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On the Diagonals of Billiards

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Abstract. A billiard is the trajectory of a mass point in a domain with ideal physical reflections in the boundary e. If e is an ellipse, then the billiard's sides are tangents of a confocal conic called caustic c. The variation of billiards in e with caustic c is called billiard motion. We recall and extend a classical result of Poncelet on the diagonals of billiards which envelope motion-invariant conics. Each billiard in e with caustic c is the flat pose of a Henrici framework. Its spatial poses define focal billiards in an ellipsoid with a fixed focal conic c. We prove that for even j the j-th diagonals are located on a motion-invariant one-sheeted hyperboloid.

Keywords: ellipse, billiard, caustic, Poncelet grid, focal billiard

1 Introduction

A *billiard* is the trajectory of a mass point in a domain with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses and their projectively equivalent counterparts have attracted the attention of mathematicians, beginning with J.-V. Poncelet [7] and C.G.J. Jacobi [5]. Computer animations carried out by Dan Reznik [8] stimulated a new vivid interest on these well studied objects. They offer an arena where problems can be attacked with analytic and algebraic methods (see, e.g., [2, 13]).

The sides of any billiard in an ellipse e are tangent to a confocal ellipse or hyperbola c called *caustic* (Fig. 1). Accordingly, we speak briefly of *elliptic* or *hyperbolic* billiards in e. It was Poncelet who proved in the projective setting [7] that if one billiard in e with caustic c closes after N reflections, then it closes for each choice of the initial vertex $P_1 \in e$. The variation of P_1 along e defines a socalled *billiard motion*, though it neither preserves angles or side lengths nor is a projective motion. However, the total length of periodic billiards remains constant, and D. Reznik [8] identified about 50 other invariants, e.g., the sum of cosines of the exterior angles θ_i (Fig. 1), which was first proved in [1].

As shown in [10], the billiards in e with caustic c can be isometrically transformed into spatial billiards in the ellipsoid $\mathcal{E}$ through e with the focal conic c. These billiards, which in the periodic case share the total length, are called *focal billiards* in $\mathcal{E}$ since their side lines are generators of confocal one-sheeted hyperboloids $\mathcal{H}_1$ and therefore focal lines of $\mathcal{E}$ (see [6, p. 284]).

The goal of this paper is to present new invariants of billiard motions for planar billiards as well as for spatial focal billiards. These invariants are related to the diagonals. In 1822, Poncelet proved in the projective setting that the envelopes of diagonals are conics [7]. A few years later, in 1828 Jacobi showed in [5, p. 388] that in the *concyclic* case, where the circumscribed and inscribed conics are circles, the diagonals envelope circles, too. It must be noted that the computation of billiards is not as easy as one might expect. The vertices of billiards can either be determined iteratively or, due to Jacobi's brilliant disclosure, be explicitely represented only in terms of Jacobian elliptic functions (see, e.g., [12]).

Structure of the article. We begin with planar billiards. In Section 2 we extend Poncelet's result by presenting formulas for the envelopes of the j-th diagonals of

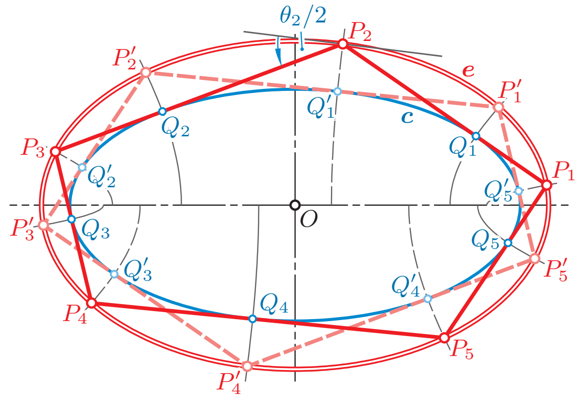

Fig. 1. Periodic billiard $P_1P_2\dots P_5$ in e with caustic c and its conjugate $P_1'P_2'\dots P_5'$.

elliptic billiards, and we determine the contact points. Similar results for hyperbolic billiards follow in Section 3, while Section 4 focuses on a general equivalence in the projective setting: We prove for polygons with circumconic and inconic that there exists a polarity which sends the vertices to the first diagonals. Finally in Section 5, we show for the spatial case and even j, that the j-th diagonals of focal billiards are generators of one-sheeted hyperboloids. For the sake of completeness, we repeat at the beginning a few theorems and proofs from the author's paper [11].

2 Diagonals of elliptic billiards

The extended sides of a billiard $P_1P_2\dots$ intersect at points which define the associated *Poncelet grid* and are located on confocal ellipses and hyperbolas. We follow the notation in [9] and define[1]

$$S_i^{(j)} := \begin{cases} [P_{i-k-1}, P_{i-k}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k, \\ [P_{i-k}, P_{i-k+1}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k-1 \end{cases} \tag{1}$$

where $i, j = 1, 2, \dots$ (Fig. 2). For fixed j, the points $S_i^{(j)}$ are located on a confocal ellipse $e^{(j)}$, which remains invariant under the billiard motion [9, Theorem 3.6]. For example, the principal semiaxes of $e^{(1)}$ and $e^{(2)}$ are

$$a_{e|1} = \frac{a_c(a_e^2b_e^2 - d^2k_e)}{a_c^2b_c^2 - k_e^2}\,, \quad a_{e|2} = \frac{a_e\left[(a_e^2b_e^2 - d^2k_e)^2 + 4d^2b_e^2k_e^2\right]}{(b_c^2a_e^2 - 3b_e^2k_e)(a_e^2b_e^2 - d^2k_e) - 4d^2b_e^2k_e^2}$$

with (a_c, b_c) and (a_e, b_e) as respective semiaxes of the confocal ellipses c and e, and moreover

$$k_e = a_e^2 - a_c^2 \quad\text{and}\quad d^2 = a_c^2 - b_c^2 = a_e^2 - b_e^2\,. \tag{2}$$

For fixed i, the points $S_i^{(1)}, S_i^{(3)}, \dots$ belong to the confocal hyperbola through Q_i (but not necessarily to the same branch), while $S_i^{(2)}, S_i^{(4)}, \dots$ are located on the confocal hyperbola through P_i (Fig. 2).

As introduced in [9], for each elliptic billiard $P_1P_2\dots$ in e with the contact points $Q_i \in c$ exists a *conjugate billiard* $P_1'P_2'\dots$ in e with contact points $Q_i' \in c$. It can be defined in the following way (see Fig. 1). There is an affine transformation

$$\alpha\colon\ (x, y) \mapsto \left(\frac{a_c}{a_e}\,x,\ \frac{b_c}{b_e}\,y\right) \quad\text{with}\quad e \mapsto c\,, \tag{3}$$

[1] Note that XY denotes the segment bounded by the points X and Y, while $[X, Y]$ denotes the connecting line.

which sends P'_i to Q_i and P_i to Q'_{i-1}.[2] Thus, the relation between the original billiard $P_1P_2\dots$ and its conjugate $P'_1P'_2\dots$ is symmetric. Below we use the symbol $S_i'^{(j)}$ for the vertices of the Poncelet grid associated with the conjugate billiard. Then for fixed i, the points $S_i'^{(1)}, S_i'^{(3)}, \dots$ belong to the confocal hyperbola through Q'_i and P_{i+1}, while the points $S_i'^{(2)}, S_i'^{(4)}, \dots$ are located on the confocal hyperbola through P'_i and Q_i (Fig. 2).

Theorem 1. *Let $P_1P_2P_3\dots$ be an elliptic billiard in the ellipse e with the caustic c. Then for fixed $j = 1, 2, \dots,$ the envelope of the diagonals $[P_i, P_{i+j+1}]$ is a coaxial ellipse $h_{e|j}$, provided that in the particular case of N-periodic billiards with even N holds $j \le [\frac{N-3}{2}]$. The ellipse $h_{e|j}$ has the semiaxes*

$$a_j = \frac{a_e a_c}{a_{e|j}} \quad \text{and} \quad b_j = \frac{b_e b_c}{b_{e|j}}\,, \tag{4}$$

where $(a_{e|j}, b_{e|j})$ are the semiaxes of the ellipse $e^{(j)}$ (Fig. 2). The ellipses $h_{e|2}$, $h_{e|3}$, $\dots$ belong to the pencil spanned by c and e.

Proof. We focus on the j-th diagonal P_iP_{i+j+1} (with j vertices between P_i and P_{i+j+1}). For the case of N-periodic billiards with even N we assume $0 < j \le [\frac{N-3}{2}]$ in order to exclude main diagonals passing through the center O.
The affine scaling α, as defined in (3), sends P_iP_{i+j+1} to $Q'_{i-1}Q'_{i+j}$, where Q'_{i-1} and Q'_{i+j} are contact points of the conjugate billiard $P'_1P'_2\dots$ with the caustic c. The pole of $[Q'_{i-1}, Q'_{i+j}]$ w.r.t. c is the point of intersection between $[P'_{i-1}, P'_i]$ and $[P'_{i+j}, P'_{i+j+1}]$. This point belongs to the ellipse $e^{(j)}$ included in the associated Poncelet grid. According to the notation explained in (1), we obtain the point $S_{i+k}'^{(j)}$ for $j = 2k$ as well as for $j = 2k + 1$ (Fig. 2).
For further details see [11], where it is also proved that the standard equation of $h_{e|j}$ is an affine combination of the standard equations of c and e. $\square$

According to (4), the envelope $h_{e|j}$ of the j-th diagonals can also be determined as the image of e under an affine scaling with $e^{(j)} \to c$. The following lemma holds for elliptic and hyperbolic billiards.

Lemma 1. *Let $P_1P_2P_3\dots$ be a billiard in the ellipse e with $Q_1, Q_2, Q_3, \dots$ as contact points with the caustic c and with $S_i^{(j)} \in e^{(j)}$ for $j = 1, 2, 3, \dots$ as points of the associated Poncelet grid according to* (1).
If there is an affine scaling

$$\beta\colon\ e^{(j)} \to c \ \ \textit{with}\ \ S_i^{(j)} \mapsto Q_i \ \ \textit{or}\ \ S_i'^{(j)} \mapsto Q_i\,, \tag{5}$$

then β sends e to the envelope $h_{e|j}$ of the j-th diagonals, and the diagonal $[P_i, P_{i+j+1}]$ contacts $h_{e|j}$ at the point of intersection between the j-th diagonals $[Q_{i-1}, Q_{i+j}]$ and $[Q_i, Q_{i+j+1}]$ of the polygon $Q_1Q_2Q_3\dots$.

Proof. We distinguish two cases.
1. j is odd, say $j = 2k - 1$: Then, according to the properties of the Poncelet grid, β sends $S_i^{(j)}$ to Q_i. We proceed in two steps: We determine the β-image of $P_i \in e$ and we show that the j-diagonal $[P_{i-k}, P_{i+k}]$ is the β-image of a tangent to e. This confirms the claim.
(i) Due to (1), the extensions of the two sides $P_{i-1}P_i$ and P_iP_{i+1} through the vertex $P_i \in e$ are the lines $[S_{i-k-1}^{(j)}, S_{i+k-1}^{(j)}]$ and $[S_{i-k}^{(j)}, S_{i+k}^{(j)}]$. Hence, the intersection T_i of their β-images $[Q_{i-k-1}, Q_{i+k-1}]$ and $[Q_{i-k}, Q_{i+k}]$ is the β-image of $P_i \in e$ (Fig. 2).

[2] An affine transformation which keeps the coordinate axes fixed is called *affine scaling.*

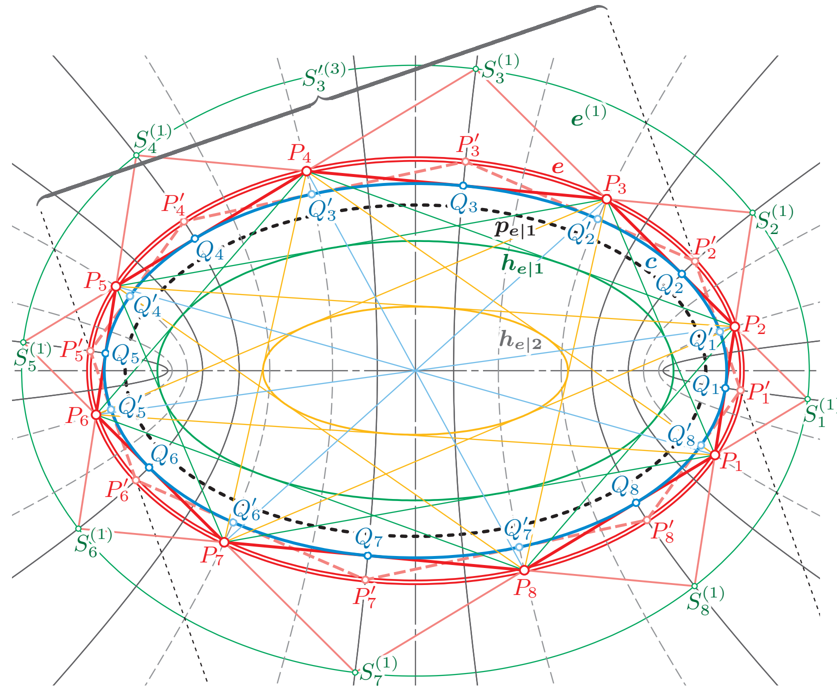

Fig. 2. Periodic elliptic billiard $P_1P_2\dots P_8$ inscribed in e with caustic c along with the conjugate billiard $P_1'P_2'\dots P_8'$ (dashed) and the envelopes $h_{e|1}$ of the first diagonals (green) and $h_{e|2}$ of the second diagonals (orange). The polarity in the ellipse $p_{e|1}$ (dotted) sends the vertices P_i to the adjacent first diagonals $[P_{i-1}, P_{i+1}]$.

(ii) The vertex P_{i+k} is the intersection of the tangents to c at Q_{i+k-1} and Q_{i+k}. Therefore, the β-preimage R_{i+k} of P_{i+k} is the intersection of the tangents to $e^{(j)}$ at $S^{(j)}_{i+k-1}$ and $S^{(j)}_{i+k}$. The four tangents drawn from $S^{(j)}_{i+k-1}$ and $S^{(j)}_{i+k}$ to c form a quadrilateral where P_i and P_{i+2k} are two opposite vertices. Now we refer to [9, Theorem 3.5] which says that this quadrilateral is concyclic; the tangents to c at P_i and P_{i+2k} and those to $e^{(j)}$ at $S^{(j)}_{i+k-1}$ and $S^{(j)}_{i+k}$ are concurrent angle bisectors. This means that the tangent to e at P_i passes through R_{i+k}.
After replacing k by $-k$, the same reasoning yields that the tangent to e at P_i also passes through the β-preimage R_{i-k} of P_{i-k} (see also [9, Fig. 6]). Thus we confirmed that β sends e to a conic that contacts the j-th diagonal $[P_{i-k}, P_{i+k}]$ at the image T_i of P_i.

2. j even, say $j = 2k$. In this case holds $\beta\colon\ S_i'^{(j)} \mapsto Q_i$. In view of the j-th diagonal $[P_{i-k}, P_{i+k+1}]$ we note that

$$P_i' = [P_{i-1}', P_i'] \cap [P_i', P_{i+1}'] = [S'^{(j)}_{i-k-1},\, S'^{(j)}_{i+k}] \cap [S'^{(j)}_{i-k},\, S'^{(j)}_{i+k+1}]$$

with the β-image $T_i' = [Q_{i-k-1},\, Q_{i+k}] \cap [Q_{i-k},\, Q_{i+k+1}]$. In order to show that β sends the tangent to e at P_i' to the j-th diagonal $[P_{i-k}, P_{i+k+1}]$, we proceed similar to the previous case. The only difference is that the quadrilateral circumscribed to c consists now of the c-tangents passing through $S'^{(j)}_{i+k+1}$ or $S'^{(j)}_{i+k}$ at the endpoint P_{i+k+1} and through $S'^{(j)}_{i-k-1}$ or $S'^{(j)}_{i-k}$ at the other endpoint P_{i-k}. $\square$

In the elliptic case the conics $e^{(j)}$ are ellipses confocal with e and c. Therefore, there exists an affine scaling $\beta\colon\ e^{(j)} \to c$ with $S_i^{(j)} \mapsto Q_i$ for odd j and $S_i'^{(j)} \mapsto Q_i$ for even j (note, e.g., [3, p. 40]). Thus, by virtue of Lemma 1 follows (see Fig. 3)

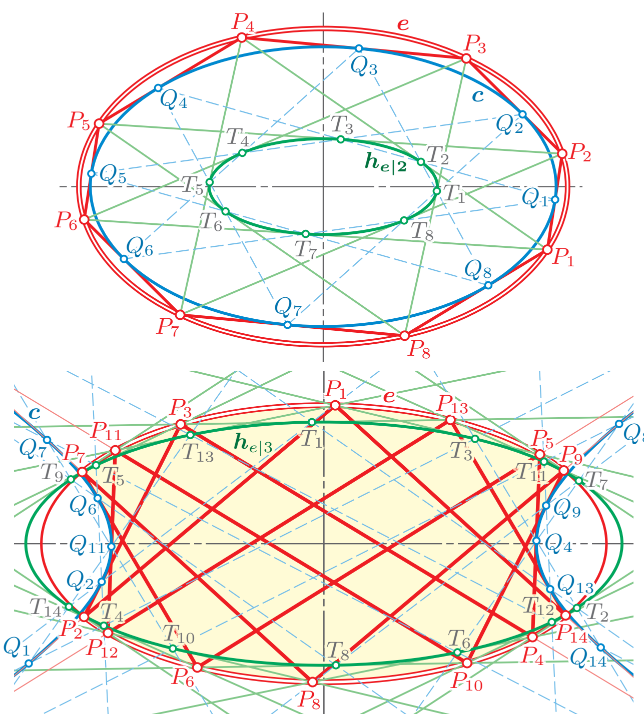

Fig. 3. Envelope $h_{e|2}$ of the second diagonals of an elliptic billiard (top) and envelope $h_{e|3}$ of the third diagonals of a hyperbolic billiard (bottom) along with the contact points T_i.

Theorem 2. *Referring to the previous notation, the envelope* $h_{e|j}$ *of the* j*-th diagonals of the elliptic billiard* $P_1P_2P_3\ldots$ *is the image of* e *under an affine scaling* β *with* $e^{(j)} \to c$. *The envelope* $h_{e|j}$ *equals the locus of the points of intersection* $T_{i+[\frac{j}{2}]} = [Q_{i-1}, Q_{i+j}] \cap [Q_i, Q_{i+j+1}]$ *between consecutive* j*-th diagonals of the polygon* $Q_1Q_2Q_3\ldots$ *inscribed in the caustic* c *(Fig. 3).*

The following theorem is a consequence of one of D. Reznik's experiments.

Theorem 3. *Let* $P_1P_2\ldots$ *be an elliptic billiard in the ellipse* e. *Then for any* $j \in \{1, 2, \ldots\}$, *the diagonal line* $[P_i, P_{i+j+1}]$ *is polar for even* j *to* $P'_{i+(j/2)}$ *and for odd* j *to* $P_{i+[(j+1)/2]}$ *w.r.t. a coaxial ellipse* $p_{e|j}$ *with semiaxes*

$$a_{p|j} = a_e\sqrt{\frac{a_c}{a_{e|j}}} \quad \textit{and} \quad b_{p|j} = b_e\sqrt{\frac{b_c}{b_{e|j}}}\,.$$

A proof can be found in [11, p. 147]; an alternative proof follows in Section 4.

3 Diagonals of hyperbolic billiards

Hyperbolic billiards in ellipses, i.e., billiards with a hyperbola as caustic c differ in various ways from their elliptic counterparts. We recall a few of them.

During the billiard motion the vertices of a hyperbolic billiard vary only either on an upper or a lower subarc of the ellipse e (note Figs. 4 or 5). The turning number τ of a periodic billiard counts how often the vertices run to and fro along these arcs. According to [9, Theorem 3.12], the points $S_i^{(1)}$, $S_i^{(3)}, \ldots$ of the Poncelet grid are located on confocal ellipses through the contact point Q_i of $[P_i, P_{i+1}]$ with c, while the points $S_i^{(2)}$, $S_i^{(4)}, \ldots$ are located on the confocal hyperbola through

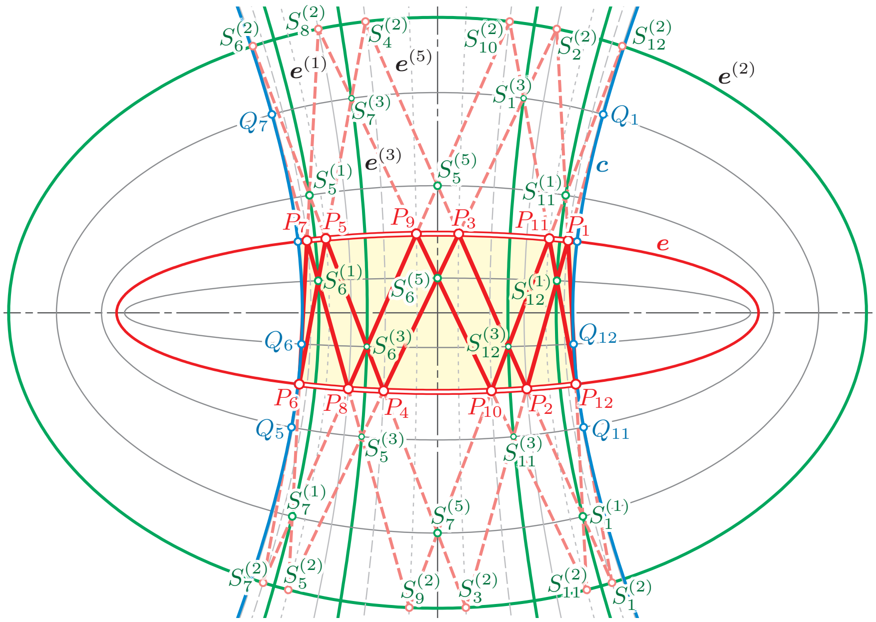

Fig. 4. Periodic billiard $P_1P_2\dots P_{12}$ with $\tau = 1$ in the ellipse e with the hyperbola c as caustic, together with the hyperbolas $e^{(1)}$, $e^{(3)}$, the secondary axis $e^{(5)}$, and the ellipse $e^{(2)}$.

P_i, but not necessarily on the same branch. For even j, the conic $e^{(j)}$ is a confocal ellipse (if finite), while for odd j we obtain confocal hyperbolas $e^{(j)}$ or an axis of symmetry.

As stated in [9, Corollary 4.3], periodic N-sided billiards with $N \equiv 0 \pmod 4$ are symmetric w.r.t. the secondary axis. For $N \equiv 2 \pmod 4$ and odd turning number τ (Fig. 5), the hyperbolic billiards are centrally symmetric, for even τ symmetric w.r.t. the principal axis of e and c. This results in an exceptional behavior of the $\frac{N-2}{2}$-th diagonals: If $N \equiv 0 \pmod 4$, then these diagonals are parallel to the principal axis, while for $N \equiv 2 \pmod 4$ and odd turning number they are diameters of e and otherwise orthogonal to the principal axis. In Theorem 4 we exclude these cases.

According to [9, Lemma 3.14], also for hyperbolic billiards there exist one or two conjugate billiards (note Fig. 5). However, there is no kind of symmetry between the contact points $Q_i \in c$ and the vertices $P'_i \in c$ since there is no affine transformation α between the ellipse e and the hyperbola c. This is the reason, why the proofs in Section 3 cannot be transferred one-to-one from elliptic to hyperbolic billiards.

Theorem 4. *Let $P_1P_2P_3\dots$ be a billiard in e with the hyperbola c as caustic. Then, apart from the exceptions listed above, the j-th diagonals envelope for odd j a coaxial ellipse $h_{e|j}$, for even j a coaxial hyperbola. These envelopes belong to the pencil spanned by e and c and have the semiaxes*

$$a_j = \frac{a_e a_c}{a_{e|j}} \quad \text{and} \quad b_j = \frac{b_e b_c}{b_{e|j}}\,,$$

where $(a_{e|j}, b_{e|j})$ are the semiaxes of $e^{(j)}$ (Fig. 2). The construction of contact points according to Lemma 1 is still valid for hyperbolic billiards.

Proof. 1. j even, say $j = 2k$, and the conic $e^{(j)}$ is an ellipse, provided that we exclude the particular case with points $S_i^{(j)}$ at infinity: There exists an affine scaling

$$\gamma\colon\ (x,y) \mapsto \left(\pm\frac{a_e}{a_{e|j}}\,x,\ \pm\frac{b_e}{b_{e|j}}\,y\right) \quad \text{with} \quad e^{(j)} \to e \quad \text{and} \quad S_i^{(j)} \mapsto P_i \tag{6}$$

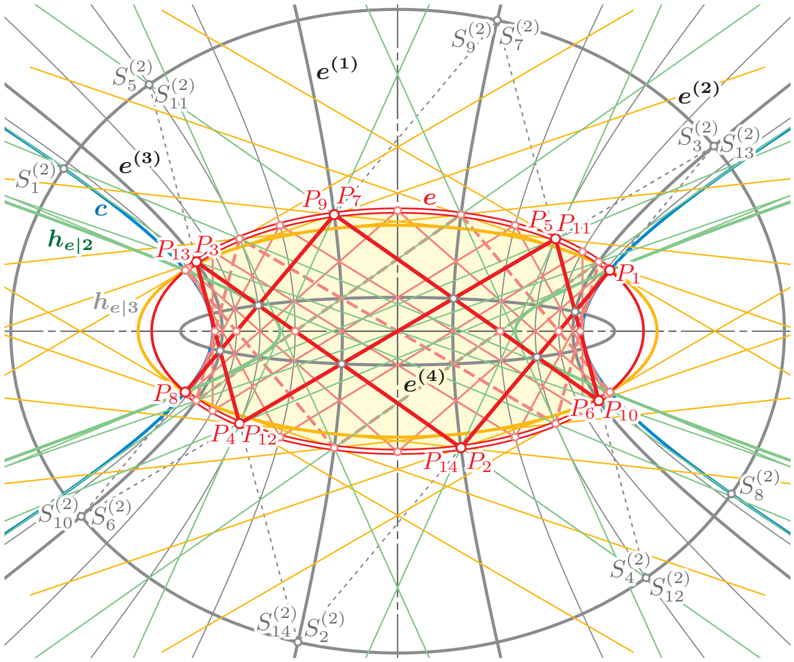

Fig. 5. Twofold covered periodic hyperbolic billiard $P_1P_2\dots P_{14}$ with $\tau = 3$ and its mirror (dashed) along with the conjugate billiard, the second diagonals (green) enveloping the hyperbola $h_{e|2}$, and the third diagonals (orange) enveloping the ellipse $h_{e|3}$.

for a particular choice of the signs. This affine scaling sends $[S_i^{(j)}, S_{i+j+1}^{(j)}]$ to the diagonal line $[P_i, P_{i+j+1}]$. The preimage is the extension of the side $P_{i+k}P_{i+k+1}$ and contacts the caustic c at the point Q_{i+k}. Consequently, the j-th diagonal $[P_i, P_{i+j+1}]$ contacts the γ-image of the caustic at the γ-image $\gamma(Q_{i+k})$ of Q_{i+k}. Thus, we obtain the hyperbola with semiaxes $a_c a_e/a_{e|j}$ and $b_c b_e/b_{e|j}$ as the envelope $h_{e|j}$.[3]

How to determine the contact point $\gamma(Q_{i+k})$ of $[P_i, P_{i+j+1}]$ with the envelope $h_{e|j}$? Let π_c denote the mapping of lines to their poles w.r.t. the caustic c. Then the coordinate representations of these mappings show that $\gamma \circ \pi_c = \pi_c \circ \gamma^{-1}$ and therefore

$$\begin{aligned}\gamma(Q_{i+k}) &= \gamma \circ \pi_c\left([P_{i+k}, P_{i+k+1}]\right) = \pi_c \circ \gamma^{-1}\left([P_{i+k}, P_{i+k+1}]\right)\\ &= \pi_c\left([S_{i+k}^{(j)}, S_{i+k+1}^{(j)}]\right) = [Q_{i-1}, Q_{i+j}] \cap [Q_i, Q_{i+j+1}].\end{aligned}$$

In the last equation we use the rule that the pole of the connection of two points is the intersection of the two respective polar lines.

2. j is odd, say $j = 2k-1$, and the conic $e^{(j)}$ is a hyperbola. There is an affine transformation

$$\beta\colon\ e^{(j)} \to c \ \text{ with } \ S_i^{(j)} \mapsto Q_i\,,$$

and the claim follows directly from Lemma 1 (note Fig. 3, right).

The affine combination in [11, p. 146] reveals that also in the case of hyperbolic billiards the conics c, e and $h_{e|j}$ belong to a pencil — in alignement with the arguments used below in the proof of Theorem 5. □

[3] Note that the axial scaling γ can also be used for proving Theorem 1.

4 An excursion to projective billiards

There is a certain converse of Theorem 3 (Fig. 2). We present a projective version which is valid in all pappian projective Fano planes.

Theorem 5. *Let $P_1P_2P_3\dots$ with $P_i \neq P_{i+1}$ for $i = 1,2,3\dots$ be a polygon inscribed in a conic e. Then the extended sides $[P_i, P_{i+1}]$ are tangent to a conic c if and only if there exists a polarity w.r.t. a conic p which sends the vertices $P_2, P_3, \dots$ to the respectively adjacent first diagonals $[P_1, P_3]$, $[P_2, P_4]$, ...*

Remark 1. The statement in Theorem 5 is trivial for N-periodic billiards with $N \leq 5$, since there always exists a second-degree curve with five given tangents or five given pairs (pole, polar).

Proof. We use homogeneous coordinates $(x_0 : x_1 : x_2)$ with $e\colon\ x_0x_2 - x_1^2 = 0$ and an inhomogeneous parameter t on e such that for the billiard's vertices holds $P_i = (t_i^2 : t_i : 1)$. The vertices are the poles of the first diagonals $[P_{i-1}, P_{i+1}]$ w.r.t. $p\colon \sum p_{ik}x_ix_k = 0$ with $p_{ik} = p_{ki}$ if and only if for all $i \in \{1,2,3,\dots\}$ consecutive vertices P_i and P_{i+1} are conjugate w.r.t. p. This is equivalent to

$$p_{00}t_i^2t_{i+1}^2 + p_{11}t_it_{i+1} + p_{22} + p_{01}t_it_{i+1}(t_i + t_{i+1}) + p_{02}(t_i^2 + t_{i+1}^2) + p_{12}(t_i + t_{i+1}) = 0. \quad (7)$$

On the other hand, the line $[P_i, P_{i+1}]$ with homogeneous line coordinates $(u_0 : u_1 : u_2) = (1 : -(t_i + t_{i+1}) : t_it_{i+1})$ is tangent to c given by the tangential equation $\sum c_{ik}u_iu_k = 0$ with $c_{ik} = c_{ki}$ iff

$$c_{00} + c_{11}(t_i + t_{i+1})^2 + c_{22}t_i^2t_{i+1}^2 - 2c_{01}(t_i + t_{i+1}) + 2c_{02}t_it_{i+1} - 2c_{12}t_it_{i+1}(t_i + t_{i+1}) = 0. \quad (8)$$

The conditions in (7) and (8) are equivalent iff

$$p_{00} : p_{01} : p_{02} : p_{11} : p_{12} : p_{22} = c_{22} : -2c_{12} : c_{11} : 2(c_{11} + c_{02}) : -2c_{01} : c_{00}\,. \quad (9)$$

Clearly, for a given conic c with coefficient matrix (c_{ik}) the second-degree curve p with the coefficient matrix (p_{ik}) is uniquely defined, and vice versa. The geometric meaning guarantees that c and p are irreducible, i.e., $\det c_{ik} \neq 0$ and $\det p_{ik} \neq 0$, which proves the claim. □

Theorem 5 provides a new approach to Poncelet's classical result.

Corollary 6 *If a polygon has a circumconic e and an inconic c, then the first diagonals envelope the conic $h_{e|1}$ which is polar to e w.r.t. p. The conic $h_{e|1}$ belongs to the pencil spanned by e and c.*

Proof. The envelope $h_{e|1}$ is polar to e w.r.t. the conic p, since the polarity in p sends points of e to tangents of $h_{e|1}$ and tangents of p to points of $h_{e|1}$.
If we specify the vertex $P_i \in e$ at a point of intersection with c, then the neigbouring vertices $P_{i-1}, P_{i+1} \in e$ coincide and the first diagonal $[P_{i-1}, P_{i+1}]$ becomes a tangent of e. Hence, the p-pole of this tangent, namely the point $P_i \in e$, must be located on $h_{e|1}$. If all points of intersection between e and c have the multiplicity 1, then in the complex extension of the real projective plane and in all projective planes over an algebraically closed field $\mathbb{K}$, the conics e, c and $h_{e|1}$ share four points and belong to a pencil.
In all other projective planes we proceed in the following way. Let $\mathbf{E}$, $\mathbf{C}$ and $\mathbf{P}$ denote the respective symmetric coefficient matrices of the conics e, c and p as used above in the proof of Theorem 5. The points $\mathbb{K}\mathbf{x}$ of e satisfy $\mathbf{x}^T\mathbf{E}\mathbf{x} = 0$, and their polars w.r.t. p have the line coordinates $\mathbf{u} = \mathbf{P}\mathbf{x}$, or conversely, $\mathbf{x} = \mathbf{P}^{-1}\mathbf{u}$. Thus, the envelope $h_{e|1}$ has the tangential equation $\mathbf{u}^T\mathbf{P}^{-1}\mathbf{E}\mathbf{P}^{-1}\mathbf{u} = 0$. The points $\mathbb{K}\mathbf{y}$ of $h_{e|1}$ satisfy

$$\mathbf{y}^T\mathbf{P}\mathbf{E}^{-1}\mathbf{P}\mathbf{y} = 0\,.$$

It remains to prove that there exist $\lambda, \mu \in \mathbb{K}$ such that

$$\mathbf{P}\,\mathbf{E}^{-1}\,\mathbf{P} = \lambda\,\mathbf{E} + \mu\,\mathbf{C}\,.$$

We substitute $\mathbf{C} = \big(c_{ik}\big)^{-1}$,

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix},\,,\ \ \mathbf{E}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix},\ \ \mathbf{P} = \begin{pmatrix} c_{22} & -2c_{12} & c_{11} \\ -2c_{12} & 2(c_{11}+c_{02}) & -2c_{01} \\ c_{11} & -2c_{01} & c_{00} \end{pmatrix},$$

and obtain as their product $\mathbf{P}\,\mathbf{E}^{-1}\,\mathbf{P}$ the matrix

$$\begin{pmatrix} 2c_{12}c_{22} - 2c_{12}^2 & -2c_{01}c_{22} + 2c_{02}c_{12} & c_{00}c_{22} - 2c_{01}c_{12} + c_{11}^2 \\ -2c_{01}c_{22} + 2c_{02}c_{12} & 8c_{01}c_{12} - 4c_{02}c_{11} - 2c_{02}^2 - 2c_{11}^2 & -2c_{00}c_{12} + 2c_{01}c_{02} \\ c_{00}c_{22} - 2c_{01}c_{12} + c_{11}^2 & -2c_{00}c_{12} + 2c_{01}c_{02} & 2c_{00}c_{11} - 2c_{01}^2 \end{pmatrix},$$

and this equals $\lambda\,\mathbf{E} + \mu\,\mathbf{C}$ with $\lambda = c_{00}c_{22} - 4c_{01}c_{12} + 2c_{02}c_{11} + c_{11}^2$, $\mu = 2\,\det\big(c_{ik}\big)$, which confirms the claim. □

Remark 2. Theorem 5 holds not only for the first diagonals of any elliptic billiard in an ellipse e, but also for the j-th diagonals with $j = 3, 5\ldots$. This follows iteratively, when we replace c by one of the envelopes of diagonals $h_{e|1}$, $h_{e|3}$, …. In the billiard case, there is also another extension: We can replace e with one of the conics $e^{(1)}$, $e^{(2)}$, … of the Poncelet grid and apply an affine scaling γ with $e^{(j)} \to e$.

5 Diagonals of focal billiards

We follow the notation in [9] and [10]. Let $\mathcal{E}$ be the ellipsoid with the semiaxes $a_e > b_e > c_e$. Then, its focal ellipse c' has the semiaxes a'_c, b'_c with

$$a_c'^2 = a_e^2 - c_e^2 \ \text{ and } \ b_c'^2 = b_e^2 - c_e^2\,.$$

The quadrics which are confocal with $\mathcal{E}$ satisfy the equations

$$\frac{x^2}{a_c'^2 + k} + \frac{y^2}{b_c'^2 + k} + \frac{z^2}{k} = 1 \quad \text{with} \quad k \in \mathbb{R} \setminus \{-a_c'^2, -b_c'^2, 0\} \tag{10}$$

as a parameter called *elliptic coordinate*. The family of confocal central quadrics contains

$$\text{for} \begin{cases} \ \ 0 \ < \ k = k_0 \ < \infty & \text{triaxial ellipsoids,} \\ -b_c'^2 < \ k = k_1 \ < \ 0 & \text{one-sheeted hyperboloids,} \\ -a_c'^2 < \ k = k_2 \ < -b_c'^2 & \text{two-sheeted hyperboloids,} \end{cases} \tag{11}$$

the focal ellipse c' as the limit for $k = 0$, and the focal hyperbola c'' in the plane $y = 0$ as the limit for $k = -b_c'^2$ with the semiaxes $a''_c = \sqrt{a_c'^2 - b_c'^2} = d'$ and $b''_c = b'_c$.

We recall that the family of confocal quadrics sends through each point $P = (x, y, z)$ with $xyz \neq 0$ three mutually orthogonal surfaces, one of each type (see, e.g., [6, p. 279]). The parameters (k_0, k_1, k_2) of these quadrics are the *elliptic coordinates* of P and satisfy

$$x^2 = \frac{(a_c'^2 + k_0)(a_c'^2 + k_1)(a_c'^2 + k_2)}{(a_c'^2 - b_c'^2)a_c'^2}\,,\ y^2 = \frac{(b_c'^2 + k_0)(b_c'^2 + k_1)(b_c'^2 + k_2)}{b_c'^2(b_c'^2 - a_c'^2)}\,,\ z^2 = \frac{k_0k_1k_2}{a_c'^2 b_c'^2}\,.$$

Conversely, eight points in space, symmetrically placed w.r.t. the coordinate frame, share the elliptic coordinates (k_0, k_1, k_2). The given ellipsoid $\mathcal{E}$ has the elliptic coordinate $k_0 = c_e^2 > 0$.

As described in [10], each elliptic billiard with circumellipse e' and caustic c' in the plane $z = 0$ can be isometrically transformed into a focal billiard on the ellipsoid $\mathcal{E}$ through e' with focal ellipse c'. The sides of this spatial billiard lie on generators of a confocal hyperboloid $\mathcal{H}_1$, which intersects $\mathcal{E}$ along a line of curvature e, the common trajectory of the vertices during a billiard motion (Fig. 6).

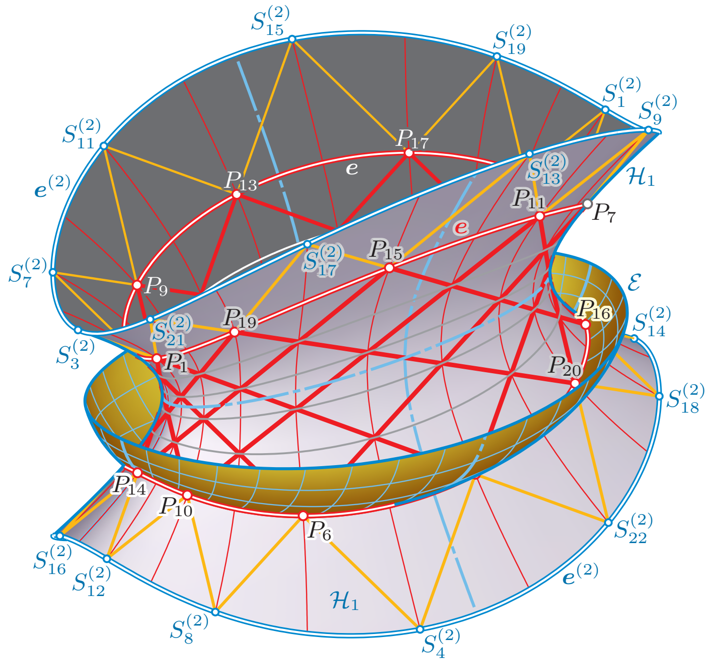

Fig. 6. N-periodic focal billiard (red) with $N = 22$ and turning number $\tau = 5$ in the ellipsoid $\mathcal{E}$ along with the line $e^{(2)}$ (blue) of the associated spatial Poncelet grid and the inscribed focal billiard (orange) consisting of two 11-gons $S_i^{(2)} S_{i+3}^{(2)} S_{i+6}^{(2)} \dots$ with $\tau = 5$ each on the confocal one-sheeted hyperboloid $\mathcal{H}_1$.

Theorem 7. *Let $P_1 P_2 \dots$ be a focal billiard on the ellipsoid $\mathcal{E}$ with vertices on the line of curvature e being the intersection between $\mathcal{E}$ and the confocal one-sheeted hyperboloid $\mathcal{H}_1$ with semiaxes a_{h_1}, b_{h_1} and c_{h_1} . Then for even $j = 2k$, the diagonals $[P_i, P_{i+j+1}]$ are generators of a coaxial one-sheeted hyperboloid $\mathcal{D}_j$ which belongs to the pencil of quadrics spanned by $\mathcal{E}$ and $\mathcal{H}_1$. The hyperboloid $\mathcal{D}_j$ has the semiaxes*

$$a_{d|j} = \frac{a_e a_{h_1}}{a_{e|j}}, \ \ b_{d|j} = \frac{b_e b_{h_1}}{b_{e|j}}, \ \ c_{d|j} = \frac{c_e c_{h_1}}{\sqrt{a_{e|j}^2 - a_c'^2}} = \frac{c_e c_{h_1}}{c_{e|j}} \, ,$$

where $a_{e|j}$, $b_{e|j}$, $c_{e|j}$ are the semiaxes of the confocal ellipsoid through the ellipse $e'^{(j)}$ of the planar Poncelet grid. Only for N-periodic focal billiards and $j = \frac{N}{2} - 1$, the diagonals belong to a quadratic cone.
In the plane $z = 0$ of the focal ellipse c', the trace points of the j-th diagonals form a polygon where the extended sides coincide with the j-th diagonals of the polygon formed by the trace points of the original focal billiard. The same holds for the plane $y = 0$ of the focal hyperbola c''.

Remark 3. It is worth to be noted that at focal billiards any two consecutive j-th diagonals $[P_i, P_{i+j+1}]$ and $[P_{i+1}, P_{i+j+2}]$ are intersecting when j is even.

Proof. The j-th diagonals form a spatial polygon with vertices $P_i, P_{i+j+1}, P_{i+2(j+1)}, P_{i+3(j+1)}, \dots$ on e, provided that the billiard is not $(2j+2)$-periodic. In a similar way, the associated spatial Poncelet grid on the hyperboloid $\mathcal{H}_1$ (note Fig. 6 or [10, Figs. 6 and 7]) contains the quartic $e^{(j)}$, and the points $S_i^{(j)}$, $S_{i+j+1}^{(j)}$, $S_{i+2(j+1)}^{(j)}, \dots$ on $e^{(j)}$ are vertices of a spatial polygon with sides along the rulings of $\mathcal{H}_1$, provided that j is even.[4] There is an axial scaling of the form

$$\delta\colon\ (x,y,z)\mapsto\left(\pm\frac{a_e}{a_{e|j}}\,x,\ \pm\frac{b_e}{b_{e|j}}\,y,\ \pm\frac{c_e}{c_{e|j}}\,z\right)\ \text{ with }\ e^{(j)}\to e\ \text{ and }\ S_i^{(j)}\mapsto P_i\,.$$

Hence, the j-th diagonals belong to the δ-image of $\mathcal{H}_1$, which is a one-sheeted hyperboloid $\mathcal{D}_j$ with the stated semiaxes and the gorge ellipse in the plane $z=0$.

The axial scaling

$$\alpha'\colon\ (x,y,z)\mapsto\left(\frac{a'_c}{a_{h_1}}\,x,\ \frac{b'_c}{b_{h_1}}\,y,\ 0\right)$$

sends $\mathcal{H}_1$ to the exterior of the focal ellipse c' in $z=0$, the j-th diagonals of the focal billiard to j-th diagonals of e', and the gorge ellipse of $\mathcal{H}_1$ to c'. The restriction of α' to $z=0$ is bijective and maps the top views of the j-th diagonals to the j-th diagonals of e'. Thus, the envelope to the top views of the spatial diagonals, i.e., the gorge ellipse of $\mathcal{D}_j$, is bijectively related to the envelope $h'_{e|j}$, and we can transfer the construction of contact points of $h'_{e|j}$, as given in Lemma 1, to that of trace points of the spatial diagonals. □

Finally, it should be noted that several of the presented theorems can also be verified using the canonical parametrization of the billiards in terms of the Jacobian elliptic functions to the modulus d/a_c (see, e.g., [4] and [12, Theorem 4.3]). As an example, we demonstrate this for Theorem 7.

According to [10, Theorem 13], the two components of $e=\mathcal{E}\cap\mathcal{H}_1$ can be parametrized as

$$\mathbf{e}_{1,2}(\tilde u)=\left(-\frac{a_e a_{h_1}}{a'_c}\,\mathrm{sn}\,\tilde u,\ \frac{b_e b_{h_1}}{b'_c}\,\mathrm{cn}\,\tilde u,\ \pm\frac{c_e c_{h_1}}{b'_c}\,\mathrm{dn}\,\tilde u\right)\tag{12}$$

where the transition from any vertex P_i of the billiard to the next one P_{i+1} corresponds to the parameter's shift by a constant $2\varDelta\tilde u$ combined with a change of the sign of the z-coordinate. Hence, if P_i has the parameter $\tilde u$, then for even j the vertex P_{i+j+1} has the parameter $\tilde u_j:=\tilde u+2(j+1)\varDelta\tilde u$, i.e., $P_i=\mathbf{e}_1(\tilde u)$ and $P_{i+j+1}=\mathbf{e}_2(\tilde u_j)$.

In order to verify that $[P_i,P_{i+j+1}]\subset\mathcal{D}_j$, it is necessary and sufficient to show that for even j the points P_i and P_{i+j+1} are conjugate w.r.t. $\mathcal{D}_j$. From [12, Cor. 4.5] follows

$$a_{e|j}=\frac{a'_c\,\mathrm{dn}\,[(j+1)\varDelta\tilde u]}{\mathrm{cn}\,[(j+1)\varDelta\tilde u]}=\frac{a'_c\,\mathrm{dn}\,\frac{\tilde u_j-\tilde u}{2}}{\mathrm{cn}\,\frac{\tilde u_j-\tilde u}{2}},\quad b_{e|j}=\frac{b'_c}{\mathrm{cn}\,[(j+1)\varDelta\tilde u]}=\frac{b'_c}{\mathrm{cn}\,\frac{\tilde u_j-\tilde u}{2}}\,,$$

hence

$$k_{e|j}\ =\ a_{e|j}^2-a_c'^2=a_c'^2\,\frac{\mathrm{dn}^2\frac{\tilde u_j-\tilde u}{2}-\mathrm{cn}^2\frac{\tilde u_j-\tilde u}{2}}{\mathrm{cn}^2\frac{\tilde u_j-\tilde u}{2}}=\frac{b_c'^2\,\mathrm{sn}^2\frac{\tilde u_j-\tilde u}{2}}{\mathrm{cn}^2\frac{\tilde u_j-\tilde u}{2}}\,.$$

Thus, it remains to show that

$$\frac{a_{e|j}^2}{a_e^2a_{h_1}^2}\,\frac{a_e^2a_{h_1}^2}{a_c'^2}\,\mathrm{sn}\,\tilde u\,\mathrm{sn}\,\tilde u_j+\frac{b_{e|j}^2}{b_e^2b_{h_1}^2}\,\frac{b_e^2b_{h_1}^2}{b_c'^2}\,\mathrm{cn}\,\tilde u\,\mathrm{cn}\,\tilde u_j+\frac{k_{e|j}}{c_e^2c_{h_1}^2}\,\frac{c_e^2c_{h_1}^2}{b_c'^2}\,\mathrm{dn}\,\tilde u\,\mathrm{dn}\,\tilde u_j=1\,,$$

hence

$$\mathrm{dn}^2\frac{\tilde u_j-\tilde u}{2}\,\mathrm{sn}\,\tilde u\,\mathrm{sn}\,\tilde u_j+\mathrm{cn}\,\tilde u\,\mathrm{cn}\,\tilde u_j+\mathrm{sn}^2\frac{\tilde u_j-\tilde u}{2}\,\mathrm{dn}\,\tilde u\,\mathrm{dn}\,\tilde u_j=\mathrm{cn}^2\frac{\tilde u_j-\tilde u}{2}\,.$$

This is an identity due to the addition theorems and half-angle theorems of elliptic functions [4].

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[4] For odd j the points P_i and P_{i+j+1} belong to the same component of $e^{(j)}$, and the line $[S_i^{(j)},S_{i+j+1}^{(j)}]$ is no generator of $\mathcal{H}_1$.

6 Conclusion

We extended Poncelet's classical results and discussed the diagonals of elliptic and hyperbolic billiards $P_1P_2\dots$ in an ellipse e as well as that of focal billiards in an ellipsoid $\mathcal{E}$. We proved that the envelopes of the j-th diagonals in the plane belong to a pencil of conics, and we disclosed a remarkable relation between the j-th diagonals of the original billiard $P_1P_2\dots$ and that of the contact points $Q_1Q_2\dots$. In space, the j-th diagonals form ruled quadrics contained in a pencil through $\mathcal{E}$, but only for even j. Periodic billiards yield periodic polygons of j-th diagonals. All obtained results can immediately be projectively generalized to statements on planar polygons with circumconic and inconic as well as to spatial counterparts.

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