This is a pre-print of a contribution published in L.-Y. Cheng (ed): ICGG 2020 - Proceedings of the 19th International Conference on Geometry and Graphics, Sao Paulo/Brazil 2021, pp. 3-14, published by Springer Nature Switzerland AG 2021 (ISBN 978-3-030-63402-5).
The final authenticated version is available online at https://doi.org/10.1007/978-3-030-63403-2_1

String Constructions of Quadrics Revisited

Hellmuth Stachel

Vienna University of Technology, 1040 Vienna, Austria,
stachel@dmg.tuwien.ac.at,
WWW home page: http://www.dmg.tuwien.ac.at/stachel

Abstract. The role of quadrics in Euclidean 3-space is similar to that of conics. Therefore, it is natural to ask for string constructions of quadrics, as spatial analogues of the gardener's construction of ellipses. The first solution given in 1882 by O. Staude is based on an ellipse e and its focal hyperbola h. A string of a given length, fixed with one end at a focal point of h, is passed behind the nearest branch of h and in front of e and finally attached to the vertex of the second branch of h. If the string is stretched at a point P, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with e and h. Later, Staude presented a second type of string constructions where e and h are replaced by an ellipsoid $\mathcal{E}_0$ and a confocal hyperboloid $\mathcal{H}_0$. Here the ends of the string follow the two branches of the curvature line $\mathcal{E}_0 \cap \mathcal{H}_0$. We provide a synthetic approach to these constructions and extend them to paraboloids.

Keywords: Quadric, string construction, focal conics, confocal quadrics

1 Introduction

In 1882, Otto Staude [9] presented a string construction for ellipsoids, based on a pair of focal conics e and h (Fig. 1). It was proposed as a spatial analogue of the gardener's construction and Graves's construction of ellipses (see, e.g., [4, Figs. 1.8 and 2.29]). Some years later, Staude [10] came up with a second version: Instead of the pair of focal conics, an ellipsoid $\mathcal{E}_0$ and a confocal hyperboloid $\mathcal{H}_0$ are used. A string which is stretched at the point P follows at its ends the two branches of the curve of intersection $\mathcal{E}_0 \cap \mathcal{H}_0$, which are curvature lines of both quadrics. Then the string continues along geodesics on $\mathcal{E}_0$ or $\mathcal{H}_0$, while point P traces a portion of an ellipsoid $\mathcal{E}$ being confocal with $\mathcal{E}_0$ and $\mathcal{H}_0$.

Staude's string constructions of ellipsoids are subject of two models in Schilling's famous collection of mathematical models (listed in [8]), namely the models VII, no. 191 and 192 (see https://mathematical-models.org/index.php/models/view/345 and https://mathematical-models.org/index.php/models/view/279, Digitales Archiv Mathematischer Modelle, TU Dresden). According to D. Hilbert, Staude's string constructions of quadrics were one of the great mathematical results of the 19$^{\text{th}}$ century [1, p. 236].

We present a synthetic approach to these constructions, thus reducing the proof to uniqueness theorems of first order differential equations. Moreover, we discuss the case of focal parabolas and, similar to the second version mentioned above, that of confocal paraboloids. For historical remarks, generalizations an additional references see [2], [3, p. 11], [5, p. 19], [6], or [9, Theorem 4.3]. Since the string constructions result from properties of quadrics in a confocal family, we start recalling the relevant ones.

2 Confocal central quadrics

Let $\mathcal{E}$ be a triaxial ellipsoid with semiaxes a, b, and c. The one-parameter family of quadrics being *confocal* with $\mathcal{E}$ is given as

$$F(x,y,z;k) := \frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} - 1 = 0, \tag{1}$$

where $k \in \mathbb{R} \setminus \{-a^2, -b^2, -c^2\}$ serves as a parameter within the family. In the case $a > b > c > 0$, this family includes

$$\text{for} \begin{cases} -c^2 < k < \infty & \text{triaxial ellipsoids,} \\ -b^2 < k < -c^2 & \text{one-sheeted hyperboloids,} \\ -a^2 < k < -b^2 & \text{two-sheeted hyperboloids.} \end{cases} \tag{2}$$

Confocal quadrics intersect their common planes of symmetry along confocal conics. As limits for $k \to -c^2$ and $k \to -b^2$ we obtain 'flat' quadrics, i.e., the focal ellipse e and the focal hyperbola h, satisfying

$$e\colon\ \frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1,\ z = 0\,, \quad h\colon\ \frac{x^2}{a^2-b^2} - \frac{z^2}{b^2-c^2} = 1,\ y = 0\,. \tag{3}$$

They form a pair of *focal conics*.[1]

The confocal family sends through each point $P = (\xi, \eta, \zeta)$ outside the coordinate planes, i.e., with $\xi\eta\zeta \neq 0$, exactly one ellipsoid, one one-sheeted hyperboloid, and one two-sheeted hyperboloid. The respective parameters (k_1, k_2, k_3) define the three *elliptic coordinates* of P, where

$$-a^2 < k_3 < -b^2 < k_2 < -c^2 < k_1\,. \tag{4}$$

For given Cartesian coordinates (ξ, η, ζ) of any point P, we obtain the elliptic coordinates by solving $F(\xi, \eta, \zeta; k) = 0$ in (1) for k. Conversely, if the tripel (k_1, k_2, k_3) of elliptic coordinates is given, then the Cartesian coordinates (ξ, η, ζ) of the corresponding points $P \in \mathcal{E}$ satisfy

$$\begin{gathered} \xi^2 = \frac{(a^2+k_1)(a^2+k_2)(a^2+k_3)}{(a^2-b^2)(a^2-c^2)}, \quad \eta^2 = \frac{(b^2+k_1)(b^2+k_2)(b^2+k_3)}{(b^2-c^2)(b^2-a^2)}, \\ \zeta^2 = \frac{(c^2+k_1)(c^2+k_2)(c^2+k_3)}{(c^2-a^2)(c^2-b^2)}. \end{gathered} \tag{5}$$

[1] Conics of a pair of focal conics lie in orthogonal planes and share the principal axis. The focal points of one conic coincide with vertices of the other (see Fig. **??** and, e.g., [4, Sect. 4.2]).

There exist eight such points, symmetric w.r.t. the coordinate planes.

At each point P outside the coordinate planes, the surface normals

$$\mathbf{v}_i := \left(\frac{\xi}{a^2+k_i},\ \frac{\eta}{b^2+k_i},\ \frac{\zeta}{c^2+k_i}\right),\ i=1,2,3 \tag{6}$$

to the three quadrics through P are mutually orthogonal. Therefore, confocal quadrics form a triply orthogonal system of surfaces. Due to a classical theorem of Dupin, they intersect each other along lines of curvature.

Lemma 1. *The tangent cones from any point P to the quadrics of a confocal family are confocal with the cones connecting P with the focal conics. Their common planes of symmetry are tangent to the quadrics passing through P. The tangent cones are coaxial cones of revolution if and only if P is a point of a focal conic.*

For the definition of confocal quadric cones see, e.g., [7, p. 284]. A proof of Lemma 1 can be found in [7, p. 286]

Given a confocal family, each line other than a generator of any contained ruled quadric contacts exactly two surfaces of the family, and the tangent planes at the corresponding points of contact are orthogonal. This results in the lemma below, which dates back to Jacobi (1839) and Chasles (note [7, p. 291]).

Lemma 2. *On each quadric $\mathcal{Q}$, the geodesics are curves with tangents contacting another fixed quadric $\mathcal{Q}'$ that is confocal with $\mathcal{Q}$ (see* [7, Fig. 7.7]*).*

3 String constructions of central quadrics

Theorem 1 (Staude's first string construction). *Let e be an ellipse with the focal hyperbola h. Let F_1 denote a vertex of e and focal point of h and F_2 the focal point of e and vertex of h at a greater distance to F_1. A string of given length, fixed with one end at F_1, is passed behind the nearest branch of h and in front of e and finally attached to F_2. If the string is stretched at a point P such that it forms a spatial polygon with vertices F_1, $G_1 \in h$, P, $G_2 \in e$, and F_2, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with e and h (see Fig. 1).*

The presented proof is based on two lemmas.

Lemma 3. *Let a string with fixed endpoints F_1 and F_2 be stretched over a given curve c. Then, the corner-point $G \in c$ of the string satisfies two conditions:*
(i) *The tangent t_G to c at G subtends congruent angles with the straight segments F_1G and F_2G, and*
(ii) *the normal plane to c at G either passes through both endpoints or separates F_1 and F_2. In the latter case, the lines $[F_1, G]$ and $[F_2, G]$ are generators of a cone of revolution with apex G and axis t_G (Fig. 2).*

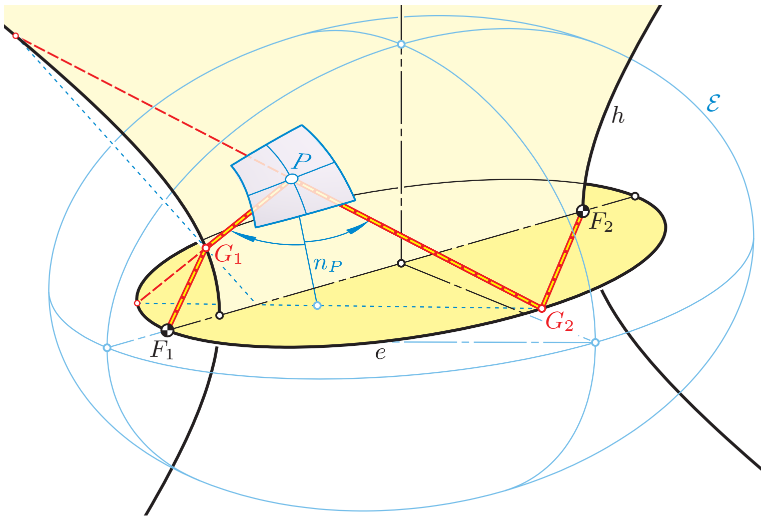

Fig. 1. Staude's first string construction of an ellipsoid uses its focal conics e and h.

Proof. When the string has reached its equilibrium at $G \in c$, the stress along the string induces two forces of equal quantity which act along the segments GF_1 and GF_2 and result in a force orthogonal to c. Therefore, the components of the two forces in direction of the tangent t_G must be opposite in order to compensate each other (see Fig. 2). This implies congruent angles between t_G and the two segments of the strengthened string. □

It is noteworthy that G needs not be unique. If, for example, the curve c is an ellipse with focal points F_1 and F_2, then each point $G \in c$ satisfies the claimed equilibrium condition, since the sum of distances $\overline{GF_1}$ and $\overline{GF_2}$ is stationary. Other examples can be found below in Fig. 7 (with $c = o$) or in [4, p. 143, Fig. 4.17]).

Lemma 4. *Let a strengthened string of given length with one fixed endpoint F_1 be bent over a curve c while the second endpoint P traces a smooth curve p. Then, at each pose P, the curve p is orthogonal to the final segment of the string.*

Proof. With respect to F_1 as the origin of a coordinate frame, the curve c can be parametrized as $\mathbf{c}(t) = \lambda(t)\mathbf{e}_1(t)$ with $\|\mathbf{e}_1(t)\| = 1$ for t in some interval J. After being bent over c, the upper segments form a ruled surface, and the position vector of the trajectory of P can be written as

$$\mathbf{f}(t) = \mathbf{c}(t) + (k - \lambda(t))\,\mathbf{e}_2(t) \text{ with } \|\mathbf{e}_2(t)\| = 1 \text{ for } k = \text{const.}$$

The angle condition claimed in Lemma 3 implies for all $t \in J$

$$\langle \dot{\mathbf{c}}, \mathbf{e}_2 \rangle = \langle \dot{\mathbf{c}}, \mathbf{e}_1 \rangle, \text{ and hence } \langle \dot{\lambda}\mathbf{e}_1 + \lambda\,\dot{\mathbf{e}}_1,\, \mathbf{e}_2 \rangle = \dot{\lambda}$$

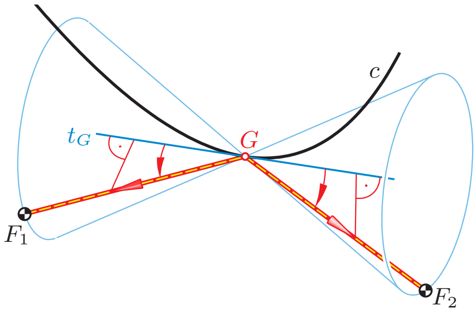

Fig. 2. A string with fixed endpoints F_1, F_2 and stretched over the curve c makes equal angles with the tangent to c at the vertex G (Lemma 3).

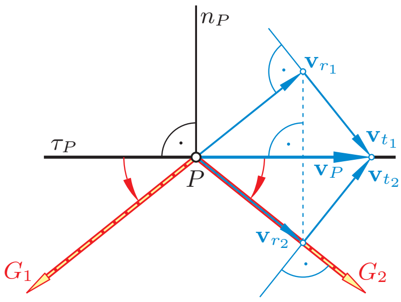

Fig. 3. Decomposition of the velocity vector $\mathbf{v}_P$ at P, while the length of the strengthened string $F_1G_1PG_2F_2$ is kept fixed.

if the dot indicates differentiation by t. Consequently, we obtain

$$\langle \dot{\mathbf{f}}, \mathbf{e}_2 \rangle = \langle \dot{\mathbf{c}} - \dot{\lambda}\mathbf{e}_2 + (k-\lambda)\,\dot{\mathbf{e}}_2,\ \mathbf{e}_2 \rangle = \dot{\lambda} + \langle -\dot{\lambda}\mathbf{e}_2 + (k-\lambda)\,\dot{\mathbf{e}}_2,\ \mathbf{e}_2 \rangle = \dot{\lambda} - \dot{\lambda} = 0.$$

This proves Lemma 4. □

Proof. [Theorem 1] By virtue of Lemma 3, point $G_1 \in h$ is the apex of a cone of revolution which passes through F_1 and P and has the tangent t_{G_1} to h as its axis (Fig. 1). We learned in Lemma 1 that both conics e and h are the locus of apices of cones of revolution which pass through the other focal conic, and the axes of these cones are tangents to the conic. Therefore, since the segment G_1F_1 meets the focal ellipse e, the same must hold for the extension of the segment G_1P.

Lines through the point P meeting e and h are common generators of two confocal cones (Lemma 1). Thus, if there exists one transversal, then there are four that are mutually symmetric w.r.t. the tangent planes to the three confocal quadrics through P. A 180° rotation about the surface normal n_P of the ellipsoid $\mathcal{E}$ through P takes the line $[P, G_1]$ to a line $[P, G_2]$ which again meets the two focal conics e and h. The traces of the plane $[P, G_1, G_2]$ in the planes of e and h reveal that, starting from P, the line $[P, G_2]$ meets first e and then h. Due to the properties of a pair of focal conics, the bent portion PG_2F_2 of the string is in equilibrium because of Lemma 3.

Let point P move in such a way that the string remains strengthened. We are going to prove that in this case the tangents to all possible trajectories of P are orthogonal to n_P.

If the point P is fixed on the moving string, then, by Lemma 4, the tangent vector $\mathbf{v}_{t_1}$ of the point P is orthogonal to PG_1. Similarly, for the point P being fixed on the final portion of the string, the velocity vector $\mathbf{v}_{t_2}$ would be orthogonal to PG_2. Because of the constant total length of the string, the relative velocities of P with respect to the two parts of the string must be equal; when the length of the initial part increases, that of the final part must decrease about the same rate, and vice versa. This implies for the vector $\mathbf{v}_P$ of the absolute velocity of P

$$\mathbf{v}_P = \mathbf{v}_{t_1} + \mathbf{v}_{r_1} = \mathbf{v}_{t_2} + \mathbf{v}_{r_2} \tag{7}$$

that the vectors $\mathbf{v}_{r_1}$ and $-\mathbf{v}_{r_2}$ are symmetric w.r.t. n_P. The orthogonal projection of the involved vectors into the plane $[P, G_1, G_2]$ reveals (see Fig. 3) that $\mathbf{v}_P$ must be orthogonal to n_P.
Consequently, at all poses in some neighbourhood, point P moves tangentially to the confocal ellipsoid through P, or in other words, if we set up the wanted trajectory of P by $F(x, y, z) = 0$, then with an appropriate scalar $\lambda(x, y, z)$

$$\mathbf{grad}\,\lambda F = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right).$$

Now we conclude, based on theorems from the theory of differential equations, that P traces a patch of the unique ellipsoid $\mathcal{E}$ through the given initial pose.

Conversely, if P remains on the ellipsoid $\mathcal{E}$, then $\mathbf{v}_P$ is orthogonal to n_P. This implies equal relative velocities $\|\mathbf{v}_{r_1}\| = \|\mathbf{v}_{r_2}\|$ in appropriate directions, and therefore, a constant length of the string. □

We find the total length L of the string by specifying the point P at one of the vertices of the ellipsoid $\mathcal{E}$. This yields $L = 2a + a_e - a_h$, where a, a_e, and a_h are the respective principal semiaxes of the ellipsoid $\mathcal{E}$, the focal ellipse e, and the focal hyperbola h. It should be noted that W. Böhm [2] used Ivory's Theorem (see, e.g., [7, Sect. 7.2]) to prove that, for all poses of P, the sum of distances equals L.

Remark 1. 1. One cannot obtain the complete ellipsoid with the string construction described in Theorem 1, since the string, starting at F_1 and coming from behind, has to be bent around the hyperbola h. This does not work if point P also lies behind the plane spanned by h. With regard to the other end of the string, the point P cannot lie under the plane of the ellipse e.
2. The same ellipsoid can be generated by using the remaining two common generators of the confocal cones which connect P with the pair of confocal conics. The two strengthened strings could even be bound together at P by a small ring through which the two strings can glide independently from each other, while P remains on the quadric.

Corollary 1. *The string construction of Theorem 1 for the triaxial ellipsoid $\mathcal{E}$ remains valid if the fixed endpoints F_1 and F_2 are replaced by other sufficiently close points of the respective conics. This variation affects only the total length L of the string and the boundary of the domain, that is traced by the point P on the ellipsoid $\mathcal{E}$.*

Proof. The condition stated in Lemma 3 remains valid when F_1 is replaced by a sufficiently close point $F_1' \in e$ (Fig. 4). On the other hand, since t_{G_1} forms

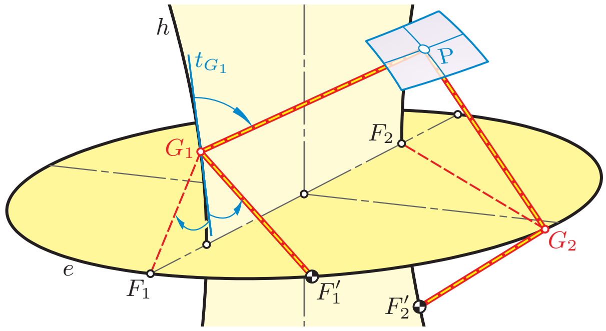

Fig. 4. The fixed points F_1 and F_2 can be replaced by $F_1' \in e$ and $F_2' \in h$.

congruent angles with G_1F_1 and G_1F_1', while F_1 and F_1' lie on the same side of the normal plane to t_{G_1} at G_1, the difference of distances

$$d := \overline{G_1F_1'} - \overline{G_1F_1}$$

remains constant. Therefore, the difference d must be added to the total length L of the string in order to keep P on the same ellipsoid. The same is valid for the other fixed endpoint $F_2 \in h$. □

In [10], Staude presented another string construction, which is also documented as a historical model in [8, p. 139]. It generalizes the version of Graves's construction on an ellipsoid $\mathcal{E}_0$, as displayed in [7, Fig. 7.14].

Theorem 2 (Staude's second string construction). *Let a string of appropriate length with both ends be attached to a pair of antipodal curvature lines e_1, e_2 of an ellipsoid $\mathcal{E}_0$ and kept taut so that it follows a geodesic crossing from e_1 to e_2. If we elongate this string to a fixed length and keep it taut at a point P between the two curves e_1 and e_2, then P traces a patch of an ellipsoid $\mathcal{E}$ confocal with $\mathcal{E}_0$. Conversely, for a point P moving locally on $\mathcal{E}$, the length of the described taut string connecting e_1 via P with e_2 remains fixed.*

If the string is sufficiently short, then if follows, from the two antipodal curvature lines $e_1, e_2 \subset \mathcal{E}_0$ on, geodesic arcs and furtheron respective tangents meeting at the point P (see Fig. 5). By virtue of Lemma 2, the two tangents $[P, T_1]$ and $[P, T_2]$ contact $\mathcal{E}_0$ and a confocal hyperboloid $\mathcal{H}_0$, which is the second confocal quadric through e_1 and e_2. Due to Lemma 1, they are common to two confocal tangent cones with apex P, and consequently, in symmetric position w.r.t. the normal at P to one of the three confocal quadrics through P. In order to prove Theorem 2, we need a statement similar to Lemma 4.

Lemma 5. *Let one end of a strengthened string of given length be attached to a line e of curvature, which is the intersection of an ellipsoid $\mathcal{E}_0$ with a confocal hyperboloid $\mathcal{H}_0$. Suppose that, in each pose, the string is a C^1-composition of*

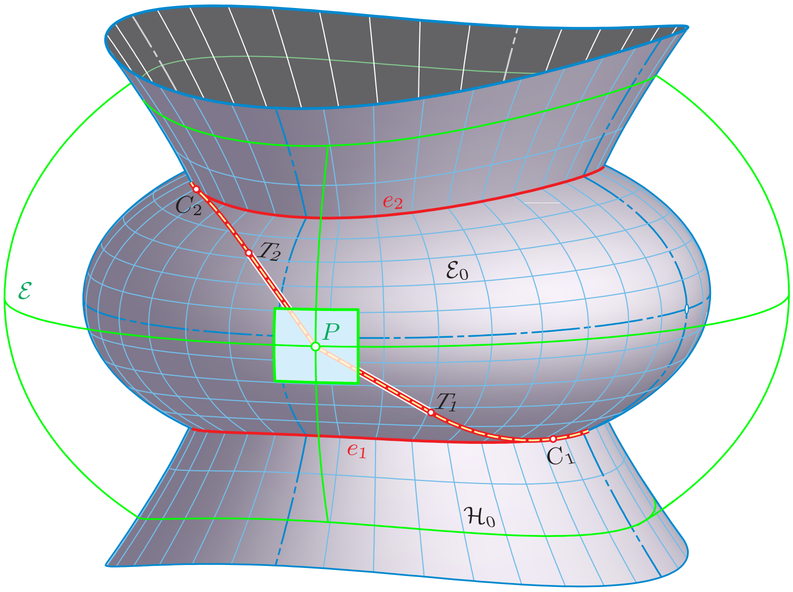

Fig. 5. According to Staude's generalized string construction, the two ends of the string have to be attached to two antipodal curvature lines e_1, e_2 of an ellipsoid $\mathcal{E}_0$, while the point P moves on a confocal ellipsoid $\mathcal{E}$.

three arcs. It begins along the curved edge e, continues from a point $C \in e$ on along a geodesic $c \subset \mathcal{E}_0$ until point T. Finally, there is a straight segment tangent to c at T. Then, in which way ever, the second endpoint P of the string moves smoothly in space, its trajectory p is orthogonal to the straight segment TP.

Proof. Any point Q that is attached to the string between C and T runs on an orthogonal trajectory of the geodesic c. There is a local parametrization $\mathbf{x}(u,v)$, $(u,v) \in I \times J$, of $\mathcal{E}_0$ with geodesics tangent to c as u-lines and its orthogonal trajectories as v-lines. By virtue of a theorem by Gauss, u can be assumed as common arc length along the geodesics. This implies for the partial derivatives

$$\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1, \quad \langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle = \langle \mathbf{x}_u, \mathbf{x}_{uv} \rangle = \langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle = 0$$

for all $(u,v) \in I \times J$. The equations $\langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle = \langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle = 0$ confirm that the osculating planes of the u-lines are orthogonal to the tangent plane.

Now we distinguish between two cases:
(i) If, under the motion of the string's endpoint P in space, the point C and the geodesic c remain fixed, i.e., T runs along c, then the point P traces an involute, which is an orthogonal trajectory of the generators on the tangent surface of c.
(ii) Otherwise, the v-coordinate of T varies. Let T trace the curve $\mathbf{p}(t)$ on $\mathcal{E}_0$ given by $u = u(t)$ and $v = t$ for $t \in J$. If the point P is supposed to be attached to the string, then we obtain for its path the parametrization

$$\mathbf{p}(t) = \mathbf{x}\,(u(t),\, t) + (k - u(t))\,\mathbf{x}_u(t) \quad \text{with} \quad k = \text{const.}$$

From

$$\dot{\mathbf{p}}(t) := \frac{\mathrm{d}\,\mathbf{p}(t)}{\mathrm{d}t} = \dot{u}\,\mathbf{x}_u + \mathbf{x}_v - \dot{u}\,\mathbf{x}_u + (k-u)(\dot{u}\,\mathbf{x}_{uu} + \mathbf{x}_{uv})$$

follows the stated orthogonality, since

$$\langle \mathbf{x}_u,\, \dot{\mathbf{p}} \rangle = \langle \mathbf{x}_u,\, \mathbf{x}_v \rangle + (k-u)\langle \mathbf{x}_u,\, \dot{u}\,\mathbf{x}_{uu} \rangle + (k-u)\langle \mathbf{x}_u,\, \mathbf{x}_{uv} \rangle = 0.$$

The same holds when we replace the ellipsoid $\mathcal{E}_0$ by the hyperboloid $\mathcal{H}$. □

Proof. [Theorem 2] Based on Lemma 5, the proof is similar to that of Theorem 1. With respect to the part of the string attached to the line of curvature e_i, $i \in \{1,2\}$, a point P which is fixed on the moving string has a tangent vector $\mathbf{v}_{t_i}$ orthogonal to the segment PT_i. If, additionally, the point P is moving relative to the string with velocity vector $\mathbf{v}_{r_i}$ in direction of PT_i, we obtain for the absolute velocity of P the decomposition (7). When the total length of the string remains constant, the relative velocities $\|\mathbf{v}_{r_1}\|$ and $\|\mathbf{v}_{r_2}\|$ must be equal. This implies, as depicted in Fig. 3, that $\mathbf{v}_P$ is orthogonal to the interior angle bisector of $\angle T_1 P\, T_2$ and tangent to the confocal ellipsoid $\mathcal{E}$ passing through P.
Conversely, if P remains on the ellipsoid, then we obtain equal relative velocities in appropriate directions, and therefore, a constant length of the string. □

4 Paraboloids

The quadrics confocal with an elliptic paraboloid can be represented as

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} - 2z - k = 0 \quad \text{for} \quad k \in \mathbb{R} \setminus \{-a^2, -b^2\}. \tag{8}$$

In the case $a > b > 0$, this one-parameter family contains

$$\text{for} \begin{cases} -b^2 < k < \infty & \text{elliptic paraboloids,} \\ -a^2 < k < -b^2 & \text{hyperbolic paraboloids,} \\ k < -a^2 & \text{elliptic paraboloids.} \end{cases} \tag{9}$$

For each k, the vertex of the corresponding paraboloid has the coordinates $(0,0,-k/2)$. The point $(0,0,b^2/2)$ is the common focal point of the principal sections in the plane $x=0$, and $(0,0,a^2/2)$ is the focus for sections in $y=0$.

The limits for $k \to -b^2$ or $k \to -a^2$ define the pair of *focal parabolas*

$$p_1\colon\; \frac{x^2}{a^2-b^2} - 2z + b^2 = 0,\; y = 0\,, \quad p_2\colon\; \frac{y^2}{a^2-b^2} + 2z - a^2 = 0,\; x = 0\,. \tag{10}$$

within the family of confocal paraboloids. The vertex of each focal parabola coincides with the focal point of the other parabola. Therefore, this pair is the same as shown in [4, Fig. 4.15]: each parabola is the locus of apices of cones of revolution passing through the other parabola. This holds since the results stated in Lemmas 1 and 2 are also true for confocal paraboloids.

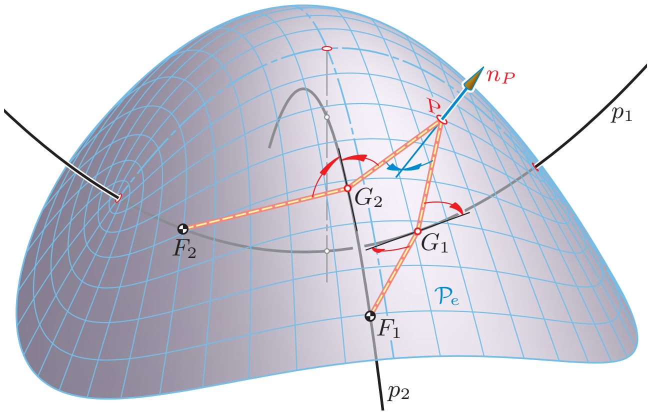

Fig. 6. A string construction based on focal conics fails at the elliptic paraboloid $\mathcal{P}_e$: Lemma 3,(ii) is not satisfied at the point G_2.

When the first string construction, as displayed in Fig. 4, is modified and applied to two focal parabolas p_1, p_2, then it fails for two reasons:

(i) At an elliptic paraboloid $\mathcal{P}_e$ (Fig. 6), the strengthened string does not satisfy the second condition of Lemma 3, i.e., the normal plane of p_2 at G_2 does not separate the two adjacent segments G_2F_2 and G_2P.

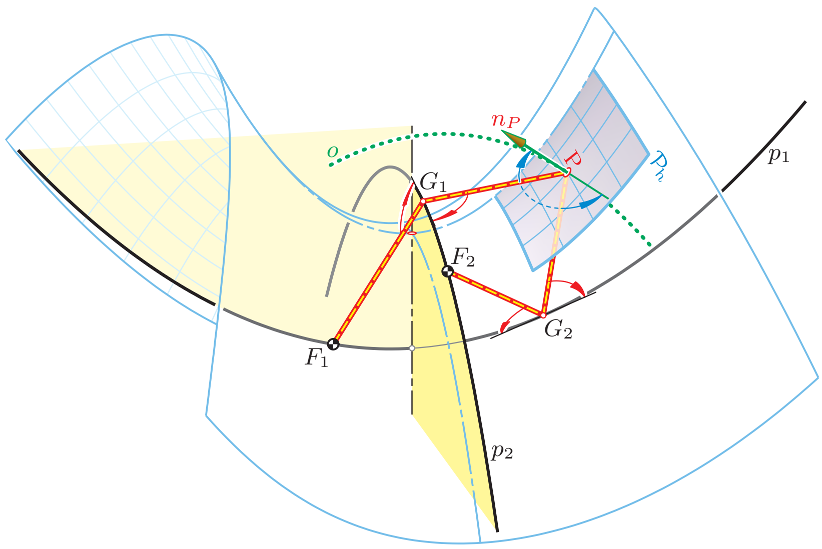

Fig. 7. The difference $(\overline{PG_1} + \overline{G_1F_1}) - (\overline{PG_2} + \overline{G_2F_2})$ of the strings' lengths remains constant if point P moves on the hyperbolic paraboloid $\mathcal{P}_h$. For points P on the orthogonal trajectory o (dotted green line) of the confocal hyperbolic paraboloids the total length of the strengthened string with fixed endpoints F_1, F_2 remains constant.

(ii) At a hyperbolic paraboloid $\mathcal{P}_h$ (Fig. 7), the surface normal n_P of $\mathcal{P}_h$ is not the interior angle bisector of $\angle G_1PG_2$, but the exterior. Therefore, we can state for points $P \in \mathcal{P}_h$ that the difference of lengths of the two parts F_1G_1P and F_2G_2P of the string,

$$(\overline{PG_1} + \overline{G_1F_1}) - (\overline{PG_2} + \overline{G_2F_2})$$

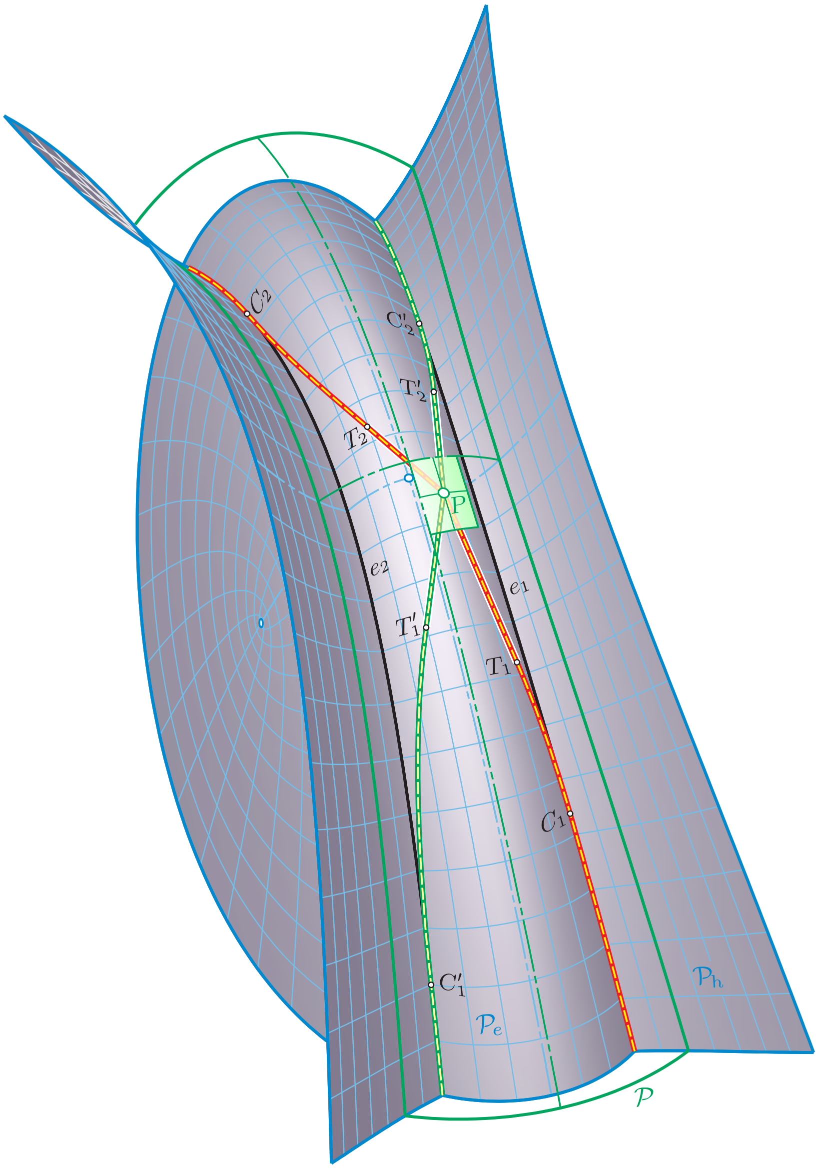

Fig. 8. String construction of the elliptic paraboloid $\mathcal{P}$.

remains constant. Otherwise, for points P running on the orthogonal trajectory o of the family of confocal hyperbolic paraboloids, the sum of these two lengths remains constant, i.e., the string remains strengthened. However, the string does not define a constrained motion of P but admits two degrees of freedom.[2]

On the other hand, Staude's second string construction remains valid for paraboloids. Figure 8 shows a string of fixed length with both ends attached to the two connected components e_1, e_2 of the line of curvature that is shared by the confocal paraboloids $\mathcal{P}_e$ and $\mathcal{P}_h$. If this string is strengthened at the point P, then P is movable on an elliptic paraboloid $\mathcal{P}$ that is confocal with $\mathcal{P}_e$ and $\mathcal{P}_h$. The proof is the same as that for Theorem 2, when the points C_i on the string denote for $i = 1, 2$ the endpoints of the sections along the line of curvature, while T_iP are the straight segments tangent to the geodesics at T_i.

If conversely point P moves locally on the elliptic paraboloid $\mathcal{P}$, then the string remains strengthened, because n_P is the interior angle bisector of $\angle T_1PT_2$. There are even two possibilities for this string (Fig. 8) since the four common tangents from P to $\mathcal{P}_e$ and $\mathcal{P}_h$ consist of two pairs of lines which are symmetric w.r.t. the surface normal n_P of $\mathcal{P}$ at P (note Lemma 1). We summarize:

Theorem 3. *Staude's second string construction, as explained in Theorem 2 for triaxial ellipsoids, works similar for elliptic paraboloids, when the two ends of the string are attached to different components of the intersection curve between confocal elliptic and hyperbolic paraboloids (Fig. 8).*

References

1. M. Berger: *Geometry Revealed: A Jacob's Ladder to Modern Higher Geometry.* Springer, Berlin, Heidelberg 2010.
2. W. Böhm: Die Fadenkonstruktionen der Flächen zweiter Ordnung. Math. Nachr. **13**, 151–156 (1955).
3. I.I. Dincă: *Thread configurations for ellipsoids.* arXiv:0902.1421v2 [math.DG] 10 Feb 2010
4. G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics.* From the ancient Greeks to 21st century developments. Springer Spectrum, Berlin, Heidelberg 2016.
5. D. Hilbert, St. Cohn-Vossen: *Anschauliche Geometrie.* 2nd ed., Springer, Berlin, 1996. English translation: *Geometry and the Imagination.* Chelsea Publishing, reprinted by American Mathematical Society, Providence, RI, 1999.
6. I. Izmestiev, S. Tabachnikov: *Ivory's Theorem revisited.* Journal of Integrable Systems **2**/1 (2017), xyx006 (2017).
7. B. Odehnal, H. Stachel, G. Glaeser: *The Universe of Quadrics,* Springer Verlag Berlin Heidelberg 2020, 606 p., 365 Figs., ISBN 978-3-662-61052-7
8. M. Schilling: *Catalog mathematischer Modelle.* 7. Auflage, Martin Schilling, Leipzig 1911.
9. O. Staude: *Über Fadenconstructionen des Ellipsoides.* Mathematische Annalen **20**, 147–185 (1882).
10. O. Staude: *Die Focaleigenschaften der Flächen zweiter Ordnung.* B.G. Teubner, Leipzig 1896.

[2] For the sake of brevity, proofs for the cases (i) and (ii) are left for a future paper.