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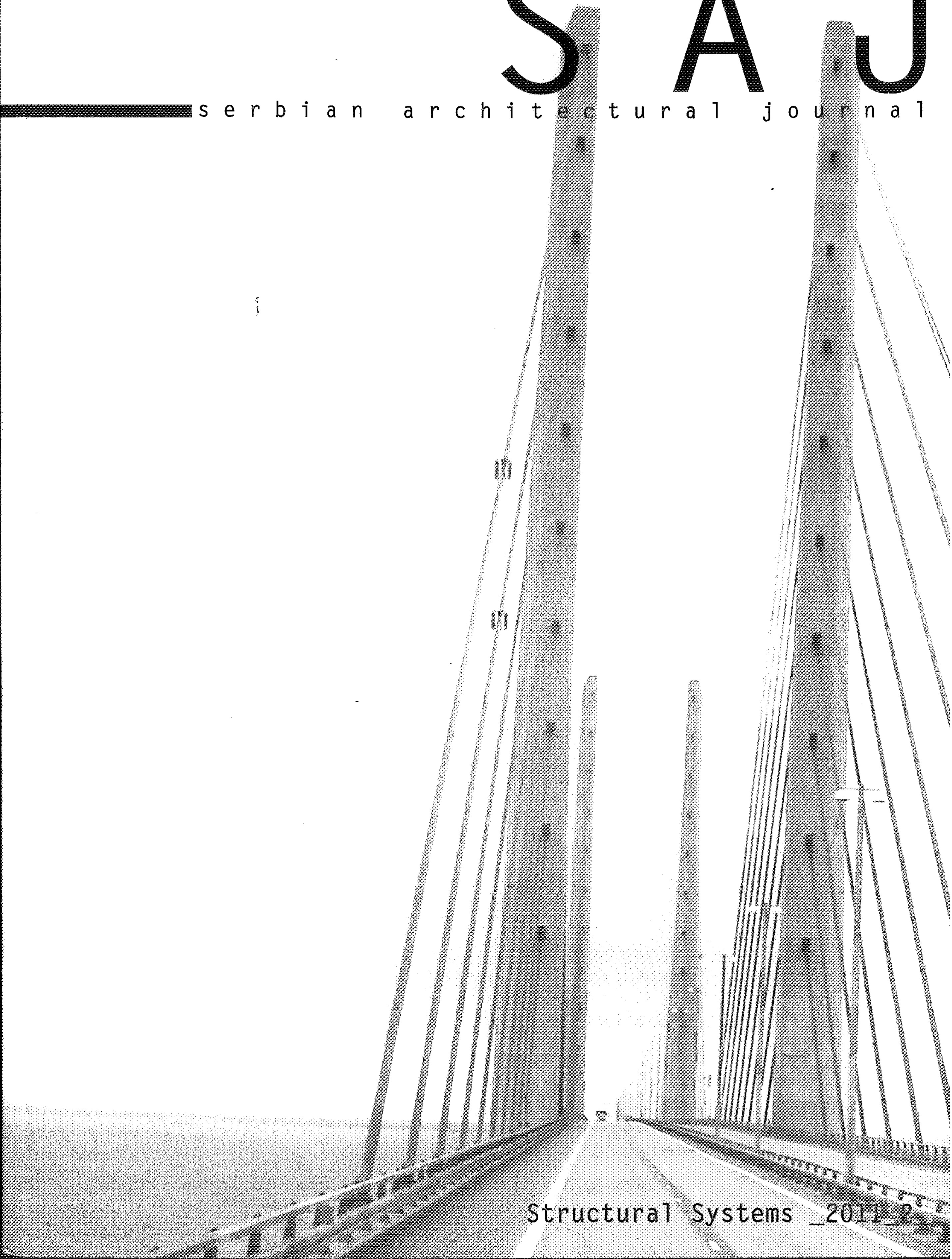

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WHAT LIES BETWEEN RIGIDITY AND FLEXIBILITY OF STRUCTURES

A B S T R A C T

The borderline between continuous flexibility and rigidity of structures like polyhedra or frameworks is not strict. There can be different levels of infinitesimal flexibility. This article presents the mathematical background and some examples of structures which under particular conditions are flexible or almost flexible and otherwise rigid.

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KEY WORDS

RIGIDITY
FLEXIBILITY
INFINITESIMAL FLEXIBILITY
FLEXIBLE POLYHEDRA
SNAPPING POLYHEDRA
KOKOTSAKIS MESHES
ORIGAMI MECHANISMS

INTRODUCTION

A framework or a polyhedron will be called *"rigid"* when the edge lengths determine its planar or spatial shape uniquely; under the term "shape" we mean its spatial form – apart from movements in space.

More generally and under inclusion of smooth or piecewise linear surfaces, a structure is called rigid when its *intrinsic metric* defines its spatial shape uniquely. In this sense, the intrinsic metric of a polyhedron is defined by its net (unfolding), i.e., the coplanar set of faces with identified pairs of edges originating from the same edge of the spatial form. After cutting out this net from paper or cardboard, a paper- or cardboard-model of this polyhedron can be built in the usual way. Does such a net really define the shape of a polyhedron uniquely?

Think of a cube where one face is replaced by a four-sided pyramid with a small height. Then, obviously two different polyhedra can be built, one convex form with the pyramid erected towards outside, the other with the apex of the pyramid inside the cube (Fig. 1). So, there are two polyhedra, two *realizations*, stemming from the same net, i.e., with the same intrinsic metric. In the convex case the internal dihedral angles along the edges of the pyramid are < 180°, in the other case they are > 180°.

If the height is sufficiently small, distort the convex polyhedron can be distorted by applying slight force and change to the concave form. In this case we speak of *"snapping"* polyhedra. Both realizations are *"locally rigid"*, i.e., there is no other realization of the same intrinsic metric sufficiently close to the one under consideration. *"Globally rigid"* is a structure for which the intrinsic metric defines its spatial shape uniquely, apart from displacements as a rigid body. Each three-sided pyramid (tetrahedron) is globally rigid.

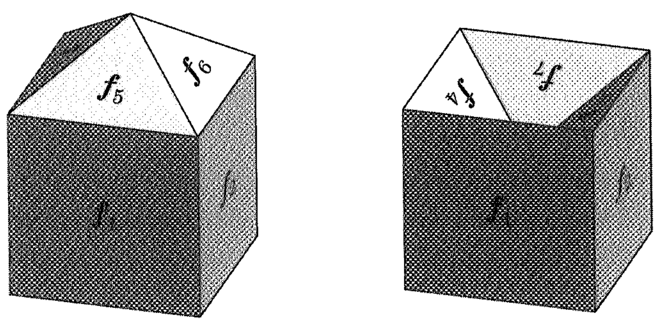

Figure 1 Two realizations of the same net

The first important result in the theory of rigidity claims that every convex polyhedron is globally rigid. This is due to A. L. Cauchy, 1813 [3].

A real-world model of a snapping polyhedron might look like a flexible one, but theoretically it is not flexible. The model admits small bendings of the faces and has some clearances at the hinges along the edges, and this causes the seeming flexibility which in any case is somehow limited within a certain neighborhood. A famous example is described in W. Wunderlich's article [18] on a polyhedron exhibited at the science exposition "Phänomena" in 1984 in Zürich. At that time it was falsely stated that this polyhedron is flexible, but it was only snapping between two different snap poses and one spatial shape.

Let us still think of a polyhedron made from cardbord with planar faces, but with variable dihedral angles between any two faces sharing a common edge. A polyhedron is called *"continuously flexible"*, when the dihedral angles of the polyhedron can vary continuously while the intrinsic metric remains invariant. Sometimes, this is called a *selfmotion* of the structure. R. Bricard classified in 1897 [2] all flexible octahedra, i.e. all flexible four-sided double-pyramids. However, all these polyhedra have self-intersections. Flexible polyhedral structures can be extracted from Bricard's octahedra only when either some faces are omitted [19] or when the polyhedron with 8 triangular faces is seen as a framework with 12 edges.

The first continuously flexible polyhedron without self-intersections was detected in 1977 by R. Connelly [4]. A "flexing sphere" with 9 vertices only was found in 1978 by K. Steffen [13] as a combination of two flexible octahedra.

At the first glance, it might be surprising that even for a *regular* octahedron there exists a continuously flexible realization with the same intrinsic metric.

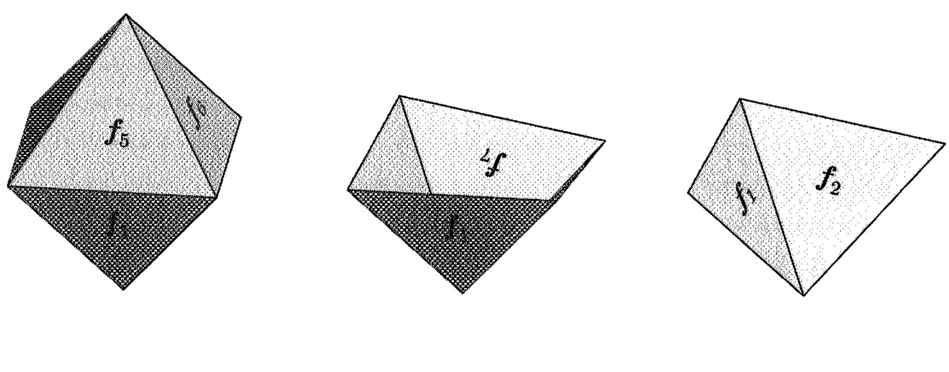

Figure 2 The regular octahedron and its re-assembled and continuously flexible versions

The structure can be re-assembled by putting one four-sided pyramid into the other. This gives a twofold covered quadratic pyramid without basis, which of course is flexible (Fig. 2). This is no contradiction with Cauchy's result, because there we have the restriction to *convex* spatial shapes only, and the regular octahedron is indeed locally rigid.

It turns out that the computation of the spatial from of any four-sided double-pyramid, i.e., of any general octahedron with given unfolding is an algebraic problem of degree 8. Hence, up to 8 different realizations are possible. Apart from particular cases, each of these realizations is locally rigid.

The question whether the edge lengths of a polyhedral structure or framework determine its planar or spatial shape uniquely, is also important for many engineering applications, e.g., for mechanical or constructional engineers, for biologists in protein modeling or for the analysis of isomers in chemistry.

In the following text some flexible examples have been presented, and it has been emphasized that there is something between "continuously flexibility" and "rigidity", the *"infinitesimal flexibility"* which can also be seen as a limiting case of snapping structures after the two different realizations converge. But first of all the terminology and the mathematical background will be clarified.

DEFINITION OF RIGIDITY AND FLEXIBILITY

In the following definitions polyhedra is not seen not as piecewise linear surfaces, but as frameworks. This means one should concentrate on its edges only. If there are faces with more than three vertices, it must be replaced by some face-to-face tetrahedra erected over this face in order to keep the original face planar. Finally, it should be noted that technical problems like stiffness of edges and clearances along the hinges are not to be of a concern. The focus is just on geometry.

Definition 1.

A framework F in R^d, d = 2, 3,..., consists of a set $V = \{x_1, ..., x_v\}$ of vertices and a set E of edges, i.e., $E = \{(i, j) \mid 0 < i < j \leq v\}$. The length of the edge $x_i x_j$ of F is denoted by l_{ij}, and the functions

$$f_{ij}(y, z) := \|y - z\|^2 - l^2_{ij} \text{ for } y, z \in R^d. \qquad (1)$$

are defined.

The framework $\mathcal{F}$ is called *continuously flexible*, if there is a continuous family $\mathcal{F}_t$ of frameworks with vertices $x_1(t), ..., x_d(t)$ for $0 \leq t \leq 1$ with $\mathcal{F}_0 = \mathcal{F}$ and $f_{ij}(x_i(t), x_j(t)) = 0$ for all $(i, j) \in E$, provided there are at least two vertices x_k, x_l which do not keep their distance constant.

The family $\mathcal{F}_t$, $0 \leq t \leq 1$ is called a *flection* or *self-motion* of $\mathcal{F}$; each single $\mathcal{F}_t$ for a fixed t is called a pose of this flection.

It is said that, the edge set E defines *the combinatorial structure* of $\mathcal{F}$. By the request that at least one distance between vertices does not remain constant during the flection, *trivial* flections are excluded, i.e., pure motions of the framework as a rigid body, expressible in matrix form by

$$x_i(t) = a(t) + A(t)x_i \text{ for each } i \in \{1, ..., v\}$$

with $a(t) \in R^d$ and an orthogonal $(d \times d)$-matrix $a(t)$, i.e., $A^T = A^{-1}$.

The conditions for keeping the lengths of edges constant, are of algebraic nature. Hence, in the case of a continuously flexible $\mathcal{F}$ the flection as a function of t is not only continuous but *analytic* in t. Therefore each $x_i(t)$ can be expanded into Taylor series. This is the basis for the following definition.

Definition 2.

A framework $\mathcal{F}$ in R^d is called *infinitesimally flexible* or - more precisely - infinitesimally flexible *of order n*, $n \geq 1$, if for each $i \in \{1, ..., v\}$ there is a polynomial function

$$z_i(t) = x_i + x_{i,1}t + ... + x_{i,n}t^n,\ x_{i,j} \in R^d \text{ for } j \in \{1, ..., v\} \qquad (2)$$

such that the substitution of $z_i(t)$ in the distance functions f_{ij} gives functions with a zero at $t = 0$ of multiplicity $> n$, i.e., by using the Landau symbol

$$f_{ij}(z_i(t), z_j(t)) = o(t^n) \text{ for all } (i, j) \in E, \qquad (3)$$

provided, there is a pair (x_k, x_l) of vertices with $\| z_k - z_l \|^2 - \| x_k - x_l \|^2 \neq o(t^n)$. $Z(t) := (z_1(t), ..., z_v(t))$ is called an *infinitesimal flection* of $\mathcal{F}$ of order n.

The first derivative $x_{i,1}$ of $z_i(t)$ at $t = 0$ is called *velocity vector*. The second derivative $x_{i,2}$ in (2) is called *acceleration vector* of vertex x_i.

An infinitesimal flection would be called *trivial*, if the polynomial functions $z_i(t)$ originate from an infinitesimal motion of $\mathcal{F}$ as a rigid body, i.e., by an assignment

$$x_{i,1} = s + S\,x_i \text{ with } s \in R[t]^d,\ S \in R[t]^{d\times d} \text{ and } S^T = -S. \qquad (4)$$

This means, the components of s and the entries in S are polynomials in t and matrix S is skew-symmetric.

Remarks:

1. In [14] L.S. Velimirović and S.R. Rančić treat the analogue of first-order flexibility for smooth surfaces. In the flexible case one speaks of *infinitesimal bendings* of a surface.
2. A framework which admits only trivial flections, is called *first order rigid* order *infinitesimally rigid.*

Each *continuously flexible* framework admits a nontrivial analytic flection and is therefore also *infinitesimally flexible* of any order. Due to the algebraic character of Eq. (1), for each combinatorial type of framework there is a sufficiently high $n \in N$ such that infinitesimal flexibility of an order $\geq n$ implies continuous flexibility. This was proved by V. Alexandrov in [1]. On the other hand, T. Tarnai presented in frameworks which are infinitesimally flexible of order $2^m - 1$ for any m.

The conditions for a framework of given combinatorial structure to be infinitesimally flexible of given order can be obtained by substituting the polynomial functions $z_i(t)$ in the distance functions f_{ij} in (1) and comparing the coefficients of all powers of t up to n. This results in a series of systems of linear equations. So, checking whether a given framework is rigid or higher-order infinitesimally flexible is reduced to inspecting the solvability of these systems of linear equations step by step.

The converse, i.e., finding the geometric meaning of these conditions, is not as straight forward as one might expect. The system for first order flexibility is homogeneous. Therefore the existence of a nontrivial first-order flection is equivalent to a sufficiently high ranked efficiency of the coefficient matrix, the socalled *rigidity matrix* of F. The solution of the first system defines the values on the right-hand side in the inhomogeneous system for second-order flexibility. When this system is solvable, its solution defines the right-hand side values for the third system, and so on. This more or less technical method has been skipped and the focus in the coming section is on the underlying geometric conditions.

It should be mentioned, that there are several applications of first-order infinitesimal flexibility. In robotics, such infinitesimally flexible poses are called *singular* and usually avoided since at least one degree of freedom is missing there and the control of the robot close to singular poses becomes problematic. When in surveying the relative location of points is determined by measuring some of the mutual distances and when the underlying framework is infinitesimally flexible, then this pose is called *critical* and results in numerical instability.

INFINITESIMAL FLEXIBILITY VS. SNAPPING FRAMEWORKS

First order flexibility

The condition (3) for first-order infinitesimal flexibility means that for each $(i, j) \in E$ in the polynomial f_{ij} $(z_i(t), z_j(t))$ the coefficient of t must vanish. This is equivalent to

$$(x_i - x_j) \times (x_{i,1} - x_{j,1}) = 0 \qquad (5)$$

This vanishing scalar product means that for each edge $x_i x_j$ of F the components of the velocity vectors of x_i and x_j in direction of the edge are equal. This is called the *Projection Theorem* (see Fig. 3). To summarize:

Theorem 1.
A framework F is infinitesimally flexible if to each vertex x_i a velocity vector $x_{i,1}$ can be assigned of a type that for all edges of F the Projection Theorem (5) *is fulfilled.*

The first example in Fig. 4 shows a planar bipartite framework. Bipartite means that the vertices can be subdivided into two sets, and each edge connects

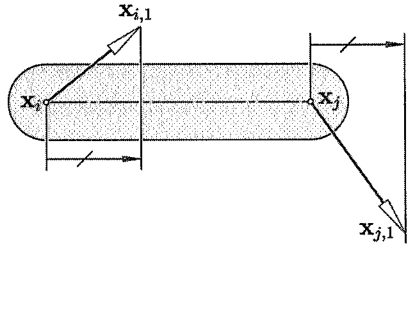

Figure 3 The Projection Theorem (Theorem 1)

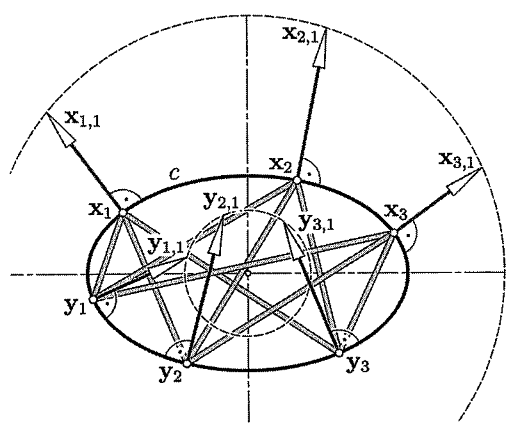

Figure 4 A planar bipartite framework is infinitesimally flexible if and only if the vertices are located on a 2nd-order curve

points from different sets. In our case there are six vertices x_i and y_j, $i, j \in \{1, 2, 3\}$, and 9 edges $x_i y_j$. It has been well known at least for one century that this framework is infinitesimally flexible if and only if the vertices are placed on a curve of degree 2, i.e., either on a conic c or on two lines. This is still true when more than 6 points x_i and y_j are specified on the same conic.

The analogous result is valid for any dimension d when the conic is replaced by any quadric in R^d. The following short proof owing to W. Whiteley [16] reveals that this condition is sufficient and that the velocity vectors can be chosen perpendicular to the quadric, as shown for $d = 2$ in Fig. 4.

Proof:

The coordinate vectors are written in columns and the equation of the quadric is set up in matrix form by $x^T Q x = k$ with a symmetric $(d \times d)$-matrix Q. Since x_i and y_j are located on the quadric, we have $x_i^T Q x_i = y_j^T Q y_j = k$. Now the velocity vectors are specified by $x_{i,1} = Q x_i$ and $y_{i,1} = -Q y_i$, and for the edge $x_i y_j$ it is verified that the Projection Theorem has been fulfilled. For this purpose the scalar product is written in matrix form and the following obtained

$$(x_i - y_j)^T (x_{i,1} - y_{j,1}) = (x_i - y_j)^T (Q x_i + Q y_j)$$
$$= x_i^T Q x_i - y_j^T Q x_i + x_i^T Q y_j - y_j^T Q y_j = 0,$$

since x_i and y_j fulfill the quadric's equation and the real number $x_i^T Q y_j$ equals its transpose $y_j^T Q^T x_i = y_j^T Q x_i$.

On the other hand, $(x_i^T Q) x = 0$ is the equation of the tangent plane of the quadric at point x_i, and hence the gradient of this plane, the vector $x_{i,1} = Q x_i$ is perpendicular to the quadric. The same holds true for point y_j (Fig. 4).

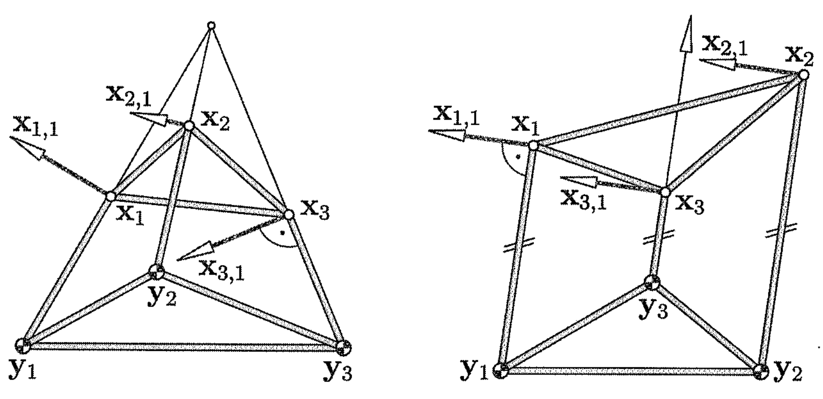

Figure 5 Another infinitesimally flexible framework with 6 vertices and 9 edges

The assignment of velocity vectors $x_{i,1}$ to the vertices x_i of an infinitesimally flexible framework F is not unique. Apart from a scaling, i.e., replacement of $x_{i,1}$ by $\alpha x_{i,1}$ for any fixed $\alpha \in R \setminus \{0\}$, an instantaneous motion can additionally be imposed according to (4). This means that to each $x_{i,1}$ the vector $s + Sx_i$ with $S^T = -S$ can be added without disturbing Eq. (5) since $(x_i - x_j)^T S(x_i - x_j)$ is the null form.

The next example displayed in Fig. 5 is again a planar framework with 6 vertices and 9 edges, but not bipartite. It is a pinned framework, i.e., vertices indicated by the black-and-white points in Fig. 5 are fixed. This framework is infinitesimally flexible if and only if the three lines $x_i y_i$ have one point in common or are parallel.

In the examples presented up to now (Figs. 4 and 5) the geometric characterization of infinitesimally flexible frameworks is of projective nature. If a collineation is applied on the flexible framework, it still remains infinitesimally flexible. This is surprising since rigidity is based on metric properties and they are changed under collinear transformations. However, the projective invariance of infinitesimal flexibility is a classical result and probably first proved in 1920 by H. Liebmann (1920) [7]. Alternative proofs can be found in [15] and [5].

Only first-order infinitesimal flexibility is projectively invariant. This follows from the examples of higher-order flexible frameworks presented in [8] and [9].

Infinitesimal flexibility can be seen as the limiting case where two realizations of a framework coincide. This was the way how W. Wunderlich studied infinitesimal flexibility. The next theorem reveals that there is a direct connection between snapping frameworks and infinitesimally flexible frameworks of the same combinatorial type. W. Whiteley [17] calls this correspondence *"averaging"*; in I. Izmestiev's paper [5] it is called *Pogorelov map*.

Theorem 2.

Let $y_1, \ldots, y_v$ *and* $y'_1, \ldots, y'_v$ *be the vertices of two incongruent realizations of a framework* F *with the same intrinsic metric, i.e., with the same edge lengths* l_{ij}. *Then the midpoints* $x_i = 1/2\ (y_i + y'_i)$ *of corresponding vertices make a framework* F *of the same combinatorial structure which is infinitesimally flexible with velocity vectors* $x_{i,1} = 1/2\ (y_i - y'_i)$.

Conversely, any infinitesimally flexible framework F *with vertices* $x^1, \ldots, x_v$ *and velocity vectors* $x_{1,1}, \ldots, x_{v,1}$ *gives rise to two incongruent realizations of a* *framework $\mathcal{F}$ of the same combinatorial type, namely that with vertices* $y_i = x_i + x_{i,1}$ *and* $y'_i = x_i - x_{i,1}$, *respectively.*

Proof:

The proof is unexpectedly short. For each edge of $\mathcal{F}$, i.e., for each $(i, j) \in E$ the equations $\| y_i - y_j \| = \| y'_i - y'_j \|$ can be rewritten as

$$(y_i - y_j)^2 - (y'_i - y'_j)^2 = 0$$

which is equivalent to

$$(y_i - y_j + y'_i - y'_j) \times (y_i - y_j - y'_i + y'_j) = 0$$

or

$$((y_i + y'_i) - (y_j + y'_j)) \times ((y_i - y'_i) - (y_j - y'_j)) = 0.$$

This is just the statement of the Projection Theorem (5) $(x_i - x_j) \times (x_{i,1} - x_{j,1}) = 0$ because of $2x_i = y_i + y'_i$ and $2x_{i,1} = y_i - y'_i$.

For any given infinitesimally flexible framework $\mathcal{F}$ the appointed velocity vectors $x_{i,1}$ can be replaced simultaneously by $\alpha x_{i,1}$ for any $\alpha \in R$. The smaller the absolute value of α, the closer the two obtained snapping poses are. The given flexible framework $\mathcal{F}$ is the limit for $\alpha \rightarrow 0$.

Figure 6 shows on the left-hand side two realizations derived from the infinitesimally flexible framework in Fig. 5, left. On the right-hand side it is illustrated how from the two realizations by the principle of *"averaging"* the infinitesimally flexible $\mathcal{F}$ is obtained. It can be proved that displacements of $\mathcal{F}'$ relative to $\mathcal{F}$ do not change the shape of $\mathcal{F}$.

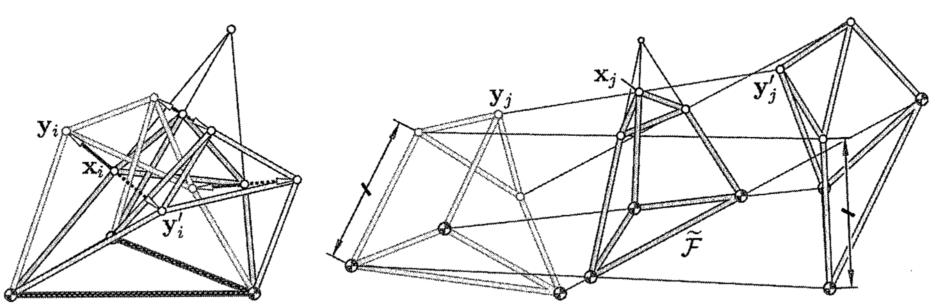

Figure 6 Left: A snapping framework obtained from that in Fig. 5. Right: An infinitesimal framework obtained by the principle of "averaging"

It could be proceeded in the same way with the infinitesimally bipartite framework obtained from Fig. 4. According to [10] it is known, that after a suitable displacement of one of the two realizations the vertices of any snapping bipartite framework are located on two confocal conics (quadrics). Then the snapping is the result of interchanging the two conics.

Flexible polygonal structures

The conclusion is with polyhedral structures which play a role in paper folding (*origami*) but also in new architectural surface design as *quad meshes*. The starting point is with a *Kokotsakis mesh* (German: Vierflach), an object named after A. Kokotsakis [6]. A quadrangular Kokotsakis mesh is the compound of 3×3 planar quadrangles. In Fig. 7, left, the scheme of a quadrangular Kokotsakis mesh is shown with a central face f_0 and a belt of 8 quadrangles around it. On the right hand side a flection of a continuously flexible version is displayed.

Though a complete classification of all flexible cases is still an open problem, some particular conditions are known which give rise to continuously flexible meshes (compare, e.g., [12]).

The start will be with the infinitesimally flexible case. The geometric characterization of these meshes is already given in [6] (see Fig. 4). Kokotsakis' arguments have been followed which are based on standard results from Kinematics. For any two faces f_i, f_j sharing an edge, the edge is the axis *ij* of the relative motion. According to the Three-Pole-Theorem for any three faces f_i, f_j, f_k with rotations as pairwise relative motions the three axes *ij*, *ik* and *jk* must

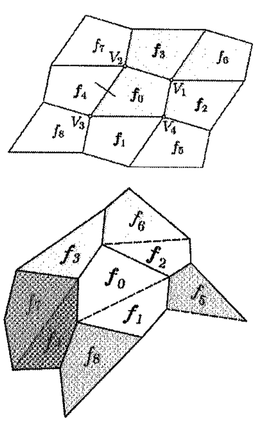

Figure 7
Up: Scheme of a Kokotsakis mesh.
Down: Pose of a continuously flexible Kokotsakis mesh with central face f_0; dashes indicate valley folds.

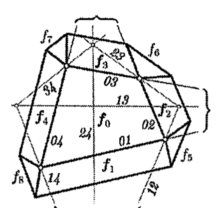

Figure 8
Infinitesimally flexible Kokotsakis mesh

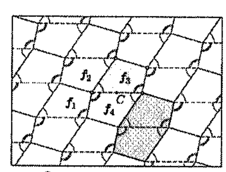

Figure 9 Kokotsakis' flexible tessellation

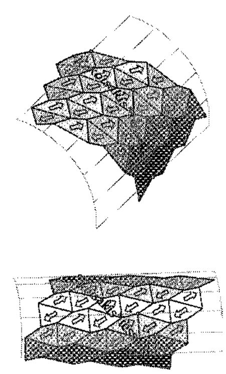

Figure 10 For each flection the vertices are located on cylinders of rotation

be coplanar and share a point. This implies, e.g., that at the Kokotsakis mesh (see Fig. 8) the axis *12* of the relative motion between f_1 and f_2 is the line of intersection between the planes spanned by f_5 and f_0. The lines of intersection with the plane are called of f_0 *traces*. In Fig. 8 the mesh is cut by a plane parallel to that of f_0.

Theorem 3.

A Kokotsakis mesh is infinitesimally flexible if and only if the following three points are collinear, the points of intersection between the traces of (f_1, f_3), (f_5, f_6) and (f_7, f_8). This is equivalent to the statement that the points of intersection between the traces of (f_2, f_4), (f_6, f_7) and (f_8, f_3) are aligned.

By the way, the equivalence between the two "collinearities" is just a consequence of Desargues' theorem.

According to Theorem 2 each infinitesimally flexible case gives rise to pairs of snapping Kokotsakis meshes. There are even examples where one realization is flat.

An interesting continuously flexible *quad mesh*, a polyhedral compound of $m \times n$ planar quadrangles, dates also back to Kokotsakis [6]. One starts with a planar tessellation by congruent non-convex quadrangles (Fig. 9) with the property, that any two quadrangles sharing a side can be interchanged by a 180°-rotation about the midpoint C of the common side. These two adjacent quadrangles form a centrally symmetric hexagon, and the tessellation can also be generated by translations of this hexagon.

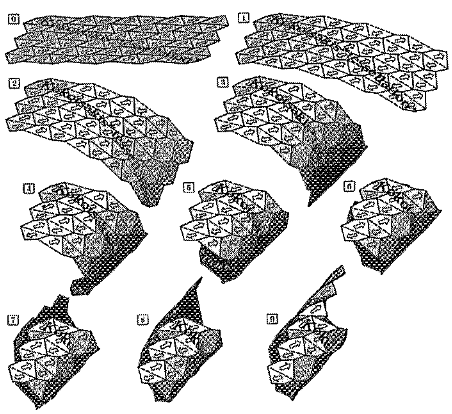

Figure 11 Flection of the Kokotsakis tesselation of the first kind

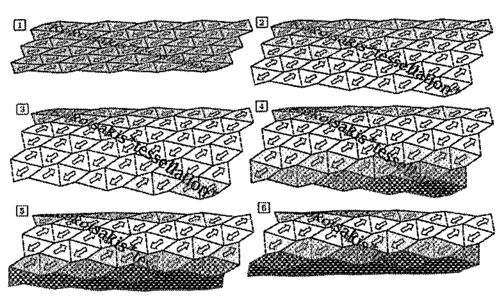

Figure 12 Flection of the Kokotsakis tesselation of second kind

Any connected portion of this tessellation is continuously flexible. As explained in [11], the flexibility can be proved by starting with a four-sided pyramid of quadrangles sharing a vertex and by continuing this flection step by step to the complete tesselation.
It turns out that starting from the flat initial pose, there are two differentiable types of bendings of this piecewise linear surface. In each non-flat realization all vertices are located on *a cylinder of revolution* (see Fig. 10). Hence the polygonal structure of each realization gives a discrete approximation of this cylinder.

Figures 11 and 12 show snapshots of these two bendings. The edges of the planar tessellation can be combined to two folds. And for each fold every second vertex lies on the same helical line. When the basic quadrangle is a trapezoid, then the folds of one family become aligned, and one family of bendings is that of a prism and therefore trivial.

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NOTES

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