Proceedings of
A Symposium Commemorating the
Legacy, Works, and Life of
Sir Robert Stawell Ball
Upon the 100th Anniversary of
A Treatise on the Theory of Screws
July 9-11, 2000
University of Cambridge, Trinity College

Instantaneous spatial kinematics and the invariants of the axodes

Hellmuth Stachel

Institute of Geometry, Vienna University of Technology
Wiedner Hauptstraße 8-10/113, A-1040 Wien
email: stachel@geometrie.tuwien.ac.at

Abstract

In this paper the dependencies between the instantaneous invariants of a spatial motion and the local invariants of the axodes are studied in a way that includes all types of ruled surfaces. New proofs for mostly wellknown formulas are given which should meet the main target of this note, namely to demonstrate anew the elegance and effectiveness that E. STUDY's dual line coordinates bring about in spatial kinematics.

MSC 1994: 53A17

Introduction

The results of this paper[1] are not really new: The spatial version of the EULER-SAVARY theorem dates back to DISTELI [6] and has since then been discussed in various papers, e.g. in [5], [18], [7], [19] (cf. the comprehensive bibliography on kinematics in [13]) and in a recent thesis [15].

A remark in O. BOTTEMA's and B. ROTH's monography [5] (page 161) says: *"The relationships between the local properties of the axodes and the higher order instantaneous invariants do not seem to have been developed"*. This paper is intended to close this gap. In [18] J. TÖLKE presented also formulas that express local motion invariants

[1]This paper is an extended version of the Technical Report [16].

in terms of invariants of the axodes. A formula for the screw parameter (Corollary 5) can be found in J. GRILL's thesis [8], eq. (5.29), too. However, the spatial EULER-SAVARY formula given in [8], eq. (7.20), deals with curvatures of enveloping surfaces. It differs from DISTELI's line-geometric version which will be discussed in the sequel.

In many of the cited papers the results are not of general validity as the axodes are restricted to skew ruled surfaces. We try to present all results in full generality including the spherical case as well as the case of cylindrical roll-slidings. As a consequence, no common natural parameter is available for all one-parameter spatial motions. However, most effective is the consequent use of dual numbers and dual vectors. In this sense this paper can also be seen as a tiny addendum to W. BLASCHKE's book *"Kinematik und Quaternionen"* [3], which ends with the touching remark: *"Es ist mir durchaus bewußt, daß dieses vorliegende Büchlein, das wohl mein letztes sein wird, manche Mängel und Lücken aufweist. Möge dieser Umstand dazu beitragen, daß junge Geometer sich dieses klassischen Gegenstandes erneut annehmen!"*.

1 Dual vectors

Let $\mathbb{D}$ denote the ring of *dual numbers*[2]

$$\underline{\lambda} = \lambda + \varepsilon\widehat{\lambda}, \quad \lambda, \widehat{\lambda} \in \mathbb{R},$$

where the dual unit ε obeys the rule $\varepsilon^2 = 0$. Only for dual numbers with the real part $\lambda \neq 0$ the inverse

$$\underline{\lambda}^{-1} = \frac{1}{\lambda^2}(\lambda - \varepsilon\widehat{\lambda})$$

exists. Pure dual numbers $\underline{\lambda} = 0 + \varepsilon\widehat{\lambda} \neq \underline{0}$ are the zero divisors in $\mathbb{D}$.

For the analytic function $f(x)$ the "dual extension" $\underline{f}(\underline{x})$ is defined as

$$\underline{f}(x + \varepsilon\widehat{x}) := f(x) + \varepsilon\,\widehat{x} f'(x). \tag{1}$$

This can be seen as the beginning of a TAYLOR series; due to $\varepsilon^2 = \varepsilon^3 = \ldots = 0$ all terms of higher order vanish.

The *dual vectors*[2]

$$\underline{\mathbf{v}} := \mathbf{v} + \varepsilon\widehat{\mathbf{v}} \text{ with } \mathbf{v}, \widehat{\mathbf{v}} \in \mathbb{R}^3$$

[2]The underbar indicates dual numbers as well as dual vectors. We have of course $\underline{0} = 0$, $\underline{1} = 1$ and for the zero vector $\underline{\mathbf{o}} = \mathbf{o}$.

constitute the module $\mathbb{D}^3$ over the ring $\mathbb{D}$. Beside the addition and multiplication with scalars $\underline{\lambda} \in \mathbb{D}$ also the following products of dual vectors are defined:

$$\begin{array}{rlll} \cdot: & (\mathbb{D}^3)^2 \to \mathbb{D}, & (\underline{\mathbf{u}},\underline{\mathbf{v}}) & \mapsto\ \underline{\mathbf{u}}\cdot\underline{\mathbf{v}} := \mathbf{u}\cdot\mathbf{v} + \varepsilon(\widehat{\mathbf{u}}\cdot\mathbf{v} + \mathbf{u}\cdot\widehat{\mathbf{v}}) \\ \times: & (\mathbb{D}^3)^2 \to \mathbb{D}^3, & (\underline{\mathbf{u}},\underline{\mathbf{v}}) & \mapsto\ \underline{\mathbf{u}}\times\underline{\mathbf{v}} := \mathbf{u}\times\mathbf{v} + \varepsilon[(\widehat{\mathbf{u}}\times\mathbf{v}) + (\mathbf{u}\times\widehat{\mathbf{v}})] \\ \det: & (\mathbb{D}^3)^3 \to \mathbb{D}, & (\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\mathbf{w}}) & \mapsto\ \det(\underline{\mathbf{u}},\underline{\mathbf{v}},\underline{\mathbf{w}}) := \underline{\mathbf{u}}\cdot(\underline{\mathbf{v}}\times\underline{\mathbf{w}}) = \det(\mathbf{u},\mathbf{v},\mathbf{w}) + \\ & & & \quad + \varepsilon\left[\det(\widehat{\mathbf{u}},\mathbf{v},\mathbf{w}) + \det(\mathbf{u},\widehat{\mathbf{v}},\mathbf{w}) + \det(\mathbf{u},\mathbf{v},\widehat{\mathbf{w}})\right]. \end{array}$$

Each dual vector $\underline{\mathbf{v}} = \mathbf{v} + \varepsilon\widehat{\mathbf{v}}$ is a multiple of a *dual unit vector* $\underline{\mathbf{g}}$, i.e.

$$\underline{\mathbf{v}} = \underline{\lambda}\,\underline{\mathbf{g}} \ \text{ with } \ \underline{\mathbf{g}}\cdot\underline{\mathbf{g}} = \mathbf{g}\cdot\mathbf{g} + 2\varepsilon\mathbf{g}\cdot\widehat{\mathbf{g}} = \underline{1}. \tag{2}$$

For $\mathbf{v} \neq \mathbf{o}$ the dual unit vector $\underline{\mathbf{g}}$ is unique – up to the factor ± 1. For $\underline{\mathbf{v}} = \varepsilon\widehat{\mathbf{v}} \neq \underline{\mathbf{o}}$ eq. (2) holds for any $\widehat{\mathbf{g}} \in \mathbb{R}^3$, provided $\underline{\lambda} = \varepsilon\widehat{\lambda}$ with $\widehat{\lambda} = \pm\|\widehat{\mathbf{v}}\|$ and $\mathbf{g} = \frac{1}{\widehat{\lambda}}\widehat{\mathbf{v}}$.

Two dual vectors $\underline{\mathbf{u}},\underline{\mathbf{v}}$ are *linearly dependent* over the ring $\mathbb{D}$ of dual numbers if and only if their real parts are linearly dependent over $\mathbb{R}$, i.e.

$$\underline{\lambda}\,\underline{\mathbf{u}} + \underline{\mu}\,\underline{\mathbf{v}} = \underline{\mathbf{o}} \ \text{ and } \ (\underline{\lambda},\underline{\mu}) \neq (\underline{0},\underline{0}) \iff \mathbf{u}\times\mathbf{v} = \mathbf{o}. \tag{3}$$

Proof: Due to $\underline{\mathbf{u}}\times\underline{\mathbf{u}} = \underline{\mathbf{v}}\times\underline{\mathbf{v}} = \underline{\mathbf{o}}$ the left side implies $\underline{\lambda}\,(\underline{\mathbf{u}}\times\underline{\mathbf{v}}) = \underline{\mu}\,(\underline{\mathbf{v}}\times\underline{\mathbf{u}}) = \underline{\mathbf{o}}$. This means for $\underline{\mathbf{w}} := \underline{\mathbf{u}}\times\underline{\mathbf{v}}$

$$\lambda\,\mathbf{w} = \mu\,\mathbf{w} = \widehat{\lambda}\,\mathbf{w} + \lambda\,\widehat{\mathbf{w}} = \widehat{\mu}\,\mathbf{w} + \mu\,\widehat{\mathbf{w}} = \mathbf{o}.$$

$\mathbf{w} = \mathbf{u}\times\mathbf{v} \neq \mathbf{o}$ implies $\lambda = \mu = \widehat{\lambda} = \widehat{\mu} = 0$.
Conversely, the dependency of the real parts can be written as

$$\lambda\mathbf{u} + \mu\mathbf{v} = \mathbf{o} \ \text{ or } \ (0 + \varepsilon\lambda)\,\underline{\mathbf{u}} + (0 + \varepsilon\mu)\,\underline{\mathbf{v}} = \underline{\mathbf{o}} \ \text{ with } \ (\lambda,\mu) \neq (0,0). \quad \square$$

More details on linear dependencies of dual vectors can be found in [17].

2 Directed lines and dual unit vectors

We continue with a brief summary[3] of STUDY's representation of directed lines (*spears*) in the Euclidean 3-space $\mathbf{E}^3$:

Let $\mathbf{a}$ be any point of the line g with given direction vector $\mathbf{g}$ obeying $\|\mathbf{g}\| = 1$. Then the corresponding *momentum vector* (2^{nd} PLÜCKERvector) $\widehat{\mathbf{g}} := \mathbf{a}\times\mathbf{g}$ is independent from the choice of point $\mathbf{a}$ at line g and it obeys $\mathbf{g}\cdot\widehat{\mathbf{g}} = 0$. Conversely, a pair $(\mathbf{g},\widehat{\mathbf{g}})$ of

[3] For more detailed information the reader is referred to [2].

vectors with $\|\mathbf{g}\| = 1$ and $\mathbf{g}\cdot\widehat{\mathbf{g}} = 0$ determines a unique directed line since $\mathbf{p} := \mathbf{g}\times\widehat{\mathbf{g}}$ is the coordinate vector of the pedal point of this line with respect to the origin. Hence there is a *bijection between the set of directed lines in* $\mathbf{E}^3$ *and the set of dual unit vectors*

$$g \;\mapsto\; \underline{\mathbf{g}} \;\text{ with }\; \underline{\mathbf{g}}\cdot\underline{\mathbf{g}} = 1. \tag{4}$$

In the following we identify directed lines with their dual unit vector.

For two directed lines $\underline{\mathbf{g}}, \underline{\mathbf{h}}$ the *dual angle* $\underline{\varphi} := \varphi + \varepsilon\widehat{\varphi}$ combines the angle φ and the shortest distance $\widehat{\varphi}$. This gives rise to geometric interpretations of the following products of the dual unit vectors $\underline{\mathbf{g}}, \underline{\mathbf{h}}$ (note the definition (1)):

$$\begin{aligned} \underline{\mathbf{g}}\cdot\underline{\mathbf{h}} &= \underline{\cos}\,\underline{\varphi} = \cos\varphi - \varepsilon\widehat{\varphi}\sin\varphi, \\ \underline{\mathbf{g}}\times\underline{\mathbf{h}} &= \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{n}} = (\sin\varphi + \varepsilon\widehat{\varphi}\cos\varphi)(\mathbf{n} + \varepsilon\widehat{\mathbf{n}})\,. \end{aligned} \tag{5}$$

Here $\underline{\mathbf{n}}$ represents a directed common perpendicular of the given lines $\underline{\mathbf{g}}, \underline{\mathbf{h}}$, and the signs of φ and $\widehat{\varphi}$ are related to the orientation of $\underline{\mathbf{n}}$.

Vanishing products of dual unit vectors characterize the following situations:

$$\begin{aligned} \underline{\mathbf{g}}\cdot\underline{\mathbf{h}} = \underline{0} \quad &\Longleftrightarrow \quad \underline{\mathbf{g}} \text{ and } \underline{\mathbf{h}} \text{ intersect perpendicularly;} \\ \underline{\mathbf{g}}\times\underline{\mathbf{h}} = \underline{\mathbf{o}} \quad &\Longleftrightarrow \quad \underline{\mathbf{g}} \text{ and } \underline{\mathbf{h}} \text{ are located on the same line, i.e. } \underline{\mathbf{g}} = \pm\underline{\mathbf{h}}; \\ \underline{\det}(\underline{\mathbf{g}}, \underline{\mathbf{h}}, \underline{\mathbf{k}}) = \underline{0} \quad &\Longleftrightarrow \quad \begin{cases} \underline{\mathbf{g}},\ \underline{\mathbf{h}} \text{ and } \underline{\mathbf{k}} \text{ are located on parallel lines or} \\ \text{they intersect a common line perpendicularly.} \end{cases} \end{aligned}$$

Let two directed lines $\underline{\mathbf{g}}, \underline{\mathbf{h}}$ be given obeying $\underline{\mathbf{g}}\cdot\underline{\mathbf{h}} = \underline{0}$: Then

$$\underline{\mathbf{k}} := \underline{\cos}\,\underline{\varphi}\,\underline{\mathbf{g}} + \underline{\sin}\,\underline{\varphi}\,\underline{\mathbf{h}} \tag{6}$$

defines a spear which is the image of $\underline{\mathbf{g}}$ under a helical displacement about the axis $(\underline{\mathbf{g}}\times\underline{\mathbf{h}})$ with the *dual screw angle* $\underline{\varphi}$.

Proof: We conclude from $(\underline{\mathbf{g}}\times\underline{\mathbf{h}})\cdot(\underline{\mathbf{g}}\times\underline{\mathbf{h}}) = (\underline{\mathbf{g}}\cdot\underline{\mathbf{g}})(\underline{\mathbf{h}}\cdot\underline{\mathbf{h}}) - (\underline{\mathbf{g}}\cdot\underline{\mathbf{h}})^2 = 1$ and

$$\underline{\mathbf{k}}\cdot\underline{\mathbf{k}} = \underline{\cos}^2\underline{\varphi} + \underline{\sin}^2\underline{\varphi} + 2\,\underline{\cos}\,\underline{\varphi}\,\underline{\sin}\,\underline{\varphi}\,(\underline{\mathbf{g}}\cdot\underline{\mathbf{h}}) = 1$$

that $(\underline{\mathbf{g}}\times\underline{\mathbf{h}})$ and $\underline{\mathbf{k}}$ are directed lines. The equation

$$\underline{\mathbf{g}}\times\underline{\mathbf{k}} = \underline{\sin}\,\underline{\varphi}\,(\underline{\mathbf{g}}\times\underline{\mathbf{h}})$$

reveals the stated property.
This proof could also be carried out using the dual unit quaternion

$$\underline{\mathbf{Q}} = \underline{\cos}\frac{\underline{\varphi}}{2} + \underline{\sin}\frac{\underline{\varphi}}{2}\,(\underline{\mathbf{g}}\times\underline{\mathbf{h}}) \;\text{ and }\; \underline{\mathbf{k}} = \underline{\mathbf{Q}}^{-1}\circ\underline{\mathbf{g}}\circ\underline{\mathbf{Q}}$$

which represents the helical displacement mentioned above. □

3 Instantaneous motion

At each moment of a spatial motion Σ_1/Σ_0 the velocity vector of any point $\mathbf{x}$ attached to the moving space Σ_1 is given by $\dot{\mathbf{x}} = \widehat{\mathbf{q}} + (\mathbf{q} \times \mathbf{x})$. We write the *instantaneous screw* $(\mathbf{q}, \widehat{\mathbf{q}})$ introduced by R.S. BALL [1] as the dual vector

$$\underline{\mathbf{q}} := \mathbf{q} + \varepsilon\widehat{\mathbf{q}} = \underline{\omega}\,\underline{\mathbf{p}} \quad \text{with} \quad \underline{\mathbf{p}} \cdot \underline{\mathbf{p}} = 1 \tag{7}$$

according to (2). Then $\underline{\mathbf{q}}$ combines the dual unit vector $\underline{\mathbf{p}}$ of the *instantaneous screw axis* (pole axis) with the *dual screw velocity* $\underline{\omega} = \omega + \varepsilon\widehat{\omega}$ where ω is the *angular velocity* and $\widehat{\omega}$ the *translation velocity*.

Proof: Let $\mathbf{s}$ denote any point of the screw axis p. Then for $\widehat{\mathbf{p}} = \mathbf{s} \times \mathbf{p}$ we obtain

$$\dot{\mathbf{x}} = \widehat{\omega}\,\mathbf{p} + \omega\,\widehat{\mathbf{p}} + \omega\,(\mathbf{p} \times \mathbf{x}) = \widehat{\omega}\,\mathbf{p} + \omega\,(\mathbf{s} \times \mathbf{p}) + \omega\,(\mathbf{p} \times \mathbf{x}) = \widehat{\omega}\,\mathbf{p} + \omega\,[\mathbf{p} \times (\mathbf{x} - \mathbf{s})]$$

which exactly expresses the stated instantaneous helical motion. □

For $\mathbf{q} \neq \mathbf{o}$ ($\Leftrightarrow \omega \neq 0$) the line $\underline{\mathbf{p}}$ is unique — up to its orientation — as

$$\omega = \pm\|\mathbf{q}\|, \quad \widehat{\omega} = \frac{\mathbf{q} \cdot \widehat{\mathbf{q}}}{\omega}, \quad \mathbf{p} = \frac{1}{\omega}\,\mathbf{q}, \quad \widehat{\mathbf{p}} = \frac{1}{\omega}\left(\widehat{\mathbf{q}} - \frac{\widehat{\omega}}{\omega}\,\mathbf{q}\right). \tag{8}$$

Let the directed line $\underline{\mathbf{g}}$ be attached to the moving space Σ_1. Then the dual "velocity vector" of $\underline{\mathbf{g}}$ under the motion Σ_1/Σ_0 reads

$$\dot{\underline{\mathbf{g}}} := \dot{\mathbf{g}} + \varepsilon\dot{\widehat{\mathbf{g}}} = \underline{\mathbf{q}} \times \underline{\mathbf{g}}. \tag{9}$$

Proof: The real part $\dot{\mathbf{g}} = \mathbf{q} \times \mathbf{g}$ is obvious. For obtaining the dual part we use any point $\mathbf{x}$ of this line with $\widehat{\mathbf{g}} = \mathbf{x} \times \mathbf{g}$:

$$\begin{aligned}
\dot{\widehat{\mathbf{g}}} &= (\dot{\mathbf{x}} \times \mathbf{g}) + (\mathbf{x} \times \dot{\mathbf{g}}) = [\widehat{\mathbf{q}} + (\mathbf{q} \times \mathbf{x})] \times \mathbf{g} \; + \; \mathbf{x} \times (\mathbf{q} \times \mathbf{g}) = \\
&= (\widehat{\mathbf{q}} \times \mathbf{g}) + (\mathbf{q} \cdot \mathbf{g})\mathbf{x} - (\mathbf{x} \cdot \mathbf{g})\mathbf{q} - (\mathbf{x} \cdot \mathbf{q})\mathbf{g} + (\mathbf{x} \cdot \mathbf{g})\mathbf{q} = \\
&= (\widehat{\mathbf{q}} \times \mathbf{g}) + [\mathbf{q} \times (\mathbf{x} \times \mathbf{g})] = (\widehat{\mathbf{q}} \times \mathbf{g}) + (\mathbf{q} \times \widehat{\mathbf{g}}). \quad \square
\end{aligned}$$

This reveals the *principle of transference:* The dual extensions of formulas from spherical kinematics are valid for the directed lines in spatial kinematics (compare Theorem 6).

4 FRENET motion for a ruled surface Φ

Let a parametrized ruled surface Φ be given by the C^2-function

$$t \in I \subset \mathbb{R} \;\mapsto\; \underline{\mathbf{g}}(t) := \mathbf{g}(t) + \varepsilon\widehat{\mathbf{g}}(t).$$

Then there is a FRENET *frame* of three pairwise perpendicularly intersecting directed lines $\underline{\mathbf{g}}$, $\underline{\mathbf{n}}$, $\underline{\mathbf{z}}$ such that the FRENET *equations* read in the notation of [3], (9.11) and (41.8),

$$\begin{array}{rcccccl} \dot{\underline{\mathbf{g}}} &=& & \underline{\lambda}\,\underline{\mathbf{n}} & &=& \underline{\mathbf{q}}\times\underline{\mathbf{g}} \\ \dot{\underline{\mathbf{n}}} &=& -\underline{\lambda}\,\underline{\mathbf{g}} & & +\underline{\mu}\,\underline{\mathbf{z}} &=& \underline{\mathbf{q}}\times\underline{\mathbf{n}} \\ \dot{\underline{\mathbf{z}}} &=& & -\underline{\mu}\,\underline{\mathbf{n}} & &=& \underline{\mathbf{q}}\times\underline{\mathbf{z}} \end{array} \tag{10}$$

for

$$\underline{\mathbf{q}} := \underline{\mu}\,\underline{\mathbf{g}} + \underline{\lambda}\,\underline{\mathbf{z}} = \underline{\omega}\,\underline{\mathbf{g}}^* \;\text{ with }\; \underline{\mathbf{g}}^*\cdot\underline{\mathbf{g}}^* = 1\,. \tag{11}$$

Again $\underline{\mathbf{q}}$ denotes the instantaneous screw of the FRENET motion along Φ. The dual screw velocity $\underline{\omega}$ obeys

$$\underline{\mathbf{q}}\cdot\underline{\mathbf{q}} = \underline{\omega}^2 = \underline{\lambda}^2 + \underline{\mu}^2\,, \;\text{ hence }\; \omega^2 = \lambda^2 + \mu^2 \;\text{ and }\; \omega\widehat{\omega} = \lambda\widehat{\lambda} + \mu\widehat{\mu}\,.$$

The axis $\underline{\mathbf{g}}^*$ of this instantaneous screw is called DISTELI-*axis* (*striction axis* or *curvature axis*). It is unique under $\omega \neq 0$.

The following theorems summarize some geometric interpretations of the dual invariants[4] $\underline{\lambda} = \lambda + \varepsilon\widehat{\lambda}$ and $\underline{\mu} = \mu + \varepsilon\widehat{\mu}$ according to the type of Φ as well as properties of the DISTELI-axis:

Theorem 1: *For $\lambda(t) \neq 0$ the common perpendicular $\underline{\mathbf{p}}(h)$ between adjacent generators $\underline{\mathbf{g}}(t)$ and $\underline{\mathbf{g}}(t+h)$ converges for $h \to 0$ towards $\underline{\mathbf{z}}(t) = \frac{1}{\underline{\lambda}}(\underline{\mathbf{g}}\times\dot{\underline{\mathbf{g}}})$, the central tangent. Hence the origin of our* FRENET *frame is the striction point* $\mathbf{s}$, *and $\underline{\mathbf{n}}(t) = \underline{\mathbf{z}}\times\underline{\mathbf{g}} = \frac{1}{\underline{\lambda}}\,\dot{\underline{\mathbf{g}}}$ is the central normal. The distribution parameter of $\underline{\mathbf{g}}(t)$ reads*

$$\delta = \frac{\dot{\mathbf{g}}\cdot\dot{\widehat{\mathbf{g}}}}{\dot{\mathbf{g}}\cdot\dot{\mathbf{g}}} = \frac{\widehat{\lambda}}{\lambda}\,.$$

Proof: From the Taylor expansion $\underline{\mathbf{g}}(t+h) = \underline{\mathbf{g}}(h) + h\,\dot{\underline{\mathbf{g}}}(t) + \underline{\mathbf{o}}(h)$ we obtain for the dual angle $\underline{\varphi}(h)$ made by the two adjacent lines

$$\underline{\sin}\,\underline{\varphi}(h)\,\underline{\mathbf{p}}(h) = \underline{\mathbf{g}}(t)\times\underline{\mathbf{g}}(t+h) = h\,[\underline{\mathbf{g}}(t)\times\dot{\underline{\mathbf{g}}}(t)] + [\underline{\mathbf{g}}(t)\times\underline{\mathbf{o}}(h)]\,.$$

[4] In [11] integral invariants of closed ruled surfaces have been represented in terms of dual unit vectors.

This leads to

$$\lim_{h\to 0}\frac{1}{h}\,\underline{\sin}\,\underline{\varphi}(h)\,\underline{\mathbf{p}}(h)=\lim_{h\to 0}\frac{\sin\varphi+\varepsilon\widehat{\varphi}\cos\varphi}{h}\,\underline{\mathbf{p}}(h)=\underline{\mathbf{g}}(t)\times\underline{\dot{\mathbf{g}}}(t)=\underline{\lambda}\,\underline{\mathbf{g}}\times\underline{\mathbf{n}}=\underline{\lambda}\,\underline{\mathbf{z}},$$

and we deduce

$$\delta=\lim_{h\to 0}\frac{\widehat{\varphi}(h)}{\varphi(h)}=\lim_{h\to 0}\frac{\widehat{\varphi}(h)\cos\varphi(h)}{\sin\varphi(h)}=\frac{\widehat{\lambda}}{\lambda}.\quad\square$$

Generators with $\lambda=0$ are cylindric. Here the FRENET equations (10) do not determine the canonical frame uniquely. Therefore at a cylindrical surface Φ ($\lambda=0$ for all $t\in I$) we proceed the other way round: We specify any transverse C^2-curve $\mathbf{s}: t\to\mathbb{R}^3$ on Φ as "striction curve", e.g. the orthogonal section given by $\mathbf{s}(t)=\mathbf{g}(t)\times\widehat{\mathbf{g}}(t)$. Then the pair $(\mathbf{z},-\mathbf{n})$ of vectors represents a canonical frame for the orthogonal section. This two-dimensional frame is right-handed, if seen against $\mathbf{g}$.

Theorem 2: *1. For $\lambda\neq 0$ the dual invariants $\underline{\lambda}$ and $\underline{\mu}$ of the parametrized ruled surface Φ obey*

$$\underline{\lambda}=v_g+\varepsilon v_s\sin\sigma,\quad \underline{\mu}=v_g\kappa_g+\varepsilon v_s\cos\sigma.$$

$v_g=\lambda$ is the velocity, $\kappa_g=\cot\gamma=\dfrac{\mu}{\lambda}$ the geodesic curvature and γ the spherical curvature radius of the spherical image $t\in I\mapsto\mathbf{g}(t)\in S^2$ of Φ. The angle σ is the striction and v_s the velocity of the striction point $\mathbf{s}$.

2. For $\lambda v_s\neq 0$ KRUPPA's curvature κ and torsion τ (cf. [12]) can be expressed as

$$\kappa=\frac{v_g}{v_s}=\frac{\lambda}{\sqrt{\widehat{\lambda}^2+\widehat{\mu}^2}},\quad \tau=\frac{v_g\kappa_g}{v_s}=\frac{\mu}{\sqrt{\widehat{\lambda}^2+\widehat{\mu}^2}}.$$

3. At a parametrized cylindrical surface Φ, i.e. $\lambda=0\ \forall t\in I$, with an arbitrary "striction curve" $\mathbf{s}(t)$ we obtain

$$\underline{\lambda}=\varepsilon v_n,\quad \underline{\mu}=v_n\kappa_n+\varepsilon v_t.$$

Here $v_n=v_s\sin\sigma$ and $v_t=v_s\cos\sigma$ are the components of the velocity vector $\dot{\mathbf{s}}$ in direction of $\mathbf{z}$ and $\mathbf{g}$ respectively. κ_n denotes the curvature of the orthogonal section of Φ.

Proof: The striction point is the origin of the canonical frame, i.e. $\mathbf{s}=\mathbf{o}$. Its velocity vector under the FRENET motion is $\dot{\mathbf{s}}=\widehat{\mathbf{q}}+(\mathbf{q}\times\mathbf{s})=\widehat{\mathbf{q}}$. With respect to this frame $\underline{\mathbf{g}},\underline{\mathbf{n}},\underline{\mathbf{z}}$ we obtain from $\underline{\mathbf{q}}=\underline{\mu}\,\underline{\mathbf{g}}+\underline{\lambda}\,\underline{\mathbf{z}}$ and $\widehat{\mathbf{g}}=\widehat{\mathbf{n}}=\widehat{\mathbf{z}}=\mathbf{o}$

$$\underline{\mathbf{q}}=(\mu+\varepsilon\widehat{\mu})\begin{pmatrix}1\\0\\0\end{pmatrix}+(\lambda+\varepsilon\widehat{\lambda})\begin{pmatrix}0\\0\\1\end{pmatrix}=\mathbf{q}+\varepsilon\begin{pmatrix}v_t\\0\\v_n\end{pmatrix}=\mathbf{q}+\varepsilon\begin{pmatrix}v_s\cos\sigma\\0\\v_s\sin\sigma\end{pmatrix}.\quad\square$$

Theorem 3: *1. The* DISTELI*-axis* $\underline{\mathbf{g}}^*$ *intersects the central normal* $\underline{\mathbf{n}}$ *perpendicularly at the point*

$$\mathbf{c} := \mathbf{s} + \frac{\lambda\widehat{\mu} - \widehat{\lambda}\mu}{\lambda^2+\mu^2}\,\mathbf{n}\,.$$

2. The DISTELI*-axis obeys the equations*

$$\underline{\omega}\,\underline{\mathbf{g}}^* = \underline{\mathbf{n}} \times \underline{\dot{\mathbf{n}}} \ \text{ and } \ \underline{\lambda}^2\,\underline{\omega}\,\underline{\mathbf{g}}^* = \underline{\dot{\mathbf{g}}} \times \underline{\ddot{\mathbf{g}}}\,.$$

Therefore $\underline{\mathbf{g}}^*$ *is the central tangent of the ruled surface* Ψ *built by the central normals of* Φ.

3. For $\lambda \neq 0$ *the dual angle* $\underline{\gamma}$ *made by the generator* $\underline{\mathbf{g}}$ *and its* DISTELI*-axis* $\underline{\mathbf{g}}^*$ *matches*

$$\underline{\cot}\,\underline{\gamma} := \cot\gamma + \varepsilon\widehat{\gamma}(1+\cot^2\gamma) = \frac{\underline{\mu}}{\underline{\lambda}}.$$

This quotient could be called the "dual curvature" of Φ. *For each generator of a cylindrical surface the* DISTELI*-axis coincides with the curvature axis of the orthogonal sections.*

4. The dual angle $\underline{\rho}(t)$ *made by the generator* $\underline{\mathbf{g}}(t)$ *and the fixed* DISTELI*-axis* $\underline{\mathbf{g}}^*(t_0)$ *is stationary of* 2^{nd} *order at* $t = t_0$*, i.e.* $\underline{\dot{\rho}}(t_0) = \underline{\ddot{\rho}}(t_0) = 0$.

Proof: Ad 1.: Eq. (11) implies $\underline{\mathbf{q}}\cdot\underline{\mathbf{n}} = (\underline{\mu}\,\underline{\mathbf{g}} + \underline{\lambda}\,\underline{\mathbf{z}})\cdot\underline{\mathbf{n}} = \underline{\omega}\,\underline{\mathbf{g}}^*\cdot\underline{\mathbf{n}} = 0$ which proves the stated perpendicularity. Therefore the point $\mathbf{c}$ of intersection is at the same time the pedal point of the DISTELI-axis with respect to the striction point $\mathbf{s}$ of Φ, which serves as the origin of our canonical frame. From (8) we get

$$\mathbf{g}^* \times \widehat{\mathbf{g}}^* = \frac{1}{\omega^2}\,\mathbf{q} \times \left(\widehat{\mathbf{q}} - \frac{\widehat{\omega}}{\omega}\,\mathbf{q}\right) = \frac{1}{\omega^2}\,(\lambda\,\mathbf{z} + \mu\,\mathbf{g}) \times (\widehat{\lambda}\,\mathbf{z} + \widehat{\mu}\,\mathbf{g}) = \frac{(\lambda\widehat{\mu} - \widehat{\lambda}\mu)(\mathbf{z}\times\mathbf{g})}{\lambda^2+\mu^2}\,.$$

At cylindrical surfaces we have $\mathbf{g}^* = \mathbf{g}$ and $\widehat{\mathbf{g}}^* = \widehat{\mathbf{g}} + \dfrac{1}{\kappa_n}\,\mathbf{z}$.

Ad 2.: From (10) and (11) we get by straight-forward computation

$$\underline{\mathbf{n}} \times \underline{\dot{\mathbf{n}}} = \underline{\mathbf{n}} \times (-\underline{\lambda}\,\underline{\mathbf{g}} + \underline{\mu}\,\underline{\mathbf{z}}) = \underline{\lambda}\,\underline{\mathbf{z}} + \underline{\mu}\,\underline{\mathbf{g}} = \underline{\mathbf{q}} = \underline{\omega}\,\underline{\mathbf{g}}^*,$$
$$\underline{\dot{\mathbf{g}}} = \underline{\lambda}\,\underline{\mathbf{n}},\ \ \underline{\ddot{\mathbf{g}}} = \underline{\dot{\lambda}}\,\underline{\mathbf{n}} + \underline{\lambda}\,\underline{\dot{\mathbf{n}}},\ \ \underline{\dot{\mathbf{g}}} \times \underline{\ddot{\mathbf{g}}} = \underline{\lambda}^2(\underline{\mathbf{n}} \times \underline{\dot{\mathbf{n}}}) = \underline{\lambda}^2\,\underline{\omega}\,\underline{\mathbf{g}}^*\,.$$

From $\underline{\mathbf{g}} \times \underline{\dot{\mathbf{g}}} = \underline{\lambda}\,\underline{\mathbf{g}} \times \underline{\mathbf{n}} = \underline{\lambda}\,\underline{\mathbf{z}}$ we conclude that $(\underline{\mathbf{n}} \times \underline{\dot{\mathbf{n}}})$ gives the central tangent of Ψ.

Ad 3.: (5) implies $\underline{\cos}\,\underline{\gamma} = \underline{\mathbf{g}}^*\cdot\underline{\mathbf{g}}$ and $\underline{\sin}\,\underline{\gamma}\,\underline{\mathbf{n}} = \underline{\mathbf{g}}^* \times \underline{\mathbf{g}}$. Due to (11) this results in

$$\underline{\tan}\,\underline{\gamma}\,\underline{\mathbf{n}} = \frac{\underline{\sin}\,\underline{\gamma}\,\underline{\mathbf{n}}}{\underline{\cos}\,\underline{\gamma}} = \frac{\underline{\mathbf{g}}^* \times \underline{\mathbf{g}}}{\underline{\mathbf{g}}^*\cdot\underline{\mathbf{g}}} = \frac{\underline{\mathbf{q}} \times \underline{\mathbf{g}}}{\underline{\mathbf{q}}\cdot\underline{\mathbf{g}}} = \frac{\underline{\lambda}\,\underline{\mathbf{n}}}{\underline{\mu}}\,.$$

Ad 4.: $\underline{\cos\rho}(t) = \underline{\mathbf{g}}^*(t_0)\cdot\underline{\mathbf{g}}(t)$ implies

$$-\dot{\underline{\rho}}(t)\,\underline{\sin\rho}(t) = \underline{\mathbf{g}}^*(t_0)\cdot\dot{\underline{\mathbf{g}}}(t) = \underline{\lambda}(t)\,\underline{\mathbf{g}}^*(t_0)\cdot\underline{\mathbf{n}}(t)\,,$$

hence $\dot{\underline{\rho}}(t_0) = 0$. The second derivation reads

$$\begin{aligned} -\ddot{\underline{\rho}}\,\underline{\sin\rho} - \dot{\underline{\rho}}^2\underline{\cos\rho} &= \dot{\underline{\lambda}}(t)\,\underline{\mathbf{g}}^*(t_0)\cdot\underline{\mathbf{n}}(t) + \underline{\lambda}(t)\,\underline{\mathbf{g}}^*(t_0)\cdot\dot{\underline{\mathbf{n}}}(t) = \\ &= \dot{\underline{\lambda}}(t)\,\underline{\mathbf{g}}^*(t_0)\cdot\underline{\mathbf{n}}(t) + \underline{\lambda}(t)\,\underline{\omega}(t)\,\underline{\det}[\underline{\mathbf{g}}^*(t_0),\underline{\mathbf{g}}^*(t),\underline{\mathbf{n}}(t)]\,, \end{aligned}$$

and this results in $\ddot{\underline{\rho}}(t_0) = 0$. □

5 Product of FRENET motions along the axodes

It is well known that under any (non-translatory) one-parameter motion Σ_1/Σ_0 the moving axode and the fixed axode are always in contact at each point of the instantaneous screw axis (cf. [5], p. 161). Therefore the corresponding FRENET frames are coinciding; the motion Σ_1/Σ_0 is the composition of the FRENET motion Σ_2/Σ_0 along the fixed axode Φ_0 and the inverse Σ_1/Σ_2 of the FRENET motion Σ_2/Σ_1 along the moving axode Φ_1[5]. This holds also for cylindrical axodes, provided the "striction curves" are specified such that in each moment these curves meet at a point of the pole axis. We call them a *"mating pair"* of striction curves.

In the following we denote for $i = 0,1$ the directed lines of the FRENET frame, the screw and the time-dependent invariants of the FRENET-motion Σ_2/Σ_i along Φ_i by $\underline{\mathbf{g}}_i$, $\underline{\mathbf{n}}_i$, $\underline{\mathbf{z}}_i$, $\underline{\mathbf{q}}_i$, $\underline{\lambda}_i$, $\underline{\mu}_i$, respectively. Let

$$\underline{\mathbf{p}} := \underline{\mathbf{g}}_0 = \underline{\mathbf{g}}_1\,, \quad \underline{\mathbf{n}} := \underline{\mathbf{n}}_0 = \underline{\mathbf{n}}_1\,, \quad \underline{\mathbf{z}} := \underline{\mathbf{z}}_0 = \underline{\mathbf{z}}_1$$

denote the axes of the coinciding canonical frames of Φ_0 and Φ_1. Then due to the spatial three-pole-theorem we obtain the instantaneous screw of Σ_1/Σ_0 (compare [8], eq. (5.21) and [18], eq. (6)) as

$$\underline{\mathbf{q}} = \underline{\omega}\,\underline{\mathbf{p}} = \underline{\mathbf{q}}_0 - \underline{\mathbf{q}}_1 = (\underline{\mu}_0 - \underline{\mu}_1)\,\underline{\mathbf{p}} + (\underline{\lambda}_0 - \underline{\lambda}_1)\,\underline{\mathbf{z}}\,. \tag{12}$$

The comparison of coefficients according to (3) results in

Theorem 4: *Let Φ_0,Φ_1 be the axodes of the motion Σ_1/Σ_0. Then the dual invariants $\underline{\lambda}_0,\underline{\mu}_0$ of Φ_0 and $\underline{\lambda}_1,\underline{\mu}_1$ of Φ_1 and the dual screw velocity $\underline{\omega}$ obey*

$$\underline{\lambda} := \underline{\lambda}_0 = \underline{\lambda}_1 \;\; \textit{and} \;\; \underline{\omega} = \underline{\mu}_0 - \underline{\mu}_1\,.$$

[5] In the sense of A. KARGER (see e.g. [9]) the moving frames in Σ_1 and Σ_2 represent a *lift* for the given motion Σ_1/Σ_0.

According to Theorem 2 the equation $\underline{\lambda}_0 = \underline{\lambda}_1$ expresses the roll-sliding of the axodes.

Corollary 5: *1. For a non-vanishing distribution parameter $\delta := \delta_0 = \delta_1$ of the axodes the pitch of the instantaneous helical motion can be expressed as*

$$\frac{\widehat{\omega}}{\omega} = \delta \, \frac{\cot\sigma_0 - \cot\sigma_1}{\cot\gamma_0 - \cot\gamma_1}$$

with σ_i as striction and γ_i as apex angle of the osculating cone of revolution of the axode Φ_i, $i = 0, 1$.

2. In the cylindrical case we obtain in the notation of Theorem 2 for a pair of mating striction curves on Φ_0 and Φ_1

$$\frac{\widehat{\omega}}{\omega} = \frac{\cot\sigma_0 - \cot\sigma_1}{\kappa_{n0} - \kappa_{n1}} = \frac{v_{t0} - v_{t1}}{v_n(\kappa_{n0} - \kappa_{n1})} \, .$$

With respect to the canonical frame $\underline{\mathbf{p}}, \underline{\mathbf{n}}, \underline{\mathbf{z}}$ all higher derivations for the trajectories can be expressed in terms of $\underline{\lambda}$ and $\underline{\omega}$. We demonstrate this only for the acceleration vectors:[6] Derivation of $\dot{\mathbf{x}} = \widehat{\mathbf{q}} + (\mathbf{q} \times \mathbf{x})$ with $\underline{\mathbf{q}} = \underline{\omega}\,\underline{\mathbf{p}}$ gives

$$\ddot{\mathbf{x}} = \dot{\widehat{\mathbf{q}}} + (\dot{\mathbf{q}} \times \mathbf{x}) + (\mathbf{q} \times \dot{\mathbf{x}}) = \dot{\widehat{\mathbf{q}}} + (\mathbf{q} \times \widehat{\mathbf{q}}) + (\dot{\mathbf{q}} \times \mathbf{x}) + [\mathbf{q} \times (\mathbf{q} \times \mathbf{x})].$$

We substitute according to (10)

$$\underline{\dot{\mathbf{q}}} = \underline{\dot{\omega}}\,\underline{\mathbf{p}} + \underline{\omega}\,\underline{\dot{\mathbf{p}}} = \underline{\dot{\omega}}\,\underline{\mathbf{p}} + \underline{\omega}\,\underline{\lambda}\,\underline{\mathbf{n}}$$

an obtain with respect to the canonical frame

$$\mathbf{q} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{\mathbf{q}} = \begin{pmatrix} \widehat{\omega} \\ 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{q}} = \begin{pmatrix} \dot{\omega} \\ \lambda\omega \\ 0 \end{pmatrix}, \quad \dot{\widehat{\mathbf{q}}} = \begin{pmatrix} \dot{\widehat{\omega}} \\ \lambda\widehat{\omega} + \widehat{\lambda}\omega \\ 0 \end{pmatrix},$$

hence (compare [5], eq. (12.1))

$$\ddot{\mathbf{x}} = \begin{pmatrix} \dot{\widehat{\omega}} \\ \lambda\widehat{\omega} + \widehat{\lambda}\omega \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \lambda\omega \\ 0 & -\omega^2 & -\dot{\omega} \\ -\lambda\omega & \dot{\omega} & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{13}$$

For $\lambda\omega \neq 0$ there is an unique *acceleration pole* with vanishing acceleration

$$P_a = \left(\frac{\dot{\omega}\beta\widehat{\beta} + \omega^4\dot{\widehat{\omega}}}{\lambda^2\omega^4}, \ \frac{\beta\widehat{\beta}}{\lambda\omega^3}, \ -\frac{\dot{\widehat{\omega}}}{\lambda\omega} \right)$$

[6] Another approach — without dual vectors — can be found in [4, 14] or in [10].

for $\beta := \sqrt{\lambda^2\omega^2+\dot{\omega}^2}$ and $\beta\widehat{\beta} := \lambda\omega(\lambda\widehat{\omega}+\widehat{\lambda}\omega)+\dot{\omega}\dot{\widehat{\omega}}$. Points with vanishing tangential acceleration constitute the BRESSE *hyperboloid* obeying

$$\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}} = \omega\dot{\omega}(x_2^2+x_3^2)-\lambda\omega^2 x_1 x_2-\widehat{\lambda}\omega^2 x_3+\widehat{\omega}\dot{\widehat{\omega}} = 0\,.$$

The condition $\dot{\mathbf{x}}\times\ddot{\mathbf{x}} = \mathbf{o}$ characterizes inflection points which under $\lambda\omega\widehat{\omega}\neq 0$ form a twisted cubic. After some computations the following parameter representation of the *spatial inflection curve* can be obtained (compare [4], eq. (29)):

$$\begin{aligned} x_1(\tau) &= -\frac{\alpha\gamma\widehat{\omega}+(\alpha\lambda\omega^2\widehat{\omega}+\gamma^2+\omega^4\widehat{\omega}^2)\,\tau+2\gamma\lambda\omega^2\tau^2+\lambda^2\omega^4\tau^3}{\lambda\omega^3\widehat{\omega}^2}\\ x_2(\tau) &= \frac{\alpha\widehat{\omega}+\gamma\tau+\lambda\omega^2\tau^2}{\omega^2\widehat{\omega}}\\ x_3(\tau) &= \tau \qquad \text{for}\ \ \alpha := \lambda\widehat{\omega}+\widehat{\lambda}\omega\ \ \text{and}\ \ \gamma := \omega\dot{\widehat{\omega}}-\dot{\omega}\widehat{\omega}\,. \end{aligned} \tag{14}$$

6 Spatial version of the EULER-SAVARY formula

In the following we compute the DISTELI-axis $\underline{\mathbf{g}}^*$ for the ruled surface Γ traced by the directed line $\underline{\mathbf{g}}$ under the motion Σ_1/Σ_0.

Let $\underline{\mathbf{k}}$ denote a common perpendicular between $\underline{\mathbf{g}}$ and the instantaneous pole axis $\underline{\mathbf{p}}$. Then there exists a common perpendicular $\underline{\mathbf{h}}$ of $\underline{\mathbf{p}}$ and $\underline{\mathbf{k}}$ such that $\underline{\mathbf{k}} = \underline{\mathbf{p}}\times\underline{\mathbf{h}}$ (see Figure 1). According to (6) we obtain a representation

$$\underline{\mathbf{g}} = \underline{\cos\alpha}\,\underline{\mathbf{p}}+\underline{\sin\alpha}\,\underline{\mathbf{h}} \tag{15}$$

based on the dual angle $\underline{\alpha}$ between $\underline{\mathbf{p}}$ and $\underline{\mathbf{g}}$. On the other hand the dual angle $\underline{\psi}$ made by the central normal $\underline{\mathbf{n}}$ and the spear $\underline{\mathbf{h}}$ leads to

$$\underline{\mathbf{h}} := \underline{\cos\psi}\,\underline{\mathbf{n}}+\underline{\sin\psi}\,\underline{\mathbf{z}} \tag{16}$$

(Figure 1) and

$$\underline{\mathbf{g}} = \underline{\cos\alpha}\,\underline{\mathbf{p}}+\underline{\sin\alpha}\,\underline{\cos\psi}\,\underline{\mathbf{n}}+\underline{\sin\alpha}\,\underline{\sin\psi}\,\underline{\mathbf{z}}\,. \tag{17}$$

From (9) and (15) we get

$$\dot{\underline{\mathbf{g}}} = \underline{\mathbf{q}}\times\underline{\mathbf{g}} = \underline{\omega}\,(\underline{\mathbf{p}}\times\underline{\mathbf{g}}) = \underline{\omega}\,\underline{\sin\alpha}\,(\underline{\mathbf{p}}\times\underline{\mathbf{h}}) = \underline{\omega}\,\underline{\sin\alpha}\,\underline{\mathbf{k}}$$

which proves with Theorem 1 that the central normal of Γ coincides with the common perpendicular $\underline{\mathbf{k}}$[7]. The second derivation reads

$$\begin{aligned} \ddot{\underline{\mathbf{g}}} &= \dot{\underline{\omega}}\,(\underline{\mathbf{p}}\times\underline{\mathbf{g}})+\underline{\omega}\,(\dot{\underline{\mathbf{p}}}\times\underline{\mathbf{g}})+\underline{\omega}^2\left[\underline{\mathbf{p}}\times(\underline{\mathbf{p}}\times\underline{\mathbf{g}})\right] = \\ &= \dot{\underline{\omega}}\,(\underline{\mathbf{p}}\times\underline{\mathbf{g}})+\underline{\omega}\,(\dot{\underline{\mathbf{p}}}\times\underline{\mathbf{g}})+\underline{\omega}^2(-\underline{\mathbf{g}}+\underline{\cos\alpha}\,\underline{\mathbf{p}}) \end{aligned}$$

[7] Under $\alpha = 0$ the moved line $\underline{\mathbf{g}}$ is parallel $\underline{\mathbf{p}}$ and a cylindrical generator of its trajectory Γ. In this case the common perpendicular $\underline{\mathbf{k}}$ is not unique.

because of (15). In view of Theorem 3,2 we compute

$$\begin{aligned}
\dot{\underline{\mathbf{g}}}\times\ddot{\underline{\mathbf{g}}} &= \underline{\omega}\,(\underline{\mathbf{p}}\times\underline{\mathbf{g}})\times\left[\dot{\underline{\omega}}\,(\underline{\mathbf{p}}\times\underline{\mathbf{g}})+\underline{\omega}\,(\dot{\underline{\mathbf{p}}}\times\underline{\mathbf{g}})+\underline{\omega}^2(-\underline{\mathbf{g}}+\underline{\cos\alpha}\,\underline{\mathbf{p}})\right]=\\
&= \underline{\omega}^2\left[\underline{\det}(\underline{\mathbf{p}},\dot{\underline{\mathbf{p}}},\underline{\mathbf{g}})\,\underline{\mathbf{g}}-\underline{\det}(\underline{\mathbf{g}},\dot{\underline{\mathbf{p}}},\underline{\mathbf{g}})\,\underline{\mathbf{p}}\right]+\\
&\quad+\underline{\omega}^3\left[-\underline{\cos\alpha}\,\underline{\mathbf{g}}+\underline{\mathbf{p}}+\underline{\cos\alpha}\,\underline{\mathbf{g}}-\underline{\cos}^2\underline{\alpha}\,\underline{\mathbf{p}}\right].
\end{aligned}$$

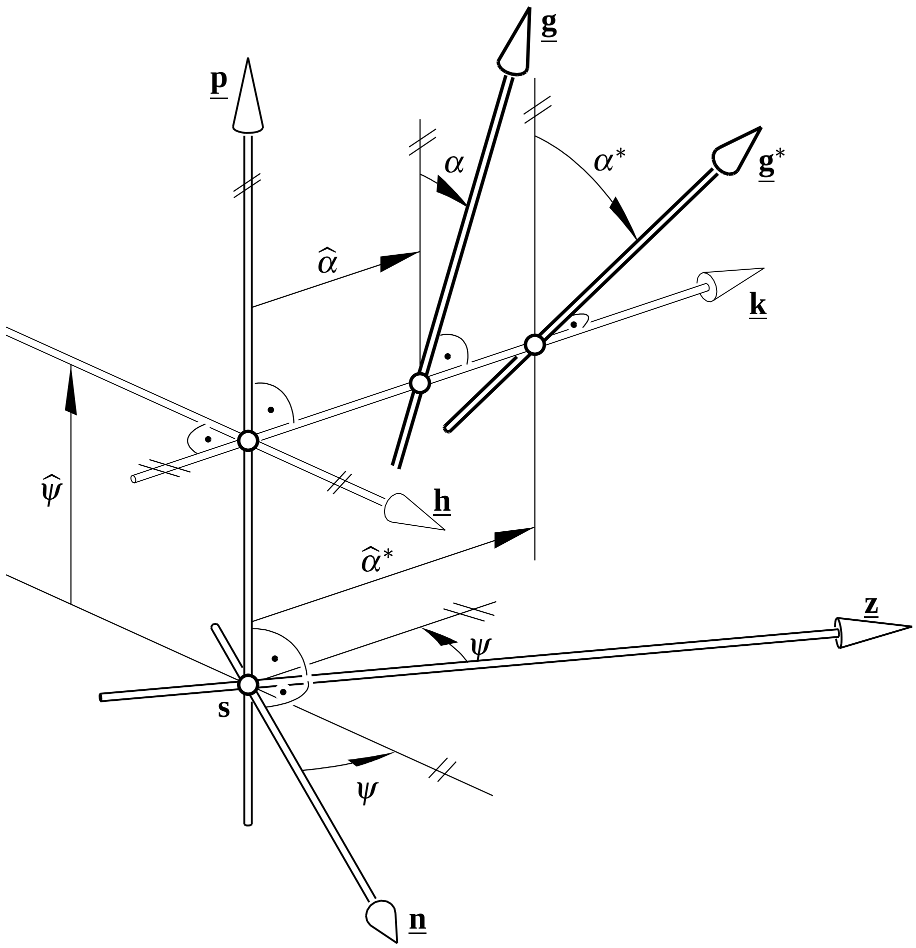

Figure 1: The moved line $\underline{\mathbf{g}}$ and its DISTELI-axis $\underline{\mathbf{g}}^*$

The FRENET equations (10) for the fixed axode Φ_0 and eqs. (15) and (17) give rise to

$$\begin{aligned}
\dot{\underline{\mathbf{g}}}\times\ddot{\underline{\mathbf{g}}} &= \underline{\omega}^2\left[\underline{\lambda}\,\underline{\det}(\underline{\mathbf{p}},\underline{\mathbf{n}},\underline{\mathbf{g}})\,\underline{\mathbf{g}}+\underline{\omega}(1-\underline{\cos}^2\underline{\alpha})\,\underline{\mathbf{p}}\right]=\\
&= \underline{\omega}^2\left[\underline{\lambda}\,(\underline{\mathbf{z}}\cdot\underline{\mathbf{g}})\,\underline{\mathbf{g}}+\underline{\omega}\,\underline{\sin}^2\underline{\alpha}\,\underline{\mathbf{p}}\right]=
\end{aligned}$$

$$= \underline{\omega}^2 [\underline{\lambda}\,\underline{\sin\alpha}\,\underline{\sin\psi}\,\underline{\mathbf{g}} + \underline{\omega}\,\underline{\sin}^2\underline{\alpha}\,\underline{\mathbf{p}}] =$$
$$= \underline{\omega}^2 \left[(\underline{\lambda}\,\underline{\sin\alpha}\,\underline{\cos\alpha}\,\underline{\sin\psi} + \underline{\omega}\,\underline{\sin}^2\underline{\alpha})\,\underline{\mathbf{p}} + \underline{\lambda}\,\underline{\sin}^2\underline{\alpha}\,\underline{\sin\psi}\,\underline{\mathbf{h}} \right]$$

and this equals $\underline{\mathbf{g}}^*$, up to a dual factor $\underline{\chi}$, i.e.

$$\underline{\dot{\mathbf{g}}} \times \underline{\ddot{\mathbf{g}}} = \underline{\chi}\,\underline{\mathbf{g}}^* = \underline{\chi}\,(\underline{\cos\alpha}^*\,\underline{\mathbf{p}} + \underline{\sin\alpha}^*\,\underline{\mathbf{h}})\,.$$

Suppose $\omega\sin\alpha \neq 0$. Then the comparison of coefficients due to (3) results in

$$\underline{\lambda}\,\underline{\sin\psi}(\underline{\cos\alpha}\,\underline{\sin\alpha}^* - \underline{\sin\alpha}\,\underline{\cos\alpha}^*) + \underline{\omega}\,\underline{\sin\alpha}\,\underline{\sin\alpha}^* = 0\,.$$

Under the additional condition $\lambda\sin\alpha^* \neq 0$ we may divide the last equation by $\underline{\lambda}\,\underline{\sin\alpha}\,\underline{\sin\alpha}^*$. Thus we obtain with Theorem 4

Theorem 6: *Spatial* EULER-SAVARY *formula:*

$$(\underline{\cot\alpha}^* - \underline{\cot\alpha})\,\underline{\sin\psi} = \frac{\underline{\omega}}{\underline{\lambda}} = \frac{\underline{\mu}_0}{\underline{\lambda}_0} - \frac{\underline{\mu}_1}{\underline{\lambda}_1} = \underline{\cot\gamma}_0 - \underline{\cot\gamma}_1\,.$$

This is exactly the dual extension of the spherical version.

References

[1] R.S. BALL: *The theory of screws: A study in the dynamics of a rigid body.* Hodges, Foster, and Co., Grafton-Street, Dublin 1876

[2] W. BLASCHKE: *Analytische Geometrie.* 2. Aufl., Verlag Birkhäuser, Basel 1954

[3] W. BLASCHKE: *Kinematik und Quaternionen.* VEB Deutscher Verlag der Wissenschaften, Berlin 1960

[4] E.H. BOKELBERG, K.H. HUNT, P.R. RIDLEY: *Spatial motion — I: Points of inflection and the differential geometry of screws.* Mech. Mach. Theory **27** (1992), 1–15

[5] O. BOTTEMA, B. ROTH: *Theoretical Kinematics.* North-Holland Publ. Comp., Amsterdam 1979

[6] M. DISTELI: *Über das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes.* Mech. Mach. Theory **62** (1914), 261–309

[7] B. DIZIOGLU: *Einfache Herleitung der Euler-Savary'schen Konstruktion der räumlichen Bewegung.* Mech. Mach. Theory **9** (1974), 247–254

[8] J. GRILL: *Zur Krümmungstheorie von Hüllflächen und ihrer Anwendung bei Werkzeugen und Verzahnungen.* Diss. TU Stuttgart 1993

[9] M. HUSTY, A. KARGER: *Flächenläufige Bewegungsvorgänge in der Geometrie der winkeltreuen Ähnlichkeiten der isotropen Ebene I.* Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **202** (1993), 1–22

[10] M. HUSTY, A. KARGER, H. SACHS, W. STEINHILPER: *Kinematik und Robotik.* Springer-Verlag, Berlin-Heidelberg 1997

[11] Ö. KÖSE: *Contributions to the theory of integral invariants of a closed ruled surface.* Mech. Mach. Theory **32** (1997), 261–277

[12] E. KRUPPA: *Analytische und konstruktive Differentialgeometrie.* Springer-Verlag, Wien 1957

[13] H. POTTMANN: *Kinematische Geometrie.* In O. Giering, J. Hoschek (ed.): Geometrie und ihre Anwendungen, Carl Hanser Verlag, München 1994, 141–175

[14] P.R. RIDLEY, E.H. BOKELBERG, K.H. HUNT: *Spatial motion — II: Acceleration and the differential geometry of screws.* Mech. Mach. Theory **27** (1992), 17–35

[15] U. SCHULZ: *Ein Analogon der* EULER-SAVARY*'schen Gleichung für Bahnregelflächen und Hüllflächenpaare aus Regelflächen unter Verwendung dualer Vektoren.* Diss. TU Dresden 1997

[16] H. STACHEL: *Instantaneous spatial kinematics and the invariants of the axodes.* Institut für Geometrie, TU Wien, Technical Report **34** (1996)

[17] H. STACHEL: *Euclidean line geometry and kinematics in the 3-space.* In N.K. Artémiadis, N.K. Stephanidis (eds.): Proc. 4th International Congress of Geometry, Thessaloniki 1996, 380–391 (1997)

[18] J. TÖLKE: *Zur Strahlkinematik I.* Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **182** (1974), 177–202

[19] J. TÖLKE: *Contribution to the theory of the axes of curvature.* Mech. Mach. Theory **11** (1976), 123–130