TEACHING SPATIAL KINEMATICS FOR MECHANICAL ENGINEERING STUDENTS

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Abstract. Spatial kinematics is a challenging field because of its real world applications at serial and parallel manipulators. However, it is not easy to teach as on the one hand the students' spatial abilities need to be well developed. And on the other hand familiarity with calculus and vector algebra is substantial. After several years of experience I learned that one can expect from students to work with dual vectors. Students soon estimate that due to this tool they can handle directed lines in space (like axes of rotations) as well as screws, i.e., instantaneous motions.

1 Dual unit vectors representing directed lines in 3-space

There is a tight connection between spatial kinematics and the geometry of lines in the Euclidean 3-space. Therefore we start with recalling the use of appropriate line coordinates (cf. [2], [7] or [8, p. 155]). Any directed line *(spear)* g with direction vector $\mathbf{g}$ and passing through point A with position vector $\mathbf{a}$, i.e., $g = \mathbf{a} + \mathbb{R}\mathbf{g}$, can be uniquely represented by the pair of vectors $(\mathbf{g}, \widehat{\mathbf{g}})$, the direction vector g and the *momentum vector* $\widehat{\mathbf{g}}$ according to the definitions

$$\mathbf{g}\cdot\mathbf{g} = 1 \ \text{ and } \ \widehat{\mathbf{g}} := \mathbf{a}\times\mathbf{g}, \ \text{ which imply } \ \mathbf{g}\cdot\widehat{\mathbf{g}} = 0.$$

Conversely, any pair $(\mathbf{g}, \widehat{\mathbf{g}})$ of vectors obeying $\mathbf{g}\cdot\mathbf{g} = 1$ and $\mathbf{g}\cdot\widehat{\mathbf{g}} = 0$ defines a unique spear g because $\mathbf{p} := \mathbf{g}\times\widehat{\mathbf{g}}$ is a point of this line, the pedal point of g with respect to the origin. It makes sense to replace the pair $(\mathbf{g}, \widehat{\mathbf{g}})$ by the *dual* vector

$$\underline{\mathbf{g}} := \mathbf{g} + \varepsilon\widehat{\mathbf{g}} \tag{1}$$

where the *dual unit* ε obeys the rule $\varepsilon^2 = 0$.

We extend the usual dot product of vectors to dual vectors and notice

$$\underline{\mathbf{g}}\cdot\underline{\mathbf{g}} = (\mathbf{g} + \varepsilon\widehat{\mathbf{g}})\cdot(\mathbf{g} + \varepsilon\widehat{\mathbf{g}}) = \mathbf{g}\cdot\mathbf{g} + 2\varepsilon\mathbf{g}\cdot\widehat{\mathbf{g}} = 1 + \varepsilon 0 = 1. \tag{2}$$

Hence we call $\underline{\mathbf{g}}$ a *dual unit vector.* In this sense *the set of directed lines in the Euclidean 3-space* $\mathbb{E}^3$ *can be seen as the dual extension of the unit sphere.*

The representation of directed lines g in $\mathbb{E}^3$ by dual unit vectors $\underline{\mathbf{g}}$ brings about several advantages, and from now on we do not distinguish between directed lines g and their representing dual vector $\underline{\mathbf{g}}$ as well as between points X and their position vector $\mathbf{x}$.

Theorem 1 *For two given directed lines* $\underline{\mathbf{g}}$, $\underline{\mathbf{h}}$ *in* $\mathbb{E}^3$ *let* $\underline{\mathbf{n}}$ *denote the common normal endowed with an arbitrary orientation. If the helical motion along* $\underline{\mathbf{n}}$ *which transforms* $\underline{\mathbf{g}}$ *into* $\underline{\mathbf{h}}$ *(see Fig. 1) has the angle* φ *of rotation and the length* $\widehat{\varphi}$ *of translation and we combine them in the dual angle* $\underline{\varphi} := \varphi + \varepsilon\widehat{\varphi}$*, then the following equations hold true:*

$$\begin{array}{rcccl} \underline{\mathbf{g}}\cdot\underline{\mathbf{h}} & = & \underline{\cos\varphi} & = & \cos\varphi - \varepsilon\widehat{\varphi}\sin\varphi \quad \textit{and} \\ \underline{\mathbf{g}}\times\underline{\mathbf{h}} & = & \underline{\sin\varphi}\,\underline{\mathbf{n}} & = & \sin\varphi\,\mathbf{n} + \varepsilon\left[\sin\varphi\,\widehat{\mathbf{n}} + \widehat{\varphi}\cos\varphi\,\mathbf{n}\right]. \end{array} \tag{3}$$

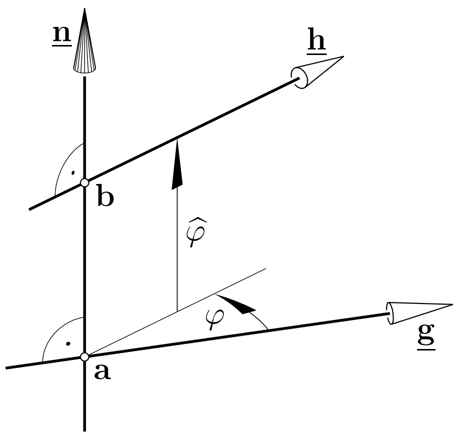

Figure 1: Dual angle $\underline{\varphi} = \varphi + \varepsilon\widehat{\varphi}$

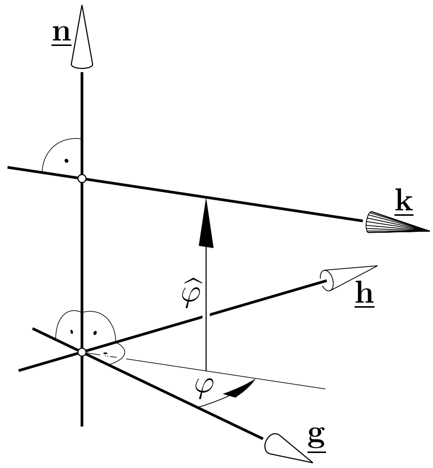

Figure 2: $\underline{\mathbf{k}} = \underline{\cos}\,\underline{\varphi}\,\underline{\mathbf{g}} + \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{h}}$

Remarks: 1) The dual extension of differentiable functions is defined by

$$\underline{f}(\underline{x}) = \underline{f}(x + \varepsilon\widehat{x}) = f(x) + \varepsilon\,\widehat{x} f'(x).$$

This is the beginning of a Taylor series where due to $\varepsilon^2 = 0$ all higher powers are vanishing. This guarantees that identities like $\cos^2 x + \sin^2 x = 1$ are preserved under the dual extension as they are valid for the power series, too.

The notation ε originates from the fact that the dual unit can be seen as such a small number that its square is neglectable. Note that only dual numbers $\underline{x} = x + \varepsilon\widehat{x}$ with non-vanishing real part, i.e., $x \neq 0$, have an inverse $\underline{x}^{-1} = \frac{1}{x}(x - \varepsilon\widehat{x})$; all others are zero divisors. Dual numbers have first been introduced 1873 by W. K. Clifford [4].

2) On the other hand we use in Theorem 1 the dual extension of the vector product according to

$$\underline{\mathbf{g}} \times \underline{\mathbf{h}} = (\mathbf{g} + \varepsilon\widehat{\mathbf{g}}) \times (\mathbf{h} + \varepsilon\widehat{\mathbf{h}}) = (\mathbf{g} \times \mathbf{h}) + \varepsilon\left[(\widehat{\mathbf{g}} \times \mathbf{h}) + (\mathbf{g} \times \widehat{\mathbf{h}})\right]. \tag{4}$$

Proof of Theorem 1: Suppose $\widehat{\mathbf{g}} = \mathbf{a} \times \mathbf{g}$ and $\widehat{\mathbf{h}} = \mathbf{b} \times \mathbf{h}$. Then the shortest distance between $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ reads

$$\begin{aligned} \widehat{\varphi} &= (\mathbf{b} - \mathbf{a})\cdot\mathbf{n} = (\mathbf{b} - \mathbf{a})\cdot\frac{1}{\sin\varphi}(\mathbf{g} \times \mathbf{h}) = \frac{1}{\sin\varphi}\left[\det(\mathbf{b}, \mathbf{g}, \mathbf{h}) - \det(\mathbf{a}, \mathbf{g}, \mathbf{h})\right] = \\ &= \frac{1}{\sin\varphi}\left[-(\mathbf{b} \times \mathbf{h})\cdot\mathbf{g} - (\mathbf{a} \times \mathbf{g})\cdot\mathbf{h}\right] = \frac{-1}{\sin\varphi}(\widehat{\mathbf{g}}\cdot\mathbf{h} + \mathbf{g}\cdot\widehat{\mathbf{h}}). \end{aligned}$$

If $\mathbf{a}$ and $\mathbf{b}$ are supposed to be the intersection points of $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ with the common normal $\underline{\mathbf{n}}$ (see Fig. 1), then

$$\sin\varphi\,\widehat{\mathbf{n}} = \mathbf{a} \times \sin\varphi\,\mathbf{n} = \mathbf{a} \times (\mathbf{g} \times \mathbf{h}) = (\mathbf{a}\cdot\mathbf{h})\mathbf{g} - (\mathbf{a}\cdot\mathbf{g})\mathbf{h} + [(\mathbf{a} - \mathbf{b})\cdot\mathbf{g}]\,\mathbf{h} = (\mathbf{a}\cdot\mathbf{h})\mathbf{g} - (\mathbf{b}\cdot\mathbf{g})\mathbf{h}.$$

The expression in brackets vanishes and could therefore be added without changing the value. On the other hand, due to standard formulas from vector algebra we see

$$\begin{aligned} (\widehat{\mathbf{g}} \times \mathbf{h}) + (\mathbf{g} \times \widehat{\mathbf{h}}) &= [(\mathbf{a} \times \mathbf{g}) \times \mathbf{h}] + [\mathbf{g} \times (\mathbf{b} \times \mathbf{h})] = \\ &= (\mathbf{a}\cdot\mathbf{h})\mathbf{g} - (\mathbf{g}\cdot\mathbf{h})\mathbf{a} + (\mathbf{g}\cdot\mathbf{h})\mathbf{b} - (\mathbf{g}\cdot\mathbf{b})\mathbf{h} = \\ &= (\mathbf{a}\cdot\mathbf{h})\mathbf{g} - (\mathbf{b}\cdot\mathbf{g})\mathbf{h} + (\mathbf{g}\cdot\mathbf{h})(\mathbf{b} - \mathbf{a}) = \sin\varphi\,\widehat{\mathbf{n}} + \widehat{\varphi}\cos\varphi\,\mathbf{n}. \qquad \square \end{aligned}$$

The dual extension makes it possible to convert formulas and theorems from spherical geometry onto the geometry of spears (cf. [9]). The following examples should illustrate this so called *'principle of transference'* (German: Übertragungsprinzip) which dates back to E. Study.

Theorem 2 *Let* $\underline{\mathbf{g}}$ *and* $\underline{\mathbf{h}}$ *be two orthogonally intersecting spears with the common perpendicular* $\underline{\mathbf{n}}$*. Then*

$$\underline{\mathbf{k}} = \underline{\cos\varphi}\,\underline{\mathbf{g}} + \underline{\sin\varphi}\,\underline{\mathbf{h}} \tag{5}$$

is the image of $\underline{\mathbf{g}}$ *under the helical motion along* $\underline{\mathbf{n}}$ *through the dual angle* $\underline{\varphi}$ *(see Fig. 2).*

Proof: From the orthogonal intersection of $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ we conclude $\underline{\mathbf{g}}\cdot\underline{\mathbf{h}} = 0$ and $\underline{\mathbf{n}} = \underline{\mathbf{g}}\times\underline{\mathbf{h}}$, hence $\underline{\mathbf{k}}\cdot\underline{\mathbf{k}} = \underline{\cos}^2\underline{\varphi}(\underline{\mathbf{g}}\cdot\underline{\mathbf{g}}) + \underline{\sin}^2\underline{\varphi}(\underline{\mathbf{h}}\cdot\underline{\mathbf{h}}) = 1$. Then Theorem 1 implies $\underline{\mathbf{g}}\times\underline{\mathbf{k}} = \underline{\sin\varphi}\,(\underline{\mathbf{g}}\times\underline{\mathbf{h}}) = \underline{\sin\varphi}\,\underline{\mathbf{n}}$ as stated. □

We iterate this procedure: Let $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$, $\underline{\mathbf{e}}_3$ be three pairwise orthogonally intersecting spears (Fig. 3). Each line $\underline{\mathbf{g}}$ has a common perpendicular $\underline{\mathbf{n}}$ with $\underline{\mathbf{e}}_3$, and using the dual angle $\underline{\beta}$ between $\underline{\mathbf{e}}_3$ and $\underline{\mathbf{g}}$ as well as $\underline{\lambda}$ between $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{n}}$ we obtain by Theorem 2

$$\underline{\mathbf{g}} = \underline{\cos\beta}\,\underline{\mathbf{e}}_3 + \underline{\sin\beta}\,\underline{\mathbf{k}} \quad \text{and} \quad \underline{\mathbf{k}} = \underline{\cos\lambda}\,\underline{\mathbf{e}}_1 + \underline{\sin\lambda}\,\underline{\mathbf{e}}_2\,.$$

Thus we get *dual sphere coordinates* $(\underline{\lambda}, \underline{\beta})$ to coordinatize directed lines (see Fig. 3) by

$$\underline{\mathbf{g}} = \underline{\cos\lambda}\,\underline{\sin\beta}\,\underline{\mathbf{e}}_1 + \underline{\sin\lambda}\,\underline{\sin\beta}\,\underline{\mathbf{e}}_2 + \underline{\cos\beta}\,\underline{\mathbf{e}}_3\,. \tag{6}$$

In the sequel we need

Theorem 3 *Any dual vector* $\underline{\mathbf{v}} = \mathbf{v} + \varepsilon\widehat{\mathbf{v}}$ *is a dual multiple of a dual unit vector, i.e.,* $\underline{\mathbf{v}} = \underline{\lambda}\,\underline{\mathbf{g}}$ *with* $\underline{\mathbf{g}}\cdot\underline{\mathbf{g}} = 1$*. In the case* $\mathbf{v} \neq \mathbf{o}$ *the dual unit vector* $\underline{\mathbf{g}}$ *is uniquely determined up to its sign.*

Proof: We have to fulfill the equation $\mathbf{v} + \varepsilon\widehat{\mathbf{v}} = (\lambda + \varepsilon\widehat{\lambda})(\mathbf{g} + \varepsilon\widehat{\mathbf{g}})$. First we note that $\underline{\mathbf{v}}\cdot\underline{\mathbf{v}} = \underline{\lambda}^2\,\underline{\mathbf{g}}\cdot\underline{\mathbf{g}}$ implies $\mathbf{v}\cdot\mathbf{v} + 2\varepsilon(\mathbf{v}\cdot\widehat{\mathbf{v}}) = \lambda^2 + 2\varepsilon\lambda\widehat{\lambda}$, hence $\mathbf{v} = \lambda\mathbf{g}$, $\widehat{\mathbf{v}} = \widehat{\lambda}\mathbf{g} + \lambda\widehat{\mathbf{g}}$ and $\mathbf{v}\cdot\widehat{\mathbf{v}} = \lambda\widehat{\lambda}$. For $\lambda = \pm\|\mathbf{v}\| \neq 0$ we get the solution

$$\mathbf{g} = \frac{1}{\lambda}\mathbf{v} \quad \text{and} \quad \widehat{\mathbf{g}} = \frac{1}{\lambda}\left(\widehat{\mathbf{v}} - \frac{\lambda\widehat{\lambda}}{\lambda^2}\mathbf{v}\right). \tag{7}$$

In the case $\lambda = 0$, i.e., $\underline{\mathbf{v}} = \varepsilon\widehat{\mathbf{v}}$, we set $\widehat{\lambda} = \|\widehat{\mathbf{v}}\|$, $\widehat{\lambda}\mathbf{g} = \widehat{\mathbf{v}}$ and choose an arbitrary $\widehat{\mathbf{g}}$ under $\widehat{\lambda} \neq 0$, otherwise the unit vector $\mathbf{g}$ can be chosen arbitrarily, too. □

2 Dual vectors representing screws

In this section we demonstrate the use of dual vectors for describing instantaneous motions (for an introduction see also [3, 5, 7, 11, 6]). Let a rigid body, which represents the moving system Σ_1, perform a one-parameter motion against the frame, the fixed system Σ_0. We assume that cartesian coordinate frames are attached to each system Σ_i, $i = 0, 1$, and we use the subscript i to indicate related coordinate vectors. Then the movement of Σ_1 against Σ_0 can analytically be described by the coordinate transformation which at each instant t gives the Σ_0-coordinate vector $\mathbf{x}_0$ of any point which with respect to the moving system Σ_1 has the coordinate vector $\mathbf{x}_1$. This coordinate transformation reads

$$\Sigma_1/\Sigma_0\colon\ \mathbf{x}_0 = \mathbf{u}_0(t) + A(t)\cdot\mathbf{x}_1 \ \text{ with } \ A(t)\cdot A(t)^T = I_3 \ \text{ and } \ \det A(t) = +1. \tag{8}$$

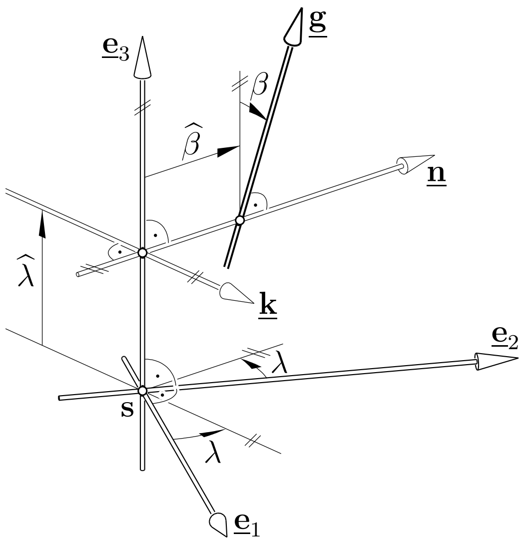

Figure 3: Dual sphere coordinates $(\underline{\lambda}, \underline{\beta})$

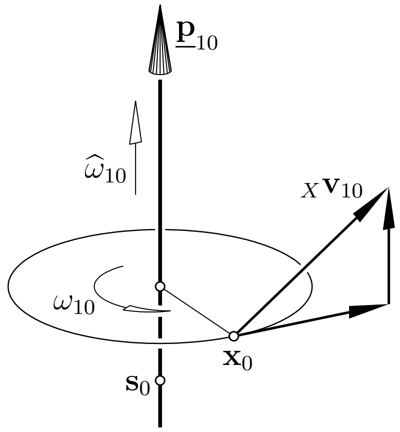

Figure 4: Instantaneous motion with the screw $\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}$

Here I_3 denotes the unit matrix, $\mathbf{u}_0(t)$ is the Σ_0-coordinate vector of the origin of Σ_1, and $A(t)$ is an orthogonal matrix, i.e., its transposed $A(t)^T$ is at the same time its inverse $A(t)^{-1}$. In order to figure out the distribution of velocity vectors ${}_X\mathbf{v}_{10}$ of points X attached to the moving system Σ_1, we differentiate and replace $\mathbf{x}_1$ by $\mathbf{x}_0$ due to (8). We thus obtain — after dropping the parameter t —

$$ {}_X\mathbf{v}_{10} = \dot{\mathbf{x}}_0 = \dot{\mathbf{u}}_0 + \dot{A}\cdot\mathbf{x}_1 = (\dot{\mathbf{u}}_0 - \dot{A}\cdot A^T\cdot\mathbf{u}_0) + \dot{A}\cdot A^T\cdot\mathbf{x}_0 \tag{9} $$

because of $\dot{\mathbf{x}}_1 = \mathbf{o}$. The matrix $\dot{A}\cdot A^T$ is skew symmetric because differentiation of $A\cdot A^T = I_3$ gives

$$ \dot{A}\cdot A^T + A\cdot\dot{A}^T = \dot{A}\cdot A^T + (\dot{A}\cdot A^T)^T = O = \text{ zero matrix.} $$

There is a dual vector $\underline{\mathbf{q}}_{10} = \mathbf{q}_{10} + \varepsilon\widehat{\mathbf{q}}_{10}$ such that

$$ \dot{A}\cdot A^T\cdot\mathbf{x}_0 = \mathbf{q}_{10}\times\mathbf{x}_0 \text{ for all } \mathbf{x}_0 \in \mathbb{R}^3 \text{ and } \widehat{\mathbf{q}}_{10} := \dot{\mathbf{u}}_0 - \dot{A}\cdot A^T\cdot\mathbf{u}_0\,. \tag{10} $$

We call this dual vector the *instantanous screw* [1] as according to (9) this vector rules the distribution of velocity vectors which the given motion Σ_1/Σ_0 assigns instantaneously to each point X attached to the system Σ_1. We have

$$ {}_X\mathbf{v}_{10} = \widehat{\mathbf{q}}_{10} + (\mathbf{q}_{10}\times\mathbf{x}_0). \tag{11} $$

Due to Theorem 3 the dual vector $\underline{\mathbf{q}}_{10}$ is a dual multiple of a dual unit vector, i.e., $\underline{\mathbf{q}}_{10} = \underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}$, or explicitely,

$$ \mathbf{q}_{10} + \varepsilon\widehat{\mathbf{q}}_{10} = (\omega_{10} + \varepsilon\widehat{\omega}_{10})(\mathbf{p}_{10} + \varepsilon\widehat{\mathbf{p}}_{10}) = \omega_{10}\,\mathbf{p}_{10} + \varepsilon(\widehat{\omega}_{10}\,\mathbf{p}_{10} + \omega_{10}\,\widehat{\mathbf{p}}_{10}) \tag{12} $$

with $\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{p}}_{10} = 1$. In order to figure out the meaning of the dual scalar $\underline{\omega}_{10}$ and the directed line $\underline{\mathbf{p}}_{10}$ we use a point $\mathbf{s}_0$ of this line and set $\widehat{\mathbf{p}}_{10} = \mathbf{s}_0\times\mathbf{p}_{10}$. Then according to (11) and (12) we get

$$ {}_X\mathbf{v}_{10} = (\widehat{\omega}_{10}\,\mathbf{p}_{10} + \omega_{10}\,\widehat{\mathbf{p}}_{10}) + (\omega_{10}\,\mathbf{p}_{10}\times\mathbf{x}_0) = \widehat{\omega}_{10}\,\mathbf{p}_{10} + \omega_{10}\,[\mathbf{p}_{10}\times(\mathbf{x}_0 - \mathbf{s}_0)] \tag{13} $$

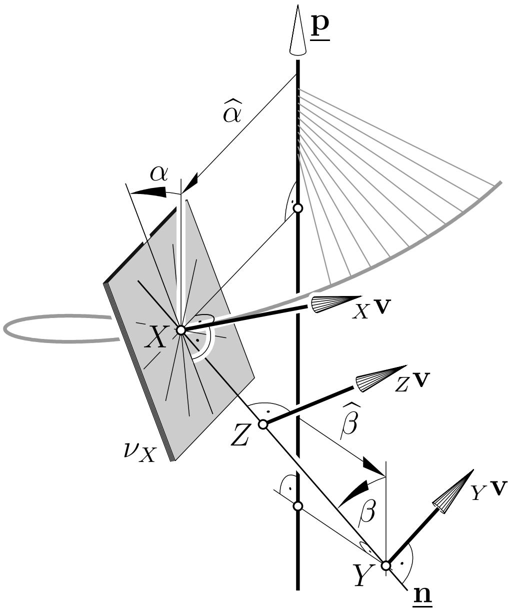

Figure 5: Linear complex of path normals

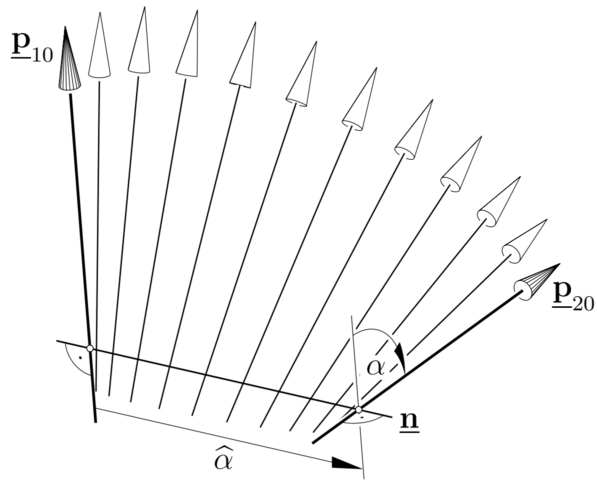

Figure 6: Portion of Plücker's cylindroid

This reveals (see Fig. 4) that ${}_X\mathbf{v}_{10}$ is the velocity vector of $\mathbf{x}_0$ under a helical motion about the instantaneous axis (ISA) $\underline{\mathbf{p}}_{10}$ with angular velocity ω_{10} and translatory velocity $\underline{\omega}_{10}$. We call $\underline{\omega}_{10}$ the *dual angular velocity* of the instantaneous motion. In this sense the dual unit vectors are at the same time the screws for instantaneous rotations with the angular velocity 1.

Now it is clear why due to Theorem 2 the axis $\underline{\mathbf{p}}_{10}$ is uniquely determined only under $\omega_{10} \neq 0$: Otherwise the instantaneous motion is a translation, and in this case only the direction of the axis is determined, but not the axis itself.

Theorem 4 *When the instantaneous screw $\underline{\mathbf{q}}_{10}$ is expressed as a multiple $\underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}$ of a dual unit vector, then $\underline{\mathbf{p}}_{10}$ is the instantaneous axis and $\underline{\omega}_{10}$ the dual angular velocity of the instantaneous helical motion.*

Theorem 5 *For any instantaneous motion with the screw $\underline{\mathbf{q}}_{10}$ the 'path-normals', i.e., the lines $\underline{\mathbf{n}}$ perpendicular to ${}_X\mathbf{v}_{10}$ and passing through X, constitute a linear line complex as $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon\widehat{\mathbf{n}}$ obeys the linear homogeneous equation*

$$\widehat{\mathbf{q}}_{10}\cdot\mathbf{n} + \mathbf{q}_{10}\cdot\widehat{\mathbf{n}} = 0 \quad (\iff \underline{\mathbf{q}}_{10}\cdot\underline{\mathbf{n}} \in \mathbb{R}). \tag{14}$$

This equation is independent of X. Hence any line $\pm\underline{\mathbf{n}}$, which is a path-normal at one of its points, is a path-normal at each point (see Fig. 5).

Proof: Due to (11) and $\widehat{\mathbf{n}} = \mathbf{x}_0 \times \mathbf{n}$ we have

$$0 = {}_X\mathbf{v}_{10}\cdot\mathbf{n} = [\widehat{\mathbf{q}}_{10} + (\mathbf{q}_{10}\times\mathbf{x}_0)]\cdot\mathbf{n} = \widehat{\mathbf{q}}_{10}\cdot\mathbf{n} + \mathbf{q}_{10}\cdot(\mathbf{x}_0\times\mathbf{n}) = \widehat{\mathbf{q}}_{10}\cdot\mathbf{n} + \mathbf{q}_{10}\cdot\widehat{\mathbf{n}}. \quad \square$$

By (3) and (12) our Equation (14) is equivalent to $\underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{n}} = \underline{\omega}_{10}\,\underline{\cos}\,\underline{\alpha} \in \mathbb{R}$ or $\widehat{\omega}_{10}\cos\alpha - \omega_{10}\widehat{\alpha}\sin\alpha = 0$, when $\underline{\alpha}$ denotes the dual angle between $\underline{\mathbf{p}}_{10}$ and $\underline{\mathbf{n}}$ (Fig. 5). Hence the path normals $\underline{\mathbf{n}}$ of the instantaneous motion are characterized by

$$\widehat{\alpha}\,\tan\alpha = \frac{\widehat{\omega}_{10}}{\omega_{10}}. \tag{15}$$

The quotient on the right hand side is the *pitch* of the helical motion.

Finally, we need the spatial Three-Pole-Theorem

Theorem 6 *If for three given systems* Σ_0*,* Σ_1*,* Σ_2 *the dual vectors* $\underline{\mathbf{q}}_{10}$ *and* $\underline{\mathbf{q}}_{20}$ *are the instantaneous screws of the motions* Σ_1/Σ_0 *and* Σ_2/Σ_0*, respectively, then*

$$\underline{\mathbf{q}}_{21} := \underline{\mathbf{q}}_{20} - \underline{\mathbf{q}}_{10}, \quad \textit{i.e.,} \quad \underline{\omega}_{21}\,\underline{\mathbf{p}}_{21} = \underline{\omega}_{20}\,\underline{\mathbf{p}}_{20} - \underline{\omega}_{10}\,\underline{\mathbf{p}}_{10}, \tag{16}$$

is the instantaneous screw of the relative motion Σ_2/Σ_1*. The three corresponding linear line complexes are included in a pencil of line complexes.*

Proof: According to (11) we have

$${}_X\mathbf{v}_{21} = {}_X\mathbf{v}_{20} - {}_X\mathbf{v}_{10} = (\widehat{\mathbf{q}}_{20} - \widehat{\mathbf{q}}_{10}) + [(\mathbf{q}_{20} - \mathbf{q}_{10}) \times \mathbf{x}_0] = \widehat{\mathbf{q}}_{21} + (\mathbf{q}_{21} \times \mathbf{x})$$

for each point X. $\square$

Let a line $\underline{\mathbf{n}}$ intersect the ISAs $\underline{\mathbf{p}}_{10}$ of Σ_1/Σ_0 and $\underline{\mathbf{p}}_{20}$ of Σ_2/Σ_0 orthogonally, i.e., $\underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{n}} = \underline{\mathbf{p}}_{10}\cdot\underline{\mathbf{n}} = 0$. Then we obtain $\underline{\omega}_{21}\underline{\mathbf{p}}_{21}\cdot\underline{\mathbf{n}} = 0$, which means that the line $\underline{\mathbf{n}}$ does also intersect the axis $\underline{\mathbf{p}}_{21}$ of Σ_2/Σ_1 orthogonally, provided $\omega_{21} \neq 0$.

Let two skew axes $\underline{\mathbf{p}}_{10}$ and $\underline{\mathbf{p}}_{10}$ be given. When the corresponding dual velocities $\underline{\omega}_{10}$ and $\underline{\omega}_{20}$ vary such that the pitches $\widehat{\omega}_{10}/\omega_{10}$ and $\widehat{\omega}_{20}/\omega_{20}$ remain constant, the axes $\underline{\mathbf{p}}_{21}$ of the relative motions Σ_2/Σ_1 constitute a *cylindroid* or *Plücker conoid.* Fig. 6 gives an impression of the cylindroid by showing some generators 'between' $\underline{\mathbf{p}}_{10}$ and $\underline{\mathbf{p}}_{20}$.
Now the principle of transference can be used to convert theorems from spherical kinematics into those of spatial kinematics (see e.g. [10].

3 Applications

Example 1: *Infinitesimal forward and inverse kinematics of 6R robots:*
A serial robot is an open kinematic chain of links $\Sigma_0, \Sigma_1, \ldots, \Sigma_6$ (Fig. 7). Any two consecutive links are connected by a revolute joint with the axis $\underline{\mathbf{p}}_{10}, \underline{\mathbf{p}}_{21}, \ldots, \underline{\mathbf{p}}_{65}$, respectively.

Given: Any posture of a 6R robot together with instantaneous angular velocites $\omega_{10}, \omega_{21}, \ldots, \omega_{65} \in \mathbb{R}$ of the relative motions about each axis. All axes are given in the same coordinate system.
Wanted: What is the instantaneous motion of the *endeffector* Σ_6?

This instantaneous motion is defined by the screw $\underline{\mathbf{q}}_{60}$. From the Three-Pole-Theorem (Theorem 6) we learned that

$$\underline{\mathbf{q}}_{60} = \omega_{10}\,\underline{\mathbf{p}}_{10} + \omega_{21}\,\underline{\mathbf{p}}_{21} + \ \ldots\ + \omega_{65}\,\underline{\mathbf{p}}_{65}.$$

This can also be written in matrix form: We combine the coordinates of the axes $\underline{\mathbf{p}}_{10}, \ldots, \underline{\mathbf{p}}_{65}$ in the columns of a (6×6)-matrix J. Then the resulting screw reads

$$\underline{\mathbf{q}}_{60} = \sum_{i=1,\ldots,6} \omega_{i\,i-1}\,\underline{\mathbf{p}}_{i\,i-1} = J\cdot\Omega$$

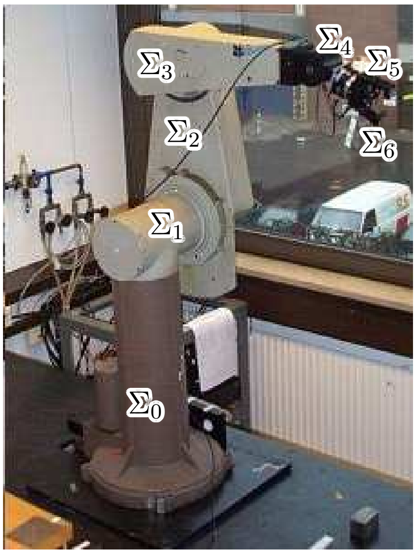

Figure 7: Serial robot

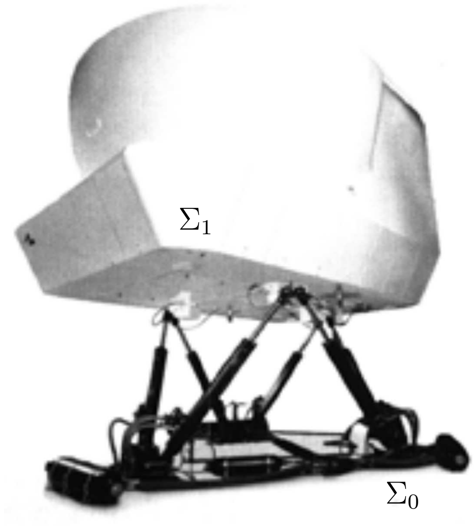

Figure 8: Parallel robot as flight simulator

with Ω as the column vector of given angular velocities. J is called *Jacobi matrix*. In the regular case ($\det J \neq 0$) we can also solve the inverse problem: For given $\underline{\mathbf{q}}_{60}$ we get the corresponding angular velocities $\omega_{10}, \ldots, \omega_{65}$ by solving a system of linear equations.

If there is a rank deficiency of J, then the instantaneous degree of freedom of Σ_6/Σ_1 is less than six. Just in this case the columns of J are linearly dependent. This is equivalent to the statement that the six axes are included in a linear line complex.

Example 2: *Calibration of Stewart-Gough-Platforms:*

Given: Any posture of a Stewart-Gough-Platform, i.e., a parallel manipulator where the platform Σ_1 is connected with the frame Σ_0 by six telescopic legs (Fig. 8). The anchor points in Σ_0 are denoted by $\mathbf{a}_0, \ldots, \mathbf{a}_6$, those in the platform Σ_1 by $\mathbf{b}_0, \ldots, \mathbf{b}_6$. We assume that for all these points the instantaneous coordinates are given in the same coordinate system.

Wanted: Suppose that by precise measurements a mislocation of the platform against the frame has been detected. How to figure out which leg is mainly responsible for this deviation.

There is a (small) helical motion which transports the actual posture into the target posture. After solving this registration problem, we get the corresponding axis $\underline{\mathbf{p}}_{10}$, the angle φ_{10} of rotation and the length $\widehat{\varphi}_{10}$ of translation. Let us assume that this movement is performed within — say — one second. This defines a screw $\underline{\mathbf{q}}_{10}$, and the instantaneous helical motion of Σ_1 assigns to each of its anchor points $\mathbf{b}_0, \ldots, \mathbf{b}_6$ a velocity vector according to (11). The component of this vector in direction of the leg $\mathbf{a}_i\mathbf{b}_i$ gives the corresponding variation in length which has to be carried out within 1 second. So, the leg with the largest velocity component should be most responsible for the mislocation. Of course, the reliability on this result needs to be fostered by iterated measurements in different postures.

How to compute these variations? Let d_i denote the distance $\|\mathbf{b}_i - \mathbf{a}_i\|$. Then the carrier line of this leg, oriented in the direction $\mathbf{a}_i\mathbf{b}_i$, has the coordinate components

$$\mathbf{l}_i = \frac{1}{d_i}(\mathbf{b}_i - \mathbf{a}_i) \ \text{ and } \ \widehat{\mathbf{l}}_i = \mathbf{b}_i \times \mathbf{l}_i\,.$$

From $d_i^2 = (\mathbf{b}_i - \mathbf{a}_i)^2$ we obtain bei differentiation $d_i\dot{d}_i = (\mathbf{b}_i - \mathbf{a}_i)\cdot\dot{\mathbf{b}}_i$, hence

$$\dot{d}_i = \mathbf{l}_i\cdot\dot{\mathbf{b}}_i = \mathbf{l}_i\cdot[\widehat{\mathbf{q}}_{10} + (\mathbf{q}_{10}\times\mathbf{b}_i)] = \mathbf{l}_i\cdot\widehat{\mathbf{q}}_{10} + (\mathbf{b}_i\times\mathbf{l}_i)\cdot\mathbf{q}_{10} = \mathbf{l}_i\cdot\widehat{\mathbf{q}}_{10} + \widehat{\mathbf{l}}_i\cdot\mathbf{q}_{10}\,.$$

We form a (6×6)-matrix $\overline{J}$ with rows consisting of the coordinates of $(\widehat{\mathbf{l}}_i, \mathbf{l}_i)$, written in this order. Then we get

$$\dot{D} = \overline{J}\cdot\underline{\mathbf{q}}_{10}$$

with $\dot{D}$ denoting the column of $\dot{d}_i$. So we obtain the variation of leg lengths by multiplying this Jacobi matrix $\overline{J}$ with the screw. In *singular* postures, which are characterized by $\det\overline{J} = 0$, there are infinitesimal self motions of the platform while the lengths of all telescopic legs remain fixed. Just in singular postures the rows in $\overline{J}$ are linearly dependent and therefore the six lines $\mathbf{a}_i\mathbf{b}_i$ included in a linear line complex.

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