Infinitesimal Flexibility of Higher Order for a Planar Parallel Manipulator

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Abstract: The problems around rigidity and flexibility of geometrical structures have a long history in mathematics. Recently obtained results like the proof of the "Bellows Conjecture" have brought them back into the center of mathematical interest. However, infinitesimal flexibility — usually of order one — has also been studied in kinematics for a long time. In particular in the field of robotics the singularity analysis of parallel manipulators is of high actuality.

The objective of this paper is to bridge the gap between rigidity theory and kinematics. On the one hand the definition and basic properties of higher-order infinitesimal flexibility are presented to kinematicians. On the other hand it is shown that for a particular planar parallel manipulator standard results from kinematics enable to characterize in a geometric way all cases of higher-order infinitesimal flexibility.

1 Infinitesimal flexibility of higher order

Let $\mathbf{F}$ be a framework in the Euclidean d-space $\mathbb{E}^d$ with *vertex set* V and *edge set* E. Each edge (rod) is given as an unordered pair of indices, hence

$$V = \{\mathbf{x}_1, \dots, \mathbf{x}_v\},\ \mathbf{x}_i \in \mathbb{R}^d \ \ \forall i \in I := \{1, \dots, v\} \quad \text{and} \quad E \subset \Big\{(i,j) \mid i < j,\ (i,j) \in I^2\Big\}.$$

Let l_{ij} denote the Euclidean length of the edge $\mathbf{x}_i\mathbf{x}_j$, $(i,j) \in E$, of our framework $\mathbf{F}$, i.e.,

$$f_{ij}(\mathbf{x}_i, \mathbf{x}_j) := \|\mathbf{x}_i - \mathbf{x}_j\| - l_{ij} = 0 \quad \forall (i,j) \in E. \tag{1}$$

We presuppose $l_{ij} > 0$ for all $e := \#E$ edges of $\mathbf{F}$. Then the *classical definition* of infinitesimal flexibility reads as follows:

Definition 1: The framework $\mathbf{F} = (V, E)$ is *infinitesimally flexible of order* n if and only if for each $k \in I$ there is a polynomial function

$$\mathbf{z}_k(t) := \mathbf{x}_k + t\,\mathbf{z}_{k,1} + \ldots + t^n \mathbf{z}_{k,n}\,, \quad n \geq 1, \tag{2}$$

such that

(i) the replacement of $\mathbf{x}_i$ and $\mathbf{x}_j$ in (1) by $\mathbf{z}_i(t)$ and $\mathbf{z}_j(t)$, resp., gives functions $f_{ij}\left(\mathbf{z}_i(t), \mathbf{z}_j(t)\right)$ with a zero of multiplicity $\geq n+1$ at $t = 0$, i.e., for $t \to 0$

$$f_{ij}\left(\mathbf{z}_i(t), \mathbf{z}_j(t)\right) = \|\mathbf{z}_i(t) - \mathbf{z}_j(t)\| - l_{ij} = o(t^n) \quad \forall (i,j) \in E; \tag{3}$$

and, in order to exclude *trivial* flexes,

(ii) the vectors $\mathbf{z}_{1,1}, \ldots, \mathbf{z}_{v,1}$ are not the velocity vectors of the vertices $\mathbf{x}_1, \ldots, \mathbf{x}_v$ under a motion of $\mathbf{F}$ as a rigid body.

Remark 1: When among $\{\mathbf{x}_1, \ldots, \mathbf{x}_v\}$ there are $d+1$ vertices forming a simplex of $\mathbb{E}^d$, then condition (ii) is equivalent to

$$\exists\ (k,l) \in I^2 \quad \text{with} \quad \frac{d}{dt}\|\mathbf{z}_k(t) - \mathbf{z}_l(t)\| \neq 0 \quad \text{at} \quad t = 0\,. \tag{4}$$

However, when the dimension of the affine span of $\{\mathbf{x}_1, \ldots, \mathbf{x}_v\}$ is smaller than d, then the distances between any two vertices can be stationary, though the vectors $\mathbf{z}_{1,1}, \ldots, \mathbf{z}_{v,1}$ differ from the velocity vectors under any motion of the whole framework. Note for example in $\mathbb{E}^2$ a four-bar linkage in a *folded* position, i.e., with aligned vertices. Of course, this position is nontrivially flexible, but all mutual distances between vertices remain instantanously constant. Compare also Footnote 2.

Because of $l_{ij} > 0$ condition (3) is equivalent to

$$(\mathbf{z}_i(t) - \mathbf{z}_j(t)) \cdot (\mathbf{z}_i(t) - \mathbf{z}_j(t)) - l_{ij}^2 = o(t^n). \tag{5}$$

For the sake of brevity we write

$$X := (\mathbf{x}_1, \ldots, \mathbf{x}_v) \in \mathbb{R}^{vd}, \quad Z_i := (\mathbf{z}_{1,i}, \ldots, \mathbf{z}_{v,i}), \quad Z(t) := (\mathbf{z}_1(t), \ldots, \mathbf{z}_v(t))\,.$$

When equ. (3) is true, then we call

$$Z(t) = X + tZ_1 + \cdots + t^n Z_n \tag{6}$$

an *n-th-order flex* of $\mathbf{F}$.

The property of being an n-th-order flex of $\mathbf{F}$ is invariant under regular parameter transformations $\varphi\colon \bar{t} \to t$ of class C^n. It is also invariant under superimposed trivial analytical flexes (motions) of $\mathbf{F}$, since all scalar products remain unchanged.

When a framework $\mathbf{F}$ admits an analytical nontrivial flex $Y(t)$, then for each $k \in I$ there is an analytical function $\mathbf{y}_k(t)$ parametrizing the trajectory of the vertex $\mathbf{x}_k$ under $Y(t)$. The functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_v(t)$ solve (1) identically with respect to the variable t, i.e.,

$$f_{ij}\left(\mathbf{y}_i(t), \mathbf{y}_j(t)\right) = \|\mathbf{y}_i(t) - \mathbf{y}_j(t)\| - l_{ij} \equiv 0 \quad \forall (i,j) \in E.$$

The TAYLOR expansions at the initial position $t = 0$,

$$\mathbf{y}_k(t) = \mathbf{x}_k + t\,\dot{\mathbf{y}}_k(0) + \frac{t^2}{2!}\,\ddot{\mathbf{y}}_k(0) + \frac{t^3}{3!}\,\dddot{\mathbf{y}}_k(0) + \cdots + \frac{t^n}{n!}\,\overset{(n)}{\mathbf{y}}_k(0) + \mathbf{o}(t^n), \quad k \in I,$$

reveal that an analytically flexing framework is also infinitesimally flexing of any order n, provided there is no stillstand at $t = 0$.[1] We obtain the flex $Z(t)$ of order n for the considered position by setting

$$Z_1 := \dot{Y}(0), \quad Z_2 := \frac{1}{2!}\,\ddot{Y}(0), \quad \ldots \quad, Z_n := \frac{1}{n!}\,\overset{(n)}{Y}.$$

In this sense, $\mathbf{z}_{k,1}$ in (2) is the velocity vector and $\mathbf{z}_{k,2}$ the acceleration vector of the vertex $\mathbf{x}_k$, when $\mathbf{F}$ is performing the flex $Z(t)$.

Suppose, the framework $\mathbf{F}$ is given by its combinatorial structure E and by the lengths l_{ij} of its edges. Then (1) represents a system of e quadratic equations

$$(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{x}_i - \mathbf{x}_j) = l_{ij}^2 \quad \forall (i,j) \in E \tag{7}$$

for the vd unknown coordinates of the vertices. With each solution $\mathbf{x}_1, \ldots, \mathbf{x}_v$ of this system also

$$\overline{\mathbf{x}}_k := \mathbf{b} + B\mathbf{x}_k \quad \forall k \in I, \quad B^T = B^{-1},$$

for any constant $\mathbf{b} \in \mathbb{R}^d$ and orthogonal $d \times d$ matrix B solves this system.

When we substitute (2) in (7), then the coefficients of $t, t^2, \ldots, t^n$ give rise to the following systems of linear equations, each for all $(i,j) \in E$:

$$\begin{aligned}
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{z}_{i,1} - \mathbf{z}_{j,1}) &= 0, && (8)\\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{z}_{i,2} - \mathbf{z}_{j,2}) &= -\tfrac{1}{2}(\mathbf{z}_{i,1} - \mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,1} - \mathbf{z}_{j,1}), && (9)\\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{z}_{i,3} - \mathbf{z}_{j,3}) &= -(\mathbf{z}_{i,1} - \mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,2} - \mathbf{z}_{j,2}), && (10)\\
(\mathbf{x}_i - \mathbf{x}_j)\cdot(\mathbf{z}_{i,4} - \mathbf{z}_{j,4}) &= -(\mathbf{z}_{i,1} - \mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,3} - \mathbf{z}_{j,3}) - \tfrac{1}{2}(\mathbf{z}_{i,2} - \mathbf{z}_{j,2})\cdot(\mathbf{z}_{i,2} - \mathbf{z}_{j,2}), &&\\
\cdots &= \cdots &&
\end{aligned}$$

The matrix M on the left side of these systems (see example in (12)) is always the same. It is called *rigidity matrix* of the given framework $\mathbf{F}$ (cf. [7]).

According to [1], the left sides in the systems (7), (8), (9), … can be expressed in terms of a symmetric bilinear map

$$\beta\colon\ (\mathbb{R}^{vd} \times \mathbb{R}^{vd}) \to \mathbb{R}^e,$$

such that the system to be solved for Z_i, $i = 1, \ldots, n$, reads

$$\beta(X, Z_1) = \mathbf{o} \quad \text{and for } i > 1: \quad \beta(X, Z_i) = -\tfrac{1}{2}\sum_{j=1}^{i-1}\beta(Z_j, Z_{i-j})$$

In particular, the velocity vectors $\mathbf{z}_{k,1}$ have to solve the homogeneous system (8) of linear equations. If there is a nontrivial solution $\tilde{\mathbf{z}}_{k,1}$ of (8), then in order to obtain other solutions,

[1] Note the counter example in [2].

we can superimpose arbitrary infinitesimal motions. According to standard results from kinematics this means that

$$\mathbf{z}_{k,1} := \tilde{\mathbf{z}}_{k,1} + \mathbf{c} + C\mathbf{x}_k\,, \quad C^T = -C,$$

with constant $\mathbf{c} \in \mathbb{R}^d$ and any skew symmetric $d \times d$ matrix C is a nontrivial solution, too.[2] This gives a free choice of

$$d + \tfrac{1}{2}d(d-1) = \tfrac{1}{2}d(d+1)$$

parameters. Therefore, for infinitesimal flexibility of order 1 it is necessary and sufficient that the rank of the rigidity matrix M obeys the inequality

$$\mathrm{rk}(M) < vd - \frac{d(d+1)}{2}. \tag{11}$$

In 1920 Liebmann proved (cf. [5], [11]) that infinitesimal flexibility of order 1 is projectively invariant.[3]

The equations (8) are equivalent to the well known *Projection Theorem* displayed in Fig. 1. For any rod AB of $\mathbf{F}$, the velocity vectors $\mathbf{v}_A, \mathbf{v}_B$ at the endpoints have equal components in the direction of $\overrightarrow{AB}$.

Each nontrivial Z_1 defines the right side in the inhomogeneous system (9). Standard criteria from linear algebra can be used to check the solvability of this system. The conditions for the acceleration vectors $2\,\mathbf{z}_{k,2}$ expressed in (9) are equivalent to a standard result from kinematics stating that for each rod AB of length l_{AB} the velocity vectors $\mathbf{v}_A$, $\mathbf{v}_B$ and acceleration vectors $\mathbf{a}_A$, $\mathbf{a}_B$ obey the condition

$$\mathbf{a}_B - \mathbf{a}_A = \mathbf{a}^n_{BA} + \mathbf{a}^t_{BA} \;\text{ with }\; \mathbf{a}^n_{BA} \perp \mathbf{a}^t_{BA} \perp AB\,, \quad \|\mathbf{a}^n_{BA}\| = \frac{\|\mathbf{v}_B - \mathbf{v}_A\|^2}{l_{AB}}\,,$$

where $\mathbf{a}_{BA}$ is pointing from B torwards A (see Fig. 2).

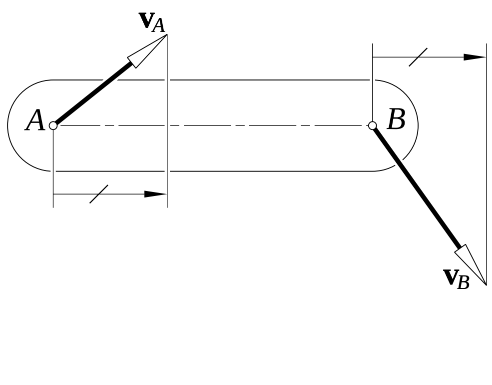

Figure 1: Projection Theorem concerning the velocity vectors $\mathbf{v}_A, \mathbf{v}_B$

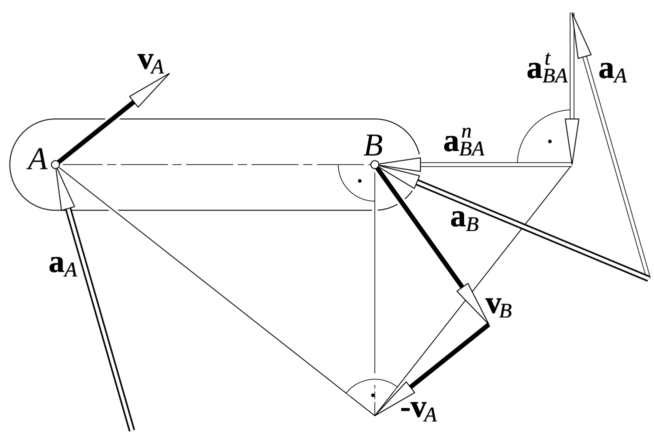

Figure 2: Relation between the acceleration vectors $\mathbf{a}_A$ and $\mathbf{a}_B$

[2] In this sense we can formulate condition (ii) in Definition 1 more precisely: There is no $\mathbf{c} \in \mathbb{R}^d$ and no skew symmetric $d \times d$ matrix C such that $\mathbf{z}_{k,1} = \mathbf{c} + C\mathbf{x}_k$ for all $k \in I$.

[3] This is not generally true for flexibility of order > 1 (see e.g. the geometric characterization of 2^{nd}-order flexibility of the framework $\mathbf{F}_{\text{pm}}$ in Fig. 6, case 1).

If also (9) is solvable, then the system (10) with the right side depending from the previous solutions is decisive whether the second-order flex of $\mathbf{F}$ can be extended to a third-order flex. Due to [1], for each framework $\mathbf{F}$ there is a sufficiently large n such that any nontrivial n-th-order flex can be extended to an analytical flex of $\mathbf{F}$. The example presented in Section 2 will demonstrate how n can vary even for frameworks with the same combinatorial structure.

Remark 2: In [2] a framework is presented which is infinitesimally flexible of first order. But only a polynomial function of type (6) starting with Z_2 is extendible to an analytical flex. Therefore I. Sabitov proposed in [6], p. 189, to denote the order of flexes $Z(t)$ by a pair (m, n) of indices, the smallest and the highest exponent of t showing up in (6).
In order to minimize m, one must require that this representation of $Z(t)$ is not redundant. This means that $Z(t)$ doesn't result from a flex $Z'(\tau)$ with smaller indices (m', n') under a non-regular parameter transformation $\tau := t^\lambda$, $\lambda \in \mathbb{N}$, $\lambda > 1$, i.e., $Z(t) \equiv Z'(t^\lambda)$.
In order to maximize n, one should prove that there is no parameter transformation $t := \tau^\mu$, $\mu \in \mathbb{N}$, $\mu > 1$, such that the new flex $Z''(\tau) := Z(\tau^\mu)$ is further extendible in a nontrivial way. "Trivial" in this sense means that e.g. the nontrivial flex $Z(t) = X + tZ_1 + t^2 Z_2$ gives rise to $Z''(\tau) := Z(\tau^2) = X + \tau^2 Z_1 + \tau^4 Z_2$ which immediately can be extended to the 5^{th}-order flex $Z'''(\tau) := Z''(\tau) + \tau^5 Z_1$.
In this more general sense, the classical definition given above covers only the flexibilities of order $(1, n)$.

2 Flexibility analysis of a particular planar framework

In the sequel a planar framework $\mathbf{F}_{\text{pm}}$ (see Fig. 3) is presented which consist of two triangles connected by three rods.[4] For this framework $\mathbf{F}_{\text{pm}}$ it is more convenient to change the previously used notation $\mathbf{x}_1, \ldots, \mathbf{x}_6$ of the vertices in such a way that $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ and $\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3$ denote the two triangles $\Delta_{\mathbf{p}}$ and $\Delta_{\mathbf{q}}$, resp., which are joined by bars $\mathbf{p}_i\mathbf{q}_i$, $i = 1, 2, 3$. Due to (11) in the case $d = 2$ this framework is infinitesimally flexible if and only if the rows in the 9×12 rigidity matrix

$$M_{\text{pm}} = \begin{pmatrix} (\mathbf{p}_1-\mathbf{p}_2) & (\mathbf{p}_2-\mathbf{p}_1) & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ (\mathbf{p}_1-\mathbf{p}_3) & \mathbf{o} & (\mathbf{p}_3-\mathbf{p}_1) & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & (\mathbf{p}_2-\mathbf{p}_3) & (\mathbf{p}_3-\mathbf{p}_2) & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_1-\mathbf{q}_2) & (\mathbf{q}_2-\mathbf{q}_1) & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_1-\mathbf{q}_3) & (\mathbf{q}_3-\mathbf{q}_1) & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_2-\mathbf{q}_3) & (\mathbf{q}_3-\mathbf{q}_2) \\ (\mathbf{p}_1-\mathbf{q}_1) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_1-\mathbf{p}_1) & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & (\mathbf{p}_2-\mathbf{q}_2) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_2-\mathbf{p}_2) & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & (\mathbf{p}_3-\mathbf{q}_3) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_3-\mathbf{p}_3) \end{pmatrix} \qquad (12)$$

are linearly dependent. Note that in this shorthand notation each entry stands for a 1×2 submatrix, so that M_{pm} actually has 12 columns.

For $\mathbf{F}_{\text{pm}}$ necessary and sufficient conditions for all levels of higher-order flexibility will be given. Also in robotics there is an interest on such "singularities" (compare e.g. [8]) since $\mathbf{F}_{\text{pm}}$ represents any posture of a planar parallel manipulator with three legs.

[4] Another planar framework with six vertices and nine edges such that at each vertex three edges are meeting is presented and analyzed in [10].

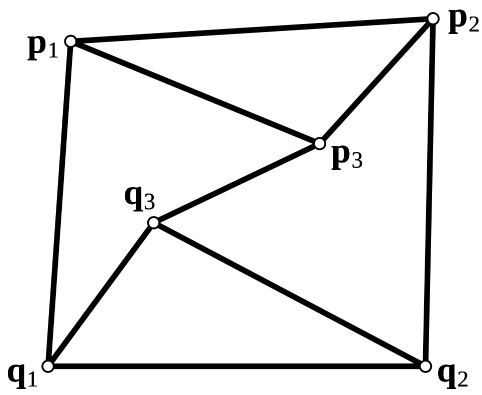

Figure 3: Framework $\mathbf{F}_{\rm pm}$

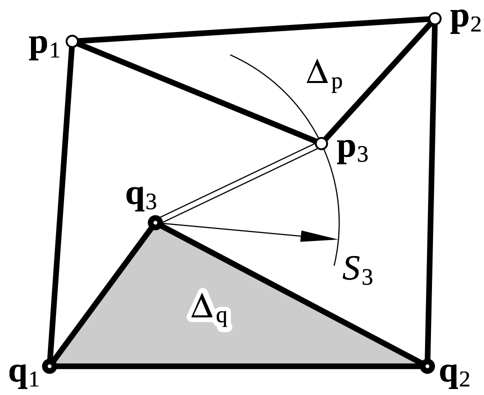

Figure 4: Four-bar linkage $\mathbf{q}_1\mathbf{q}_2\mathbf{p}_2\mathbf{p}_1$

From the standpoint of kinematics it is quite natural to analyze the framework $\mathbf{F}_{\rm pm}$ as a four-bar linkage with moving points $\mathbf{p}_1, \mathbf{p}_2$ and fixed points $\mathbf{q}_1, \mathbf{q}_2$ where the one-parameter mobility of the moving triangle $\Delta_{\mathbf{p}}$ against the fixed triangle $\Delta_{\mathbf{q}}$ is restricted by the additional bar $\mathbf{p}_3\mathbf{q}_3$ (see Fig. 4). Due to the following Lemma 1 we can apply standard results from kinematics in order to characterize frameworks $\mathbf{F}_{\rm pm}$ with higher-order flexibility.

Lemma 1: *Let $\mathbf{F} = (V, E)$ be a framework in $\mathbb{E}^d$, and let $\mathbf{F}' = (V, E')$ be the subframework where just one edge — say $\mathbf{x}_1\mathbf{x}_2$ — is missing. Suppose, $\mathbf{F}'$ admits a nontrivial analytical flex $Y'(t)$ which keeps $\mathbf{x}_1$ fixed, but has no stillstand at the beginning. Let S in $\mathbb{E}^d$ denote the hypersphere centered at $\mathbf{x}_1$ with radius $l_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$. Then the following implications hold:*

a) If the trajectory $\mathbf{y}_2'(t)$ of $\mathbf{x}_2$ under $Y'(t)$ has an n-th-order contact with S at the initial position $\mathbf{y}_2'(0) = \mathbf{x}_2$, then $\mathbf{F}$ is infinitesimally flexible of order n.

b) If $\mathbf{F}$ admits a flex $Z(t)$ of order n which — with regard to $\mathbf{F}'$ — can be extended to $Y'(t)$, then the trajectory $\mathbf{y}_2'(t)$ of $\mathbf{x}_2$ has an n-th-order contact with S.

Proof: Due to our assumption, the flex $Y'(t)$ is regularly parametrized, i.e., $\dot{Y}'(0)$ is nontrivial in the sense of (4). Then n-th-order contact between the hyersphere $S = (\mathbf{x}_1\,; l_{12})$[5] and the parametrized trajectory $\mathbf{y}_2'(t)$ at $t = 0$ means that the substitution of $\mathbf{y}_2'(t)$ in the equation

$$G(\mathbf{x}) := (\mathbf{x} - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_1) - l_{12}^2 = 0$$

of S gives a function $G\,(\mathbf{y}_2'(t))$ with a zero of multiplicity $n + 1$ at $t = 0$, i.e.,

$$(\mathbf{y}_2'(t) - \mathbf{x}_1) \cdot (\mathbf{y}_2'(t) - \mathbf{x}_1) - l_{12}^2 = o(t^n).$$

This holds also true for singular points with $\dot{\mathbf{y}}_2'(0) = \mathbf{o}$.

Ad a) The analytical functions $\mathbf{y}_1'(t), \ldots, \mathbf{y}_v'(t)$ given by $Y'(t)$ with $\mathbf{y}_k'(0) = \mathbf{x}_k$, $\mathbf{y}_1'(t) \equiv \mathbf{x}_1$, obey the equations

$$\|\mathbf{y}_i'(t) - \mathbf{y}_j'(t)\| - l_{ij} \equiv 0 \quad \forall (i,j) \in E'$$

and

$$\exists\ (k,l) \in I^2 \quad \text{with} \quad \frac{d}{dt}\|\mathbf{y}_k'(t) - \mathbf{y}_l'(t)\| \neq 0 \quad \text{at} \quad t = 0\,.$$

[5]This notation indicates the center $\mathbf{x}_1$ and the radius l_{12} of the hypersphere S in $\mathbb{E}^d$.

If there is an n-th-order contact between the trajectory $\mathbf{y}_2'(t)$ and the sphere S at $t=0$, then

$$Z(t) = X + t\,\dot{Y}' + \ldots + t^n \overset{(n)}{Y'} \tag{13}$$

is a nontrivial n-th-order flex of $\mathbf{F}$ since (3) holds also for the edge $(i,j)=(1,2)$ which was missing at $\mathbf{F}'$.

Ad b) Conversely, we suppose that the given flex $Z(\tau)$ is extendible to $Y'(t)$. This means that there is a regular C^n parameter transformation $t \to \tau$ and a superimposed trivial flex such that the initial terms of the transformed $Z(\tau)$ equal that of $Y'(t)$ as given on the right side of (13). Then equation (5) for $(i,j)=(1,2)$ expresses exactly the n-th-order contact between S and $\mathbf{y}_2'(t)$ at $t=0$. $\square$

After omitting the bar $\mathbf{p}_3\mathbf{q}_3$, the framework $\mathbf{F}_{\mathrm{pm}}$ reduces to a four-bar linkage $\mathbf{F}_{\mathrm{pm}}'$ (see Fig. 4). Then in almost every position of $\mathbf{F}_{\mathrm{pm}}'$ the moving triangle $\Delta_{\mathbf{p}}$ performs a uniquely defined *coupler motion* $Y_{(3)}'(t)$ against the frame link $\Delta_{\mathbf{q}}$. The only exceptions appear at folded positions with the four bars $\mathbf{q}_1\mathbf{p}_1$, $\mathbf{p}_1\mathbf{p}_2$, $\mathbf{p}_2\mathbf{q}_2$, $\mathbf{q}_2\mathbf{q}_1$ being aligned. Here the space of nontrivial velocity vectors Z_1' of $\mathbf{F}_{\mathrm{pm}}'$ is two-dimensional.

At any folded position there are two cases to distinguish: When $\mathbf{p}_1\mathbf{p}_2$ represents the minimal or the maximal distance between the possible path circles $(\mathbf{q}_i\,;\|\mathbf{p}_i-\mathbf{q}_i\|)$, $i=1,2$, of $\mathbf{p}_i$, then the four-bar linkage is only infinitesimally flexible of first order. Otherwise there are two analytical flexes passing through the folded position.

When in the following we address the nontrivial analytical flex $Y_{(3)}'(t)$ of $\Delta_{\mathbf{p}}$ against $\Delta_{\mathbf{q}}$, then we tacitly exclude the exceptional case of second-order rigidity. At all other folded positions we assume that one of the two analytical flexes passing through has already been specified. In this sense we can state:

Lemma 2: *The framework $\mathbf{F}_{\mathrm{pm}}$ is infinitesimally flexible of order n if and only if the trajectory of $\mathbf{p}_3$ under the analytical flex $Y_{(3)}'(t)$ of $\Delta_{\mathbf{p}}$ against $\Delta_{\mathbf{q}}$ has an n-th-order contact with the circle $S_{(3)} := (\mathbf{q}_3\,;\|\mathbf{p}_3-\mathbf{q}_3\|)$ at the initial position $t=0$.*

Remark 3: n-th-order flexibility of $\mathbf{F}_{\mathrm{pm}}$ is independent from the choice of the bar $\mathbf{p}_i\mathbf{q}_i$ to be omitted. So, n-th-order contact between the circle $S_{(3)} = (\mathbf{q}_3\,;\|\mathbf{p}_3-\mathbf{q}_3\|)$ and the path of $\mathbf{p}_3$ under $Y_{(3)}'$ is equivalent to a contact of n-th order between the circle $S_{(1)} := (\mathbf{q}_1\,;\|\mathbf{p}_1-\mathbf{q}_1\|)$ and the path of $\mathbf{p}_1$ under $Y_{(1)}'$, when $Y_{(1)}'$ denotes the coupler motion $\Delta_{\mathbf{p}}/\Delta_{\mathbf{q}}$ with coupler $\mathbf{p}_2\mathbf{p}_3$ and frame $\mathbf{q}_2\mathbf{q}_3$.

Now we make use of the following standard results concerning instantaneous kinematics of one-parameter Euclidean motions (cf. e.g. [12]):

(i) In each moment of an analytical flex $Y_{(3)}'(t) : \Delta_{\mathbf{p}}/\Delta_{\mathbf{q}}$ the lines orthogonal to the trajectories of the moving points A (attached to $\Delta_{\mathbf{p}}$) pass through a common point. This point P is called *instantaneous pole*; it can be finite (case 1 in Fig. 5) or a point at infinity (case 2).

(ii) Each moving point A has an uniquely defined curvature center A^* of its trajectory. The *curvature transformation* $A \mapsto A^*$ is either quadratic (cases 1 and 2.1) or a translation (case 2.2). The quadratic transformation is ruled by BOBILLIER's construction (see Fig. 6).

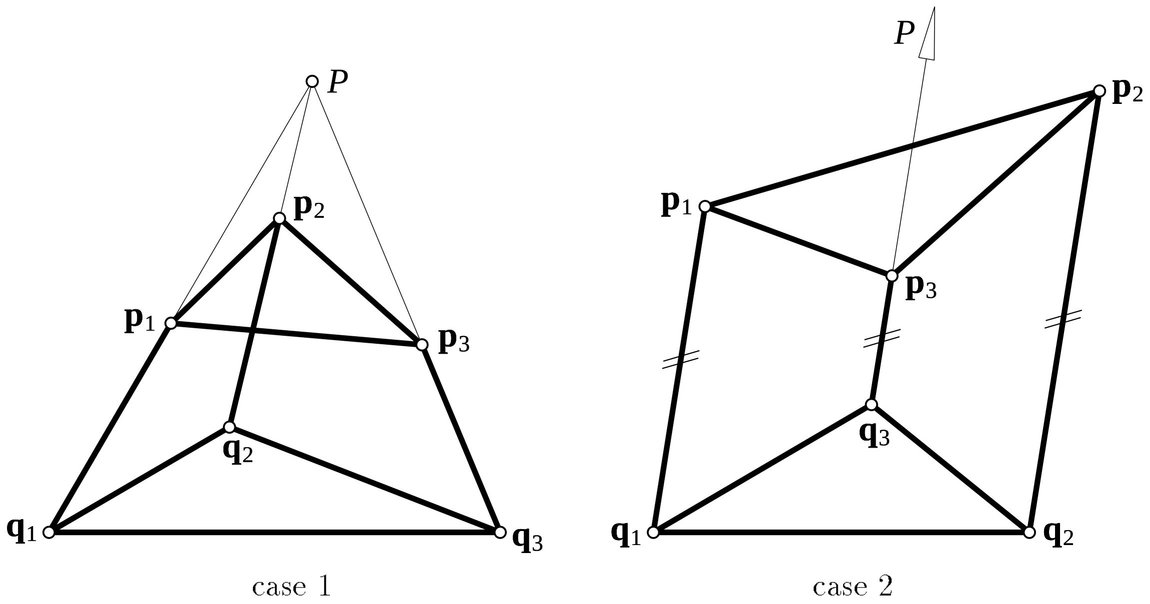

Figure 5: Infinitesimal flexibility of order 1

(iii) All moving points A with trajectories of stationary curvature are located on the *circle-point curve* c, which is either a cubic (case 1) or an orthogonal hyperbola (case 2.1). In case 2.2 this pointset is either a line or empty or the whole plane.

(iv) There are — in the algebraic sense — four points whose trajectories have a 4^{th}-order contact with their curvature circle. For a four-bar linkage, two of these BURMESTER *points* terminate the coupler. The other two need not be real points. It can happen (see [4]) that there is even a 5^{th}-order contact with the curvature circle.

(v) Coupler curves are algebraic of order six[6] with triple points at the absolute cyclic points. Therefore due to BEZOUT's theorem, a 6^{th}-order contact between a coupler curve and a circle implies that the circle is a component of the coupler curve.

Theorem 1:

1. The framework $\mathbf{F}_{\text{pm}}$ *in Fig. 3 is infinitesimally flexible of order* **1** *if and only if the lines* $\mathbf{p}_1\mathbf{q}_1$, $\mathbf{p}_2\mathbf{q}_2$, *and* $\mathbf{p}_3\mathbf{q}_3$ *are concurrent (case 1) or parallel (case 2) (see Fig. 5).*

2. $\mathbf{F}_{\text{pm}}$ *is flexible of order* **2** *if and only if* $\mathbf{p}_i \mapsto \mathbf{q}_i$, $i = 1, 2, 3$, *are corresponding under the curvature transformation of* $Y'_{(3)}(0)$. *This is equivalent in case 1 to congruent directed angles, in case 2.1 to equal directed distances, as indicated in Fig. 6. In case 2.2 the second-order flex can already be extended to an analytical flex.*

3. A second-order flexible framework $\mathbf{F}_{\text{pm}}$ *is even flexible of order* **3** *if and only if* $\mathbf{p}_3$ *is located on the circle-point curve* c *of* $Y'_{(3)}(0)$ *(Fig. 7).*[7]

[6]See e.g. [12], p. 68. The only exceptional moving points — beside $\mathbf{p}_1$ and $\mathbf{p}_2$ — are the two finite imaginary points of intersection between the zero circles $(\mathbf{p}_1\,;0)$ and $(\mathbf{p}_2\,;0)$ in the moving plane attached to $\Delta_{\mathbf{p}}$. The trajectories of these points are of order 4 with nodes at the absolute cyclic points.

[7]According to [3], the human knee is an example of a third-order infinitesimally flexible structure.

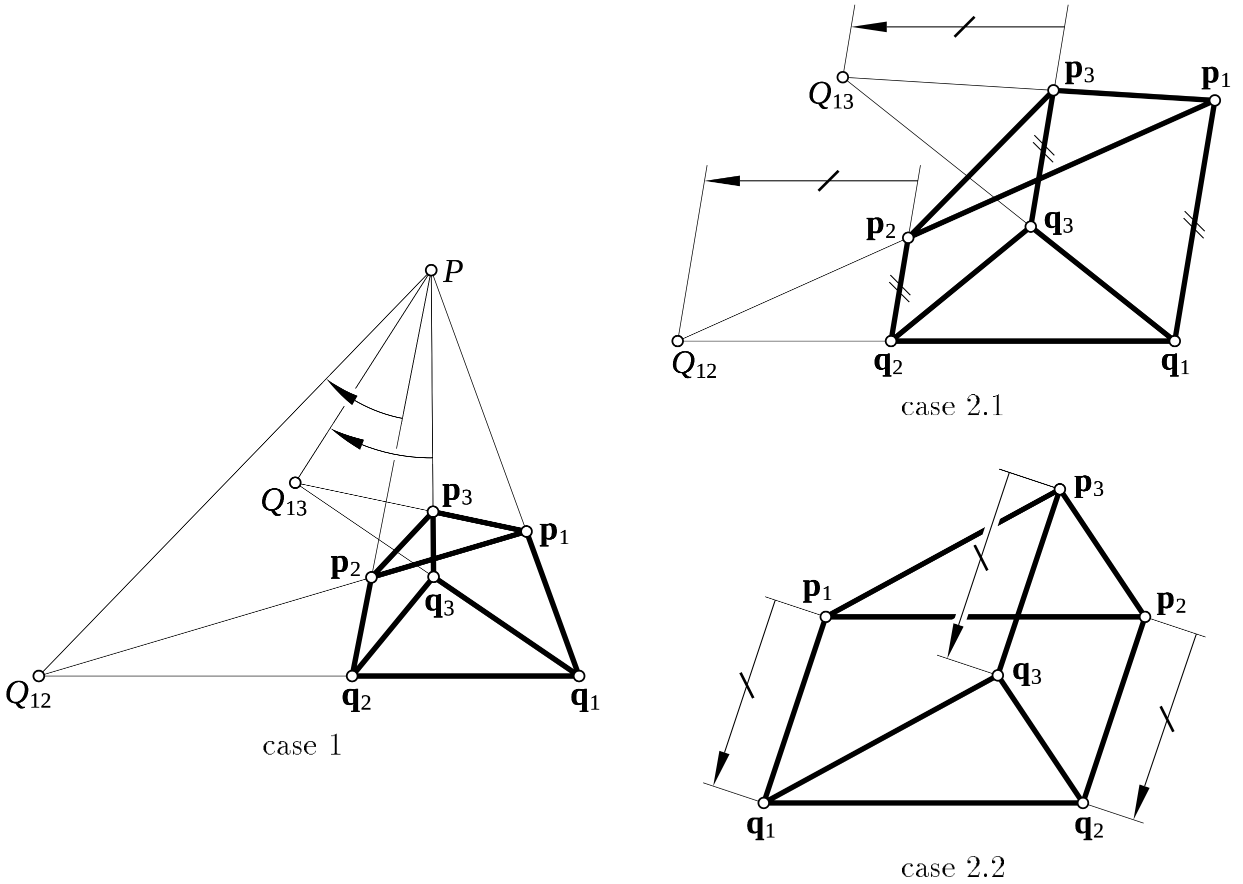

Figure 6: Infinitesimal flexibility of order 2, curvature transformation $\mathbf{p}_i \mapsto \mathbf{q}_i$

4. A third-order flexible framework $\mathbf{F}_{\mathrm{pm}}$ *is even flexible of order* **4** *if and only if* $\mathbf{p}_3$ *is one of the remaining two* BURMESTER *points of* $Y'_{(3)}(0)$*. In particular cases (see* [4]*) a fourth-order flexible framework* $\mathbf{F}_{\mathrm{pm}}$ *can even be flexible of order* **5**.
5. A framework $\mathbf{F}_{\mathrm{pm}}$ *which is infinitesimally flexible of order* **6** *must permit an analytical flex. However, beside the parallelogram linkage mentioned in item 2, case 2.2, there is no real analytically flexible framework* $\mathbf{F}_{\mathrm{pm}}$.[8] *The only non-real example comes from the antiparallelogram linkage, where the real focal points as well as the two imaginary focal points of the moving ellipse trace circles (cf.* [12]*, p. 199).*

These geometric characterizations together with Lemma 2 reveal that infinitesimally flexible frameworks $\mathbf{F}_{\mathrm{pm}}$ can preserve their flexibility, even when additional bars $\mathbf{p}_i\mathbf{q}_i$, $i = 4, \ldots$, are added with $\mathbf{p}_i$ attached to $\Delta_{\mathbf{p}}$ and $\mathbf{q}_i$ attached to $\Delta_{\mathbf{q}}$. In particular,

(i) flexibility of order 1 is preserved as long as the line spanned by each additional bar $\mathbf{p}_i\mathbf{q}_i$ passes through the instantaneous pole P.

(ii) Flexibility of order 2 is preserved for all pairs $\mathbf{p}_i \mapsto \mathbf{q}_i$ of the curvature transformation, provided $\mathbf{q}_i$ is finite.

[8] For a four-bar linkage with a moving kite, obeying e.g. $\|\mathbf{p}_1 - \mathbf{q}_1\| = \|\mathbf{q}_2 - \mathbf{q}_1\|$ and $\|\mathbf{p}_2 - \mathbf{p}_1\| = \|\mathbf{p}_2 - \mathbf{q}_2\|$, the motion $\Delta_{\mathbf{p}}/\Delta_{\mathbf{q}}$ is reducible. Through the folded position with $\mathbf{p}_1 = \mathbf{q}_2$ there passes also the pure rotation of $\Delta_{\mathbf{p}}$ about $\mathbf{q}_2$. Under this component of the coupler motion all points $\mathbf{p}_3$ attached to $\Delta_{\mathbf{p}}$ trace circles centered at $\mathbf{q}_3 = \mathbf{q}_2$. However, this case is excluded because of our general assumption $\|\mathbf{q}_3 - \mathbf{q}_2\| \neq 0$.

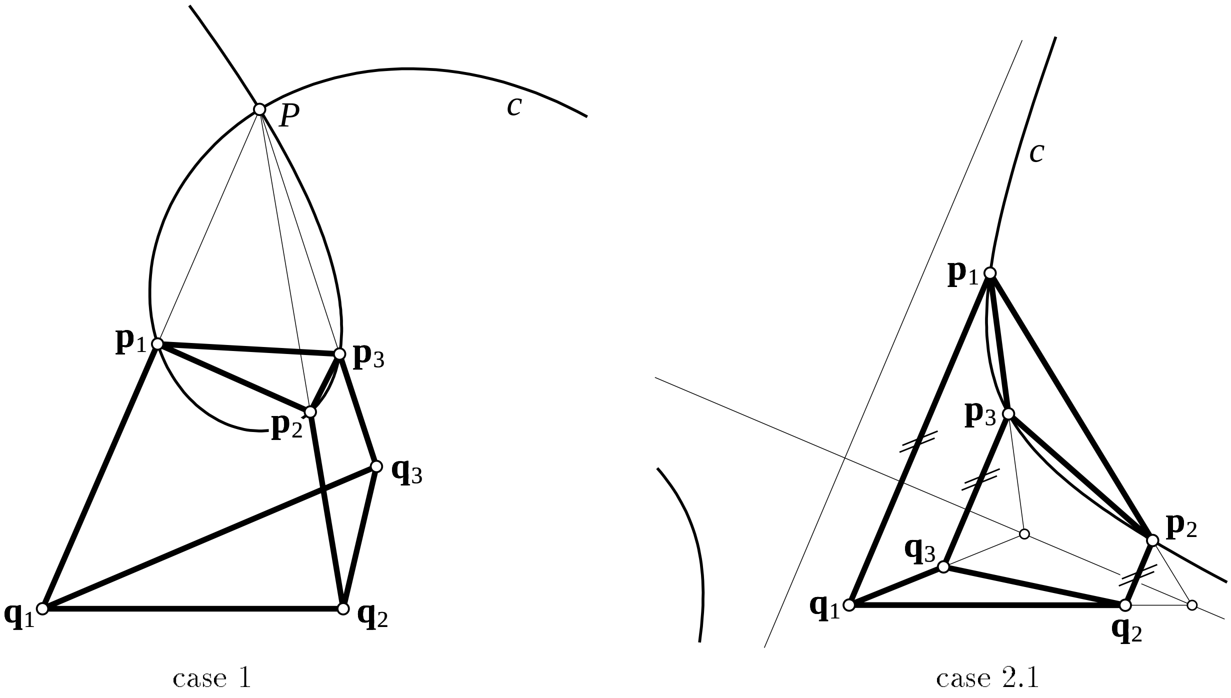

Figure 7: Flexibility of order 3, circle-point curve c

(iii) Any additional pair $\mathbf{p}_i \mapsto \mathbf{q}_i$ with $\mathbf{p}_i$ on the circle-point curve c of $Y'_{(3)}(0)$ preserves 3^{rd}-order flexibility. This reveals with Remark 3 that the circle-point curve of $Y'_{(3)}(0)$ (four-bar $\mathbf{q}_1\mathbf{q}_2\mathbf{p}_2\mathbf{p}_1$) must coincide with that of $Y'_{(1)}(0)$ (four-bar $\mathbf{q}_2\mathbf{q}_3\mathbf{p}_3\mathbf{p}_2$) and $Y'_{(2)}(0)$ (four-bar $\mathbf{q}_3\mathbf{q}_1\mathbf{p}_1\mathbf{p}_3$).[9]

(iv) In the cases 1 and 2.1 there is at most one additional bar $\mathbf{p}_4\mathbf{q}_4$ which still preserves flexibility of 4^{th} order. Again due to Remark 3, the initial positions of the analytical flexes $Y'_{(1)}(t)$, $Y'_{(2)}(t)$ and $Y'_{(3)}(t)$ must share their BURMESTER points.

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[9] Since the circle-point curve defines the curvature of the polodes (see e.g. [9]), the fixed polodes of the flexes $Y'_{(1)}(t)$, $Y'_{(2)}(t)$ and $Y'_{(3)}(t)$ must osculate each other at $t = 0$. The same holds for the moving polodes.

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