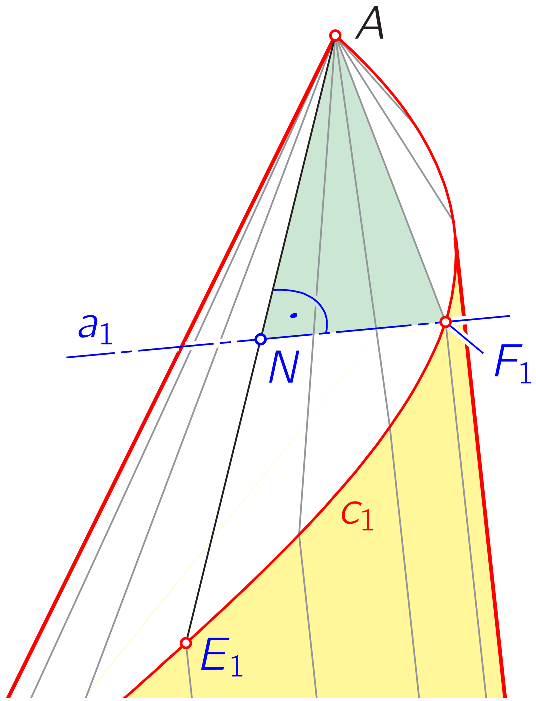
Approximation 2 shows an excellent accordance with the physical model.

. . . but there remains a contradiction.



3. Curved folding, Example 2





Due to the symmetry w.r.t. a_1, the midpoint N of AE_1 lies on a_1. The distances $\overline{A_0F_{10}}$ and $\overline{A_0E_{10}}$ are preserved, the triangle ANF_1 is congruent to its counterpart $A_0N_0F_{10}$ in the unfolding. But NF_1 is not (exactly) orthogonal to the tangent of c_1 at F_1.

Thank you for your attention!

References

[1] A.D. Alexandrov: *Convex Polyhedra*. Springer Monographs in Mathematics, Springer 2005, ISBN 9783540263401, (first Russian ed. 1950).

[2] A.I. Bobenko, I. Izmestiev: *Alexandrov's theorem, weighted Delaunay triangulations, and mixed volumes*. Annales de l'Institut Fourier **58**/(2), 447–505 (2008), arXiv:math.DG/0609447.

[3] R. Bricard: *Mémoire sur la théorie de l'octaèdre articulé*. J. math. pur. appl., Liouville **3**, 113–148 (1897).

[4] R. Bricard: *Leçon de Cinématique, Tome II, de Cinématique appliqueé*. Gauthier-Villars et C^ie^, Paris 1927.

[5] E.D. Demaine, J. O'Rourke: *Geometric folding algorithms: linkages, origami, polyhedra*. Cambridge University Press 2007.

[6] M. Kilian, S. Flöry, Z. Chen, N.J. Mitra, A. Sheffer, H. Pottmann: *Curved Folding*. ACM Trans. Graphics **27**/3 (2008), Proc. SIGGRAPH 2008.

[7] H. Stachel: *What lies between the flexibility and rigidity of structures*. Serbian Architectural J. **3**/2, 102–115 (2011).

[8] H. Stachel: *On the Rigidity of Polygonal Meshes*. South Bohemia Math. Letters **19**/1, 6–17 (2011).

[9] H. Stachel: *On the flexibility and symmetry of overconstrained mechanisms*. Phil. Trans. R. Soc. A **372**, num. 2008, 20120040 (2014).

[10] H. Stachel: *Flexible Polyhedral Surfaces With Two Flat Poses*. In Special Issue, B. Schulze (ed.): *Rigidity and Symmetry*, Symmetry **7**(2), 774–787 (2015).

[11] H. Stachel: *Two examples of curved foldings*. Proceedings of the 17th Internat. Conf. on Geometry and Graphics, Beijing 2016 (to appear).

[12] W. Whiteley: *Rigidity and scene analysis*. In J.E. Goodman, J. O'Rourke (eds.): *Handbook of Discrete and Computational Geometry*, 2nd ed., Chapman & Hall/CRC, Boca Raton 2004, pp. 1327–1354.

[13] W. Wunderlich: *Aufgabe 300*. El. Math. **22**, p. 89 (1957); author's solution: El. Math. **23**, 113–115 (1958).