Higher-Order Flexibility for a Bipartite Planar Framework

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Dedicated to Prof. M. Hiller at the occasion of his 60^{th} birthday

Abstract

Infinitesimally flexible frameworks are well known in kinematics, in particular recently as singular postures in robotics. The objective of this paper is to analyze a bipartite planar framework in view of higher-order infinitesimal flexibility. The characterization of first-order flexibility of such frameworks has been well known for a long time. Now explicit necessary and sufficient conditions are proved for the orders two, three and even for n, provided both classes of vertices are non-collinear. For bipartite frameworks with 2^{nd}-order flexibility also a geometric characterization is given.

Key Words: Infinitesimal flexibility, framework, bipartite graph

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1 Infinitesimal flexibility of higher order

Let $\mathbf{F}$ be a framework in the Euclidean plane $\mathbb{E}^2$ with *vertex set* V and *edge set* E, i.e.,

$$V = \{\mathbf{x}_1, \ldots, \mathbf{x}_v\},\ \mathbf{x}_i \in \mathbb{R}^2\ \ \forall i \in I := \{1, \ldots, v\} \quad \text{and} \quad E \subset \Big\{(i,j) \mid i < j,\ (i,j) \in I^2\Big\}.$$

For each edge $\mathbf{x}_i\mathbf{x}_j$ of $\mathbf{F}$ the Euclidean length l_{ij} obeys

$$f_{ij}(\mathbf{x}_i, \mathbf{x}_j) := \|\mathbf{x}_i - \mathbf{x}_j\| - l_{ij} = 0 \quad \forall (i,j) \in E. \tag{1}$$

We suppose $l_{ij} > 0$ for all edges of $\mathbf{F}$.

Definition: The framework $\mathbf{F} = (V, E)$ is *infinitesimally flexible of order* n (in the classical sense[1]) if and only if for each $k \in I$ there is a polynomial function

$$\mathbf{z}_k(t) := \mathbf{x}_k + t\,\mathbf{z}_{k,1} + \ldots + t^n \mathbf{z}_{k,n}\,, \quad n \geq 1, \tag{2}$$

such that

(i) the replacement of $\mathbf{x}_i$ and $\mathbf{x}_j$ in (1) by $\mathbf{z}_i(t)$ and $\mathbf{z}_j(t)$, resp., gives functions obeying

$$f_{ij}\left(\mathbf{z}_i(t), \mathbf{z}_j(t)\right) = \|\mathbf{z}_i(t) - \mathbf{z}_j(t)\| - l_{ij} = o(t^n) \quad \forall (i,j) \in E, \tag{3}$$

i.e., with a zero of multiplicity $\geq n+1$ at $t = 0$, and

[1] See Sabitov (1992) or note Remark 2 in Stachel (1999).

(ii) in order to exclude *trivial* flexes, the vectors $\mathbf{z}_{1,1},\ldots,\mathbf{z}_{v,1}$ are not the velocity vectors of the vertices $\mathbf{x}_1,\ldots,\mathbf{x}_v$, resp., under any motion of $\mathbf{F}$ as a rigid body.

The v-tupel of functions $(\mathbf{z}_1(t),\ldots,\mathbf{z}_v(t))$ is called a nontrivial *n-th-order flex* of $\mathbf{F}$.

Suppose, the framework $\mathbf{F}$ is given by its combinatorial structure E and by the lengths l_{ij} of its edges. Then (1) defines a system of $e := \#E$ quadratic equations

$$(\mathbf{x}_i-\mathbf{x}_j)\cdot(\mathbf{x}_i-\mathbf{x}_j) = l_{ij}^2 \quad \forall (i,j)\in E \tag{4}$$

for the $2v$ unknown coordinates of the vertices. After substituting (2) in (4) the comparison of coefficients of $t, t^2, \ldots, t^n$ gives rise to the following systems of linear equations, each for all $(i,j)\in E$:

$$\begin{aligned}
(\mathbf{x}_i-\mathbf{x}_j)\cdot(\mathbf{z}_{i,1}-\mathbf{z}_{j,1}) &= 0\,, && (5)\\
(\mathbf{x}_i-\mathbf{x}_j)\cdot(\mathbf{z}_{i,2}-\mathbf{z}_{j,2}) &= -\tfrac{1}{2}(\mathbf{z}_{i,1}-\mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,1}-\mathbf{z}_{j,1}), && (6)\\
(\mathbf{x}_i-\mathbf{x}_j)\cdot(\mathbf{z}_{i,3}-\mathbf{z}_{j,3}) &= -(\mathbf{z}_{i,1}-\mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,2}-\mathbf{z}_{j,2}), && (7)\\
(\mathbf{x}_i-\mathbf{x}_j)\cdot(\mathbf{z}_{i,4}-\mathbf{z}_{j,4}) &= -(\mathbf{z}_{i,1}-\mathbf{z}_{j,1})\cdot(\mathbf{z}_{i,3}-\mathbf{z}_{j,3})-\tfrac{1}{2}(\mathbf{z}_{i,2}-\mathbf{z}_{j,2})\cdot(\mathbf{z}_{i,2}-\mathbf{z}_{j,2}), && (8)\\
\cdots &= \cdots
\end{aligned}$$

The matrix M on the left side of each of these systems (see example in (10)) is called *rigidity matrix* of $\mathbf{F}$ (cf. Graver and Servatius (1993)).

In particular, the *"velocity vectors"* $\mathbf{z}_{1,1},\ldots,\mathbf{z}_{v,1}$ of the vertices have to solve the homogeneous system (5) of linear equations. Therefore infinitesimal flexibility of order 1 can be characterized by the rank condition

$$\mathrm{rk}(M) < 2v-3 \tag{9}$$

(see e.g. Stachel (1999), eq. (11)). Any nontrivial solution of (5) defines the right side in the inhomogeneous system (6), and then 2nd-order flexibility of $\mathbf{F}$ is equivalent to the solvability of this system. This can be repeated step by step for (7), (8), ... to figure out the order of flexibility for any given framework. If $(\tilde{\mathbf{z}}_{1,r},\ldots,\tilde{\mathbf{z}}_{v,r})$ is any particular solution of the r-th linear system, $r\in\{1,\ldots,n\}$, then also

$$\mathbf{z}_{k,r} := \tilde{\mathbf{z}}_{k,r}+\mathbf{c}+C\mathbf{x}_k\,,\quad C^T=-C,\quad \text{for}\quad k=1,\ldots,v$$

with constant $\mathbf{c}\in\mathbb{R}^2$ and any skew symmetric 2×2 matrix C is a solution of this system.

It has been proved by Alexandrov (1998) that for each framework there is a sufficiently large n such that any nontrivial n-th-order flex (2) can be extended to a set of analytical functions which solve (1) identically.

2 Flexibility analysis of a particular planar framework

We now focus on the planar framework $\mathbf{F}_\mathrm{b}$ (see Fig. 1) which is based on a bipartite graph.[2] We change the notation of the vertices such that the edges of $\mathbf{F}_\mathrm{b}$ can be written as $\mathbf{p}_i\mathbf{q}_j$ for all $i,j\in\{1,2,3\}$. This framework (see also Wunderlich (1983) or Graver and Servatius (1993),

[2]This means that the set V of vertices can be subdivided into two classes such that E consists of all edges connecting vertices from different classes.
In Stachel (1999) another planar framework is presented and analyzed, which consists of six vertices and nine edges with three edges meeting at each vertex — like $\mathbf{F}_\mathrm{b}$.

Fig. 4.25) is infinitesimally flexible if and only if the nine rows in the 9×12 rigidity matrix (each entry stands here for a 1×2 submatrix)

$$M_{\rm b} = \begin{pmatrix} (\mathbf{p}_1-\mathbf{q}_1) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_1-\mathbf{p}_1) & \mathbf{o} & \mathbf{o} \\ (\mathbf{p}_1-\mathbf{q}_2) & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_2-\mathbf{p}_1) & \mathbf{o} \\ (\mathbf{p}_1-\mathbf{q}_3) & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_3-\mathbf{p}_1) \\ \mathbf{o} & (\mathbf{p}_2-\mathbf{q}_1) & \mathbf{o} & (\mathbf{q}_1-\mathbf{p}_2) & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & (\mathbf{p}_2-\mathbf{q}_2) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_2-\mathbf{p}_2) & \mathbf{o} \\ \mathbf{o} & (\mathbf{p}_2-\mathbf{q}_3) & \mathbf{o} & \mathbf{o} & \mathbf{o} & (\mathbf{q}_3-\mathbf{p}_2) \\ \mathbf{o} & \mathbf{o} & (\mathbf{p}_3-\mathbf{q}_1) & (\mathbf{q}_1-\mathbf{p}_3) & \mathbf{o} & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & (\mathbf{p}_3-\mathbf{q}_2) & \mathbf{o} & (\mathbf{q}_2-\mathbf{p}_3) & \mathbf{o} \\ \mathbf{o} & \mathbf{o} & (\mathbf{p}_3-\mathbf{q}_3) & \mathbf{o} & \mathbf{o} & (\mathbf{q}_3-\mathbf{p}_3) \end{pmatrix} \tag{10}$$

are linearly dependent.

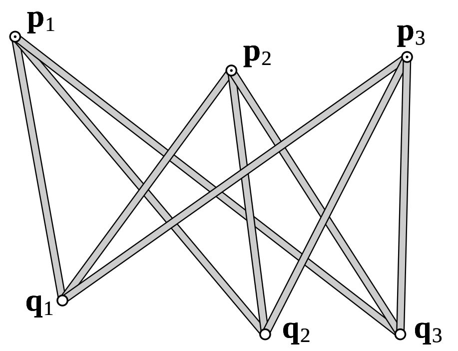


Figure 1: Framework $\mathbf{F}_{\rm b}$

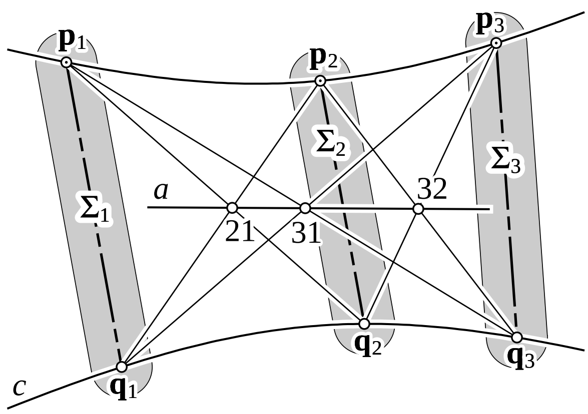


Figure 2: First-order flexing $\mathbf{F}_{\rm b}$

Infinitesimal flexibility can be seen as the limiting case of adjacent incongruent configurations of a framework with given edge-lengths l_{ij} . Due to Stachel (1982a), (1982b) or (1982c) any two incongruent configurations of $\mathbf{F}_{\rm b}$ are associated with a pair of confocal conics passing through the triples $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of vertices, respectively (see e.g. particular case in Fig. 4). Hence, one can expect any relation between higher-order flexibility of $\mathbf{F}_{\rm b}$ and linear systems of conics.

The *configuration problem* of $\mathbf{F}_{\rm b}$, i.e., the problem of determining the vertices $\mathbf{p}_1, \dots, \mathbf{q}_3$ from the nine lengths $l_{ij} = \|\mathbf{p}_i - \mathbf{q}_j\|$, is of degree 8 (Wunderlich (1977)). Therefore one can expect that *any* $\mathbf{F}_{\rm b}$ *which admits an 8*$^{\rm th}$*-order flex must be continuously flexible.* According to Dixon (1899) there are two movable versions (see figures 4 and 6 in Wunderlich (1977)): At *Dixon's first mechanism* each of the classes $\{\mathbf{p}_1, \dots\}$ and $\{\mathbf{q}_1, \dots\}$ is collinear, and the spanned lines are orthogonal; the mobility holds for any number of vertices. At *Dixon's second mechanism* the vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ are selected respectively from two rectangles with common axes of symmetry. Even the fourth vertices $\mathbf{p}_4$ and $\mathbf{q}_4$ of the rectangles can be added without disturbing the mobility.

2.1 First-order flexibility revisited

The following geometric characterization of first-order flexibility of $\mathbf{F}_{\rm b}$ is well known. According to Wunderlich (1983) it is almost impossible to figure out where it stems from. The particular orthogonal choice of the velocity vectors $\mathbf{p}_{1,1}, \dots, \mathbf{q}_{3,1}$ originates from Wunderlich (1983), but it can also be deduced by a limiting process from the results given in Stachel (1982a) or (1982b). The eqs. in (23) will reveal that this specification is actually the "simplest" one.

Theorem 1: $\mathbf{F}_\mathrm{b}$ *is flexible of first order if and only if the six vertices are located on a curve* c *of second order. This flexibility is preserved when the bipartite framework is extended by an arbitrary number of additional vertices* $\mathbf{p}_4, \ldots, \mathbf{p}_m, \mathbf{q}_4, \ldots \mathbf{q}_n \in c$.
In almost all cases there is a nontrivial first order flex with velocity vectors orthogonal to c.

Proof: For each $i = 1, 2, 3$ the edge $\mathbf{p}_i\mathbf{q}_i$ defines a system Σ_i. According to the three-pole-theorem by Aaronhold-Kennedy the existence of velocity vectors being compatible with all given lengths l_{ij} is equivalent to aligned poles 21, 31, 32 of the coupler motions Σ_2/Σ_1, Σ_3/Σ_1, Σ_3/Σ_2, resp. (see Fig. 2). However, this characterizes the vertices as points of a curve c of second order for the following reason: According to the theorem of Pappus-Pascal $\mathbf{p}_1, \ldots, \mathbf{q}_3 \in c$ implies collinearity of the three relative poles. For the converse we follow Walker (1978), p. 64, and prove it by means of Algebraic Geometry:

The triples of lines $k_1 := \{\mathbf{p}_1\mathbf{q}_2, \mathbf{p}_2\mathbf{q}_3, \mathbf{p}_3\mathbf{q}_1\}$ and $k_2 := \{\mathbf{p}_2\mathbf{q}_1, \mathbf{p}_3\mathbf{q}_2, \mathbf{p}_1\mathbf{q}_3\}$ define two reducible curves of third order, which span a linear system $\mathcal{S}$ of cubics. Each curve of this system $\mathcal{S}$ contains all points of intersection between k_1 and k_2, i.e., the six vertices and the three poles. Let a denote the line through the relative poles. Through any point $\mathbf{x} \in a \setminus \{21, 31, 32\}$ there passes a cubic $k \in \mathcal{S}$. The line a shares four points with k. Therefore k must break up into a and any curve c of second order passing through the remaining $\mathbf{p}_1, \ldots, \mathbf{q}_3$.

In order to obtain velocity vectors $\mathbf{p}_{i,1}$ and $\mathbf{q}_{j,1}$, we follow Wunderlich (1983) and parametrize the curve c according to its type:

type of c	$\mathbf{p}_i$	$\mathbf{q}_j$
ellipse	$(a\cos u_i, b\sin u_i)$	$(a\cos v_j, b\sin v_j)$
hyperbola	$(a\cosh u_i, b\sinh u_i)$	$(a\cosh v_j, b\sinh v_j)$
parabola	$(u_i^2 - a^2, 2au_i)$	$(v_j^2 - a^2, 2av_j)$
intersecting lines	(u_i, ku_i)	$(v_j, \pm kv_j)$
parallel lines	(u_i, k)	$(v_j, \pm k)$

(11)

with constant $a, b, k \neq 0$, where k and $-k$ belong to the same framework.

Now we can easily verify that the following choice gives vectors which obey (5) for all $i, j \in \{1, 2, \ldots\}$. In almost all cases they are orthogonal to c. The only exceptions arise when c splits into two lines and both sets $\{\mathbf{p}_1, \ldots\}$ and $\{\mathbf{q}_1, \ldots\}$ of vertices are collinear and the spanned lines are different. Then the orthogonal choice would give trivial flexes only, as it already has been noted in Wunderlich (1983). In the following table these exceptions are called *collinear cases*. Then only the lower sign of k in the table above is permitted. Note that in the collinear case with intersecting lines the point of intersection must not be any vertex.

type of c	$\mathbf{p}_{i,1}$	$\mathbf{q}_{j,1}$
ellipse	$(b\cos u_i, a\sin u_i)$	$-(b\cos v_j, a\sin v_j)$
hyperbola	$(b\cosh u_i, -a\sinh u_i)$	$(-b\cosh v_j, a\sinh v_j)$
parabola	$(-2a^2, 2au_i)$	$(2a^2, -2av_j)$
intersecting lines	$(-k^2u_i, ku_i)$	$(k^2v_j, \mp kv_j)$
– collinear case	$(k/u_i, 1/u_i)$	$(-k/v_j, 1/v_j)$
parallel lines	$(0, k)$	$(0, \mp k)$
– collinear case	$(-u_i, u_i^2/2k)$	$(v_j, v_j^2/2k)$

(12)

The elliptic and parabolic cases are displayed in Fig. 3. In the twofold singular case with c being a line, any choice of vectors $\mathbf{p}_{i,1}$ and $\mathbf{q}_{j,1}$ orthogonal to c gives a first-order flex of $\mathbf{F}_\mathrm{b}$. $\square$

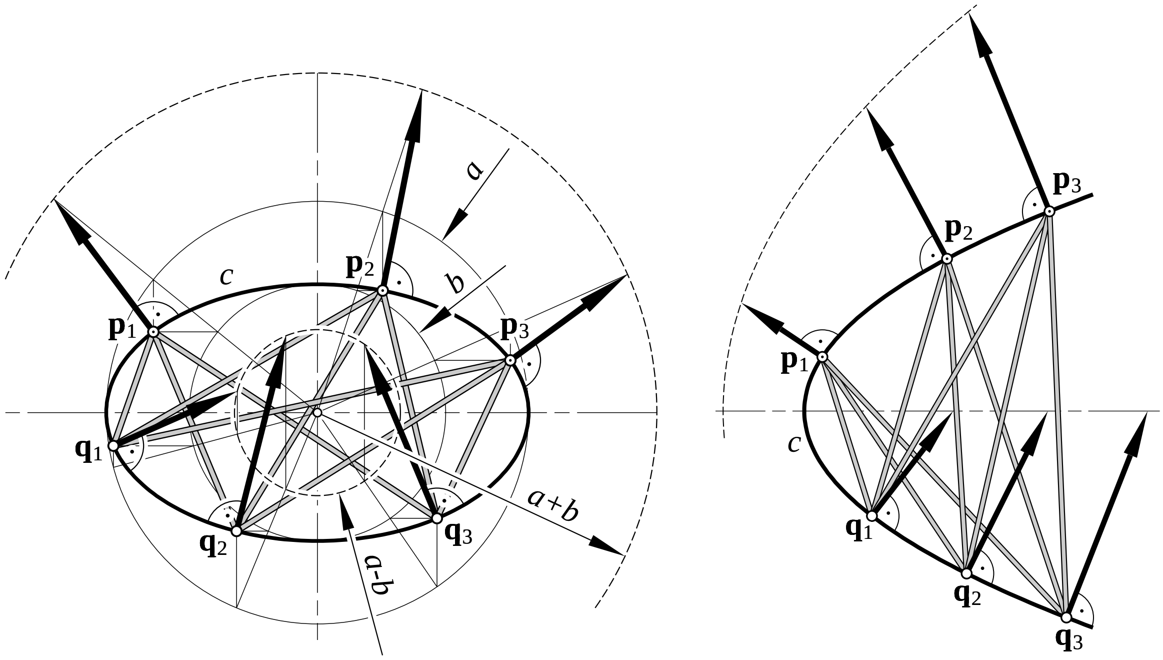
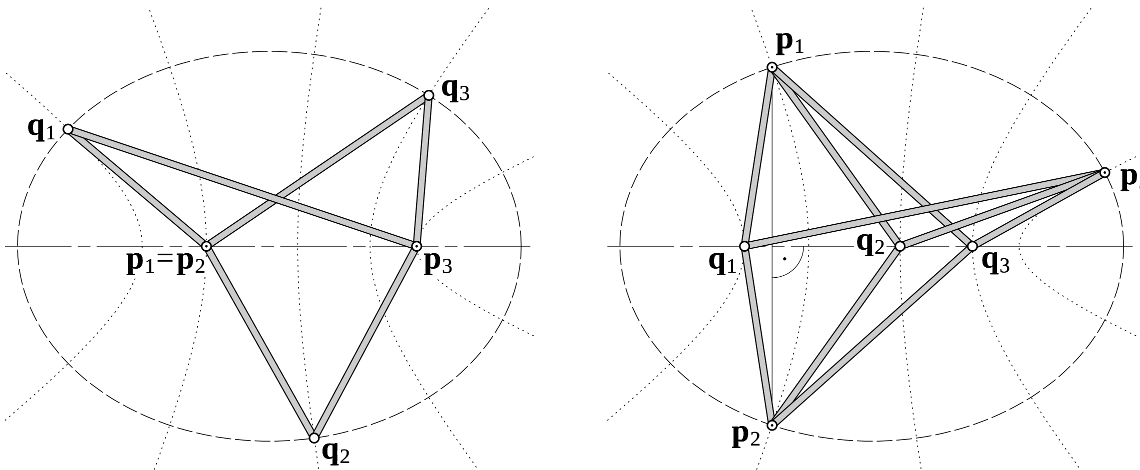


Figure 3: Two first-order flexing frameworks with velocity vectors orthogonal to the conic c

2.2 Conditions for $\mathbf{F}_\mathrm{b}$ with n-th-order flexibility

From now on we confine ourselves to the case of pairwise different vertices $\mathbf{p}_1, \ldots, \mathbf{q}_3$. The condition $\mathbf{p}_i \neq \mathbf{q}_j$ is already guaranteed by the general assumption $l_{ij} > 0$. In the case $\mathbf{p}_1 = \mathbf{p}_2$ the framework $\mathbf{F}_\mathrm{b}$ admits a continuous flex (see Fig. 4a). As a consequence of Ivory's theorem (see Stachel (1982a)) the same framework has a rigid configuration too (Fig. 4b).



a) continuously flexible configuration　　b) rigid configuration

Figure 4: Exceptional framework $\mathbf{F}_\mathrm{b}$ with $\mathbf{p}_1 = \mathbf{p}_2$

Let an n-th-order flex of $\mathbf{F}_\mathrm{b}$ be given by

$$\mathbf{p}'_i(t) = \mathbf{p}_i + \mathbf{p}_{i,1}t + \cdots + \mathbf{p}_{i,n}t^n \quad \text{and} \quad \mathbf{q}'_j(t) = \mathbf{q}_j + \mathbf{q}_{j,1}t + \cdots + \mathbf{q}_{j,n}t^n \tag{13}$$

for $i, j = 1, 2, 3$.

From now on we suppose *non-collinear* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$: Then for each $t \in \mathbb{R}$ there is an affine transformation $\alpha : \mathbf{p}_i \mapsto \mathbf{p}'_i(t)$ for $i = 1, 2, 3$. When we use matrix notation and write the coordinate vectors as columns, then we can set up

$$\alpha(t)\colon\ \mathbf{p}_i \ \mapsto\ \mathbf{p}'_i(t) = \mathbf{a}(t) + A(t)\mathbf{p}_i \quad \text{with} \quad \mathbf{a}(0) = \mathbf{o}, \quad A(0) = I_2\,, \tag{14}$$

where I_2 denotes the 2×2 unit matrix. The coordinates of $\mathbf{a}(t)$ and the entries of $A(t)$ are polynomials of degree $\le n$. In the new notation equation (3) reads

$$\left[\mathbf{p}'_i(t) - \mathbf{q}'_j(t)\right]^2 - (\mathbf{p}_i - \mathbf{q}_j)^2 = o(t^n) \quad \text{for all} \quad i, j \in \{1, 2, 3\}. \tag{15}$$

We subtract the equation for $i = 1$ and obtain

$$[\mathbf{p}'_i(t) - \mathbf{p}'_1(t)]^T \left[\mathbf{p}'_i(t) + \mathbf{p}'_1(t) - 2\mathbf{q}'_j(t)\right] - (\mathbf{p}_i - \mathbf{p}_1)^T(\mathbf{p}_i + \mathbf{p}_1 - 2\mathbf{q}_j) = o(t^n).$$

Subtracting from this the equation for $j = 1$ gives

$$-2\,[\mathbf{p}'_i(t) - \mathbf{p}'_1(t)]^T \left[\mathbf{q}'_j(t) - \mathbf{q}'_1(t)\right] + 2(\mathbf{p}_i - \mathbf{p}_1)^T(\mathbf{q}_j - \mathbf{q}_1) = o(t^n),$$

and due to (14) we get

$$-2(\mathbf{p}_i - \mathbf{p}_1)^T A(t)^T \left[\mathbf{q}'_j(t) - \mathbf{q}'_1(t)\right] + 2(\mathbf{p}_i - \mathbf{p}_1)^T(\mathbf{q}_j - \mathbf{q}_1) = o(t^n) \quad \text{for} \quad i, j \in \{2, 3\}.$$

The supposed linear independence of the difference vectors $\{\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1\}$ implies

$$A(t)^T \left[\mathbf{q}'_j(t) - \mathbf{q}'_1(t)\right] - (\mathbf{q}_j - \mathbf{q}_1) = \mathbf{o}(t^n) \quad \text{for} \quad j = 2, 3.$$

This means that for each $t \in \mathbb{R}$ there is a second affine transformation

$$\widehat{\alpha}(t)\colon\ \mathbf{q}'_j(t) \ \mapsto\ \widehat{\mathbf{a}}(t) + A(t)^T\mathbf{q}'_j(t) = \mathbf{q}_j + \mathbf{o}(t^n) \quad \text{for} \quad j = 1, 2, 3 \tag{16}$$

with $\widehat{\mathbf{a}}(0) = \mathbf{o}$. Thus we have proved

Lemma 1: *An n-th-order flex* (13) *of $\mathbf{F}_\mathrm{b}$ with non-collinear $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ can be embedded in a pair of one-parameter affine motions $\alpha(t)$ and $\widehat{\alpha}(t)$ according to* (14) *and* (16)*. These affine motions induce adjoint linear mappings for each $t \in \mathbb{R}$.*

For non-collinear $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ there exists also the inverse

$$\begin{aligned} \widehat{\alpha}(t)^{-1}\colon \quad & \mathbf{q}_j \ \mapsto\ \mathbf{b}(t) + A(t)^{-1T}\mathbf{q}_j = \mathbf{q}'_j + \mathbf{o}(t^n) \\ & \text{for} \quad \mathbf{b} := -A^{-1T}\widehat{\mathbf{a}} \quad \text{or} \quad \widehat{\mathbf{a}}^T = -\mathbf{b}^T A. \end{aligned} \tag{17}$$

Corollary 1: *If the bipartite framework with non-collinear $\{\mathbf{p}_1, \dots\}$ or $\{\mathbf{q}_1, \dots\}$ is flexible of order n, then for each r obeying $1 \le r \le n$ there are affine transformations*

$$\alpha_r\colon\ \mathbf{p}_i \mapsto \mathbf{p}_i + \mathbf{p}_{i,r}\,, \quad \widehat{\alpha}_r\colon\ \mathbf{q}_j \mapsto \mathbf{q}_j + \mathbf{q}_{j,r} \quad \text{for} \quad i, j = 1, 2, \dots,$$

mapping the vertices onto the endpoints of arrows indicating the r-th derivation vectors.

Proof: According to (14) we have

$$\mathbf{p}_i + r!\,\mathbf{p}_{i,r} = \mathbf{p}_i + \left.\frac{d^r\mathbf{p}_i'(t)}{dt^r}\right|_{t=0} = \overset{(r)}{\mathbf{a}}(0) + \left[I_2 + \overset{(r)}{A}(0)\right]\mathbf{p}_i\,.$$

In the same way we deduce from (17)

$$\mathbf{q}_j + r!\,\mathbf{q}_{j,r} = \overset{(r)}{\mathbf{b}}(0) + \left[I_2 + \left.\frac{d^r A(t)^{-1^T}}{dt^r}\right|_{t=0}\right]\mathbf{q}_j\,.$$

Obviously, Corollary 1 makes only sense, if there are more than six vertices. Note that this statement is not true in the “collinear” cases listed in (12). □

We substitute in (15) the matrix representations (14) of α and (16) of $\widehat{\alpha}$ and obtain

$$\begin{aligned}&\mathbf{a}^T\mathbf{a} + 2\mathbf{a}^T(A\,\mathbf{p}_i - \mathbf{q}_j') + \mathbf{p}_i^T A^T A\,\mathbf{p}_i - 2\mathbf{p}_i^T A^T\mathbf{q}_j' + \mathbf{q'}_j^T\mathbf{q}_j' - \\ &-\widehat{\mathbf{a}}^T\widehat{\mathbf{a}} - 2\widehat{\mathbf{a}}^T(A^T\mathbf{q}_j' - \mathbf{p}_i) - \mathbf{q'}_j^T AA^T\mathbf{q}_j' + 2\mathbf{p}_i^T A^T\mathbf{q}_j' - \mathbf{p}_i^T\mathbf{p}_i = o(t^n).\end{aligned} \tag{18}$$

Separating the terms with $\mathbf{p}_i$ from those with $\mathbf{q}_j'(t)$ leads to

$$\begin{aligned}&\mathbf{p}_i^T(A^TA - I_2)\mathbf{p}_i + 2(\mathbf{a}^TA + \widehat{\mathbf{a}}^T)\mathbf{p}_i + \mathbf{a}^T\mathbf{a} = \\ &= \mathbf{q'}_j^T(AA^T - I_2)\mathbf{q}_j' + 2(\mathbf{a}^T + \widehat{\mathbf{a}}^TA^T)\mathbf{q}_j' + \widehat{\mathbf{a}}^T\widehat{\mathbf{a}} + o(t^n).\end{aligned}$$

As this equation must hold for all $i,j \in \{1,2,3\}$, both sides of the equation must be constant for each $t \in \mathbb{R}$. This results in two equations which are quadratic in the coordinates of $\mathbf{p}_i$ and $\mathbf{q}_j'$, respectively. With (17) we obtain — after subtracting $\mathbf{b}^T\mathbf{a} = \mathbf{a}^T\mathbf{b}$ from both sides —

$$\mathbf{p}_i^T(A^TA - I_2)\mathbf{p}_i + 2(\mathbf{a}^T - \mathbf{b}^T)A\,\mathbf{p}_i + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{a} = \gamma(t), \qquad i = 1,2,3, \tag{19}$$

and

$$\mathbf{q'}_j^T(t)(AA^T - I_2)\mathbf{q}_j'(t) + 2(\mathbf{a}^T - \mathbf{b}^TAA^T)\mathbf{q}_j'(t) + \mathbf{b}^TAA^T\mathbf{b} - \mathbf{a}^T\mathbf{b} = \gamma(t) + o(t^n)$$

with a rational function $\gamma(t)$. For the sake of brevity we did no longer indicate that the matrix A as well as the vectors $\mathbf{a}$ and $\mathbf{b}$ are functions of t.

Now we apply $\widehat{\alpha}^{-1}$ to $\mathbf{q}_j'(t)$ in the second quadratic equation. Due to (17) we get finally

$$\mathbf{q}_j^T(I_2 - A^{-1}A^{-1^T})\mathbf{q}_j + 2(\mathbf{a}^T - \mathbf{b}^T)A^{-1^T}\mathbf{q}_j + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{b} = \gamma(t) + o(t^n) \tag{20}$$

for $j = 1,2,3$.

Conversely, the two equations (19) and (20) imply (18) and hence (15). We summarize:

Theorem 2: *The framework* $\mathbf{F}_\mathrm{b}$ *with non-collinear* $\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}$ *and* $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$ *is flexible of order* n *if and only if in a neighborhood of* $t = 0$ *there are functions* $\mathbf{a}(t)$, $\mathbf{b}(t)$, $A(t)$, *and* $\gamma(t)$ *of class* C^n *such that the vertices* $\mathbf{p}_1,\ldots,\mathbf{q}_3$ *obey the equations* (19) *and* (20).

From (19) and (20) we can deduce two functions

$$\begin{aligned}f(t,\mathbf{x}) &:= \mathbf{x}^T(A^TA - I_2)\mathbf{x} + 2(\mathbf{a}^T - \mathbf{b}^T)A\,\mathbf{x} + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{a} - \gamma(t),\\ g(t,\mathbf{x}) &:= \mathbf{x}^T(I_2 - A^{-1}A^{-1^T})\mathbf{x} + 2(\mathbf{a}^T - \mathbf{b}^T)A^{-1^T}\mathbf{x} + (\mathbf{a}^T - \mathbf{b}^T)\mathbf{b} - \gamma(t).\end{aligned} \tag{21}$$

Lemma 2: *For each* t_0 *sufficiently near to* 0 *the equations* $f(t_0,\mathbf{x}) = 0$ *and* $g(t_0,\mathbf{x}) = 0$ *represent two confocal curves of second order.*

Sketch of the proof: We specify the coordinate systems for $\mathbf{p}_1, \ldots$ and $\mathbf{p}'_1(t_0), \ldots$ according to the singular-value decomposition of the regular matrix $A(t_0)$. Then $A(t_0)$ becomes a diagonal matrix and the curve $f(t_0, \mathbf{x}) = 0$ gets one of the following normal forms:

$$f(t_0, \mathbf{x}) = \begin{cases} (\lambda^2 - 1)x^2 + (\mu^2 - 1)y^2 - \gamma(t_0) & \text{for} \quad (\lambda^2 - 1)(\mu^2 - 1) \neq 0 \\ 2\nu x + (\mu^2 - 1)y^2 - \rho & \text{for} \quad \mu^2 \neq 1 \end{cases}$$

with certain constants ν, ρ and λ, μ denoting the singular values of $A(t_0)$. For the corresponding second curve we obtain

$$g(t_0, \mathbf{x}) = \begin{cases} \dfrac{\lambda^2 - 1}{\lambda^2}\, x^2 + \dfrac{\mu^2 - 1}{\mu^2}\, y^2 - \gamma(t_0) \quad \text{or} \\ 2\nu x + \dfrac{\mu^2 - 1}{\mu^2}\, y^2 - \rho - \nu^2 \end{cases}$$

Now we can see: If these curves are conics, then both are of the same type and they share the focal points. In the singular case ($\gamma(t_0) = 0$ or $\nu = 0$) both are pairs of lines sharing their axes of symmetry. □

If for any t_0 the equations $f(t_0, \mathbf{p}_i) = 0$ and $g(t_0, \mathbf{q}_j) = 0$ hold for all $i, j = 1, 2, 3$, then we have found a second order curve passing through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ for which there is a confocal conic passing through $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (compare Stachel (1982c)). The conditions (19) and (20) show that in the case of higher-order flexibility there is a multiple zero at $t_0 = 0$.

2.3 Characterizing 2^{nd}- and 3^{rd}-order flexibility

According to Theorem 2 the equations (19) and (20) give necessary conditions for the framework $\mathbf{F}_\mathrm{b}$ being infinitesmally flexible of order n. In order to obtain geometric characterizations for $n = 1, 2, \ldots$ we must compare the coefficients of t^i, $i = 1, 2, \ldots$. For this purpose we set up the Taylor expansions as follows:

$$\begin{aligned} A(t) &= I_2 + A_1 t + A_2\, t^2 + \ldots & \mathbf{a}(t) &= \mathbf{a}_1 t + \mathbf{a}_2\, t^2 + \ldots \\ \gamma(t) &= \gamma_1 t + \gamma_2\, t^2 + \ldots & \mathbf{b}(t) &= \mathbf{b}_1 t + \mathbf{b}_2\, t^2 + \ldots \end{aligned}$$

The inverse matrix $A^{-1}(t)$ can be expanded in the form

$$A^{-1}(t) = I_2 + B_1 t + B_2\, t^2 + \ldots$$

with

$$\begin{aligned} B_1 &= -A_1 & B_2 &= -A_2 + A_1^2 \\ B_3 &= -A_3 + A_2 A_1 + A_1 A_2 - A_1^3 & B_k &= -A_k - A_{k-1} B_1 - \ \ldots\ - A_1 B_{k-1}\,. \end{aligned}$$

The coefficients of t in (19) and (20) give $f_1(\mathbf{p}_i) = f_1(\mathbf{q}_j) = 0$, $i, j = 1, 2, 3$, with

$$f_1(\mathbf{x}) := \mathbf{x}^T (A_1 + A_1^T)\mathbf{x} + 2(\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{x} - \gamma_1\,. \tag{22}$$

This gives *the curve c of second order mentioned in Theorem 1.* For the velocity vectors

$$\mathbf{p}_{i,1} = \mathbf{a}_1 + A_1 \mathbf{p}_i \quad \text{and} \quad \mathbf{q}_{j,1} = \mathbf{b}_1 - A_1^T \mathbf{q}_j$$

we can suppose a symmetric matrix A_1 and $\mathbf{b}_1 = -\mathbf{a}_1$. This is possible since otherwise we could superimpose a motion which appoints to each point $\mathbf{r}$ the instantaneous velocity vector

$$\mathbf{v}(\mathbf{r}) := -\tfrac{1}{2}(\mathbf{a}_1 + \mathbf{b}_1) - \tfrac{1}{2}(A_1 - A_1^T)\mathbf{r}$$

with the skew symmetric matrix $(A_1 - A_1^T)$. Then we would obtain the vectors

$$\mathbf{p}_{i,1} = \tfrac{1}{2}(\mathbf{a}_1 - \mathbf{b}_1) + \tfrac{1}{2}(A_1 + A_1^T)\mathbf{p}_i\,, \qquad \mathbf{q}_{j,1} = -\tfrac{1}{2}(\mathbf{a}_1 - \mathbf{b}_1) - \tfrac{1}{2}(A_1 + A_1^T)\mathbf{q}_j\,, \tag{23}$$

which still solve (5). These vectors indeed are orthogonal to c (see Fig. 3), as according to (22) the equation of the polar line of point $\mathbf{r}$ with respect to c reads

$$\mathbf{x}^T(A_1 + A_1^T)\mathbf{r} + \mathbf{x}^T(\mathbf{a}_1 - \mathbf{b}_1) + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{r} - \gamma_1 = 0\,.$$

The coefficients of t^2 in (19) and (20) give two equations: $f_2(\mathbf{p}_i) = 0\,,\ i = 1,2,3\,,$ with

$$\begin{aligned} f_2(\mathbf{x}) := \;& \mathbf{x}^T(A_2^T + A_1^T A_1 + A_2)\mathbf{x} + \\ & +2\left[(\mathbf{a}_1^T - \mathbf{b}_1^T)A_1 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\right]\mathbf{x} + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{a}_1 - \gamma_2\,, \end{aligned} \tag{24}$$

and $g_2(\mathbf{q}_j) = 0\,,\ j = 1,2,3\,,$ with

$$\begin{aligned} g_2(\mathbf{x}) := \;& \mathbf{x}^T(A_2 - A_1 A_1^T + A_2^T - A_1^2 - A_1^{T2})\mathbf{x} + \\ & +2\left[-(\mathbf{a}_1^T - \mathbf{b}_1^T)A_1^T + (\mathbf{a}_2^T - \mathbf{b}_2^T)\right]\mathbf{x} + (\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{b}_1 - \gamma_2\,. \end{aligned} \tag{25}$$

$f_2(\mathbf{x}) = 0$ and $g_2(\mathbf{x}) = 0$ represent two curves of second order, one passing through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, the other through $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. The difference of these equations

$$h_2(\mathbf{x}) := \mathbf{x}^T(A_1 + A_1^T)^2\,\mathbf{x} + 2(\mathbf{a}_1^T - \mathbf{b}_1^T)(A_1 + A_1^T)\mathbf{x} + (\mathbf{a}_1^T - \mathbf{b}_1^T)(\mathbf{a}_1 - \mathbf{b}_1) = 0 \tag{26}$$

depends only on the first derivatives $\mathbf{a}_1$, $\mathbf{b}_1$, A_1, but not γ_1. This difference equation represents a curve d in the pencil $[pq]$ spanned by p and q. For a nontrivial flex the polynomial $h_2(\mathbf{x})$ cannot vanish and it is neither proportional to $f_1(\mathbf{x})$, nor to $f_2(\mathbf{x})$ or $g_2(\mathbf{x})$ (note (27) and (28)).

In order to figure out the geometric relation between the curves d and c, we specify the coordinate system such that the equation (22) of c gets one of the forms listed in (11) and the velocity vectors $\mathbf{p}_{i,1}$ and $\mathbf{q}_{j,1}$ according to (23) equal those given in (12). This implies:

type of c	*reduced equation of c*	$A_1 = A_1^T$	$\mathbf{a}_1 = -\mathbf{b}_1$	γ_1
ellipse	$x^2/a^2 + y^2/b^2 = 1$	$\begin{pmatrix} b/a & 0 \\ 0 & a/b \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2ab$
hyperbola	$x^2/a^2 - y^2/b^2 = 1$	$\begin{pmatrix} b/a & 0 \\ 0 & -a/b \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2ab$
parabola	$y^2 - 4a^2x = 4a^4$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2a^2 \\ 0 \end{pmatrix}$	$8a^4$
intersecting lines	$-k^2x^2 + y^2 = 0$	$\begin{pmatrix} -k^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	0
parallel lines	$y^2 = k^2$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2k^2$

(27)

Then the equations of d and of a particular curve d' selected from the pencil $[cd]$ are

type of c	d: $h_2(\mathbf{x}) = 0$	*curve* d'
ellipse	$4b^2x^2/a^2 + 4a^2y^2/b^2 = 0$	$x^2 + y^2 = a^2 + b^2$ ... director circle of c
hyperbola	$4b^2x^2/a^2 - 4a^2y^2/b^2 = 0$	$(a^2 + b^2)x^2 = a^4$ ... parallel lines
parabola	$4y^2 + 16a^4 = 0$	$x = -2a^2$ ... director line & line at inf.
inters. lines	$4k^4x^2 + 4y^2 = 0$	$y^2 = 0$ ... symmetry axis as double line
parallel lines	$4y^2 = 0$	$y^2 = 0$ ... symmetry axis as double line

(28)

For a 2^{nd}-order flex of $\mathbf{F}_{\text{b}}$ obeying (22) it is necessary that there are curves p, q, passing through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ or $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, resp., such that the pencil $[pq]$ spanned by p and q contains d.

In the projective 5-space formed by all curves of second order those through $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ or $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ give two planes π_p, π_q, resp., which share the "point" c. The "line" (=pencil) $[pq]$ with $p, q \neq c$ passes through d if and only if d lies in the plane spanned by c, p and q (see Fig. 5).

Hence, for a 2^{nd}-order flexing $\mathbf{F}_{\text{b}}$ each pair of second-order curves (p', q') with $p' \in [cp] \setminus \{c\}$, $q' \in [cq] \setminus \{c\}$, spans a pencil $[p'q']$ which shares a curve d' with the pencil $[cd]$.

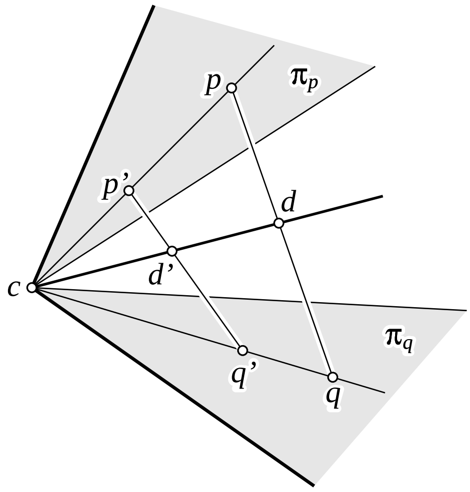


Figure 5: Looking at the projective space of second-order curves

Conversely, if such a pair (p', q') with $p', q' \neq c$ is given with p' passing through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and q' passing through $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (see examples in Fig. 6), then we can find any $p \in [cp']$ and $q \in [cq']$ such that the difference of their equations gives $h_2(\mathbf{x}) = 0$ in (26). Then from the equations of c, p and q one gets $A_2 + A_2^T$ and $(\mathbf{a}_2^T - \mathbf{b}_2^T)$.

Again, for the acceleration vectors $2\mathbf{p}_{i,2}$ and $2\mathbf{q}_{j,2}$ we can suppose a symmetric matrix A_2 and $\mathbf{a}_2 = -\mathbf{b}_2$ in

$$\mathbf{p}_{i,2} = \mathbf{a}_2 + A_2\mathbf{p}_i \quad \text{and} \quad \mathbf{q}_{j,2} = \mathbf{b}_2 - (A_2^T - A_1^{T2})\mathbf{q}_j\,,$$

because otherwise — after superimposing a suitable motion — we would obtain

$$\mathbf{p}_{i,2} = \tfrac{1}{2}(\mathbf{a}_2 - \mathbf{b}_2) + \tfrac{1}{2}(A_2 + A_2^T)\mathbf{p}_i\,, \quad \mathbf{q}_{j,2} = -\tfrac{1}{2}(\mathbf{a}_2 - \mathbf{b}_2) - \tfrac{1}{2}(A_2 + A_2^T - 2A_1^{T2})\mathbf{q}_j$$

as another solution of (6).

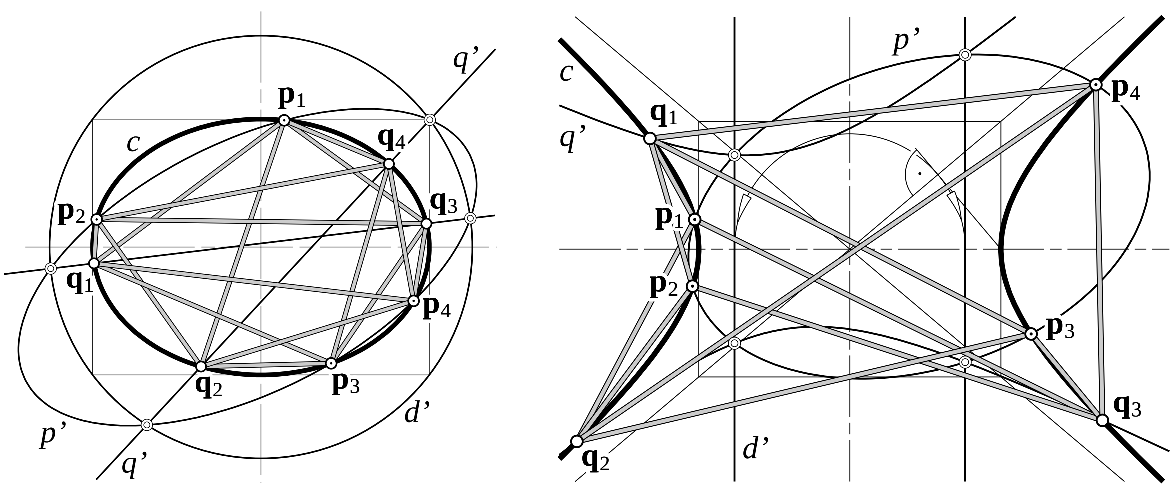


Figure 6: Two second-order flexing bipartite frameworks

We demonstrate this method at the following

<u>*Example:*</u> Let c be a pair of intersecting lines with $k = 1$ according to (27), i.e.,

$$c\colon\ -x^2 + y^2 = 0 \quad\Longrightarrow\quad d\colon\ 4(x^2 + y^2) = 0\,.$$

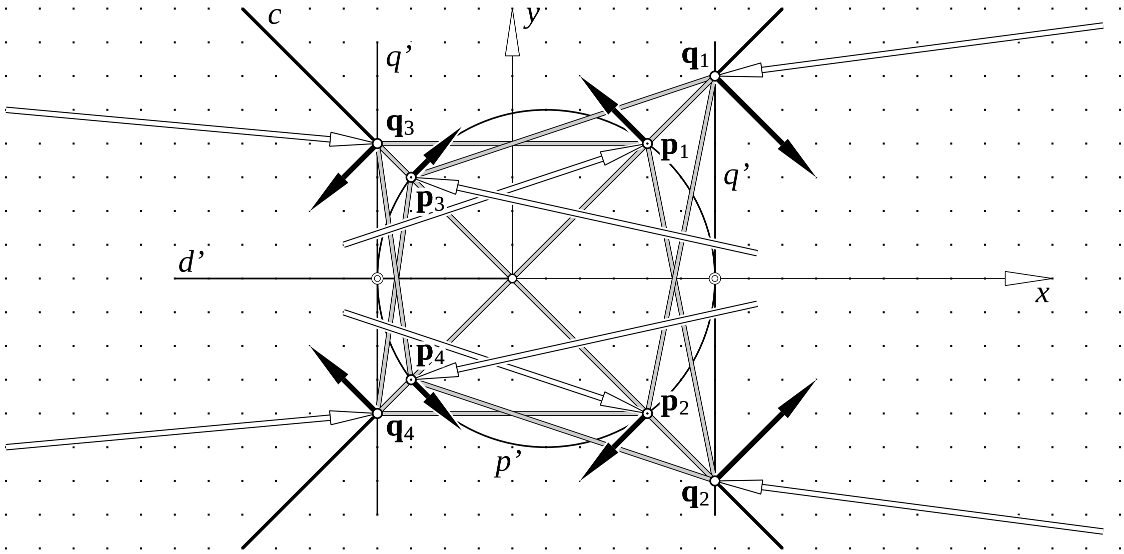


Figure 7: Computed example of a 2^{nd}-order flexible bipartite framework showing the velocity vectors $\mathbf{p}_{i,1}$, $\mathbf{q}_{j,1}$ with filled arrows (scaling factor 0.5) and the acceleration vectors $2\mathbf{p}_{i,2}$, $2\mathbf{q}_{j,2}$ with white arrows (scaling factor 0.25)

The circle p' and the pair q' of parallel lines (see Fig. 7) with equations

$$p'\colon\ x^2 - 2x + y^2 - 24 = 0 \quad \text{and} \quad q'\colon\ x^2 - 2x - 24 = 0$$

span a pencil which contains the double line $d'\colon y^2 = 0$. Now we replace p' by $p \in [cp']$ and q' by $q \in [cq']$ (we may choose $q' = q$) such that the difference of the equations gives exactly the equation (26) of d. We specify

$$p\colon\ 12x^2 - 16x + 4y^2 - 192 = 0 \quad \text{and} \quad q = q'\colon\ 8x^2 - 16x - 192 = 0\,.$$

The comparison of these equations with (24) and (25) together with (27) gives

$$A_1 = A_1^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{a}_1 = -\mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \gamma_1 = 0\,,$$

$$A_2 + A_2^T = \begin{pmatrix} 11 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \mathbf{a}_2 - \mathbf{b}_1 = \begin{pmatrix} -8 \\ 0 \end{pmatrix}, \qquad \gamma_2 = 192\,.$$

For the eight points $\mathbf{p}_i$ and $\mathbf{q}_j$ of intersection between c and p' or q', resp., we get the following velocity and acceleration vectors obeying the linear systems (5) and (6) of 16 equations, each:

vertex	*velocity vector*	*half acceleration vector*
$\mathbf{p}_1^T = (4,4)$	$\mathbf{p}_{1,1}^T = (-4,4)$	$\mathbf{p}_{1,2}^T = (18,6)$
$\mathbf{p}_2^T = (4,-4)$	$\mathbf{p}_{2,1}^T = (-4,-4)$	$\mathbf{p}_{2,2}^T = (18,-6)$
$\mathbf{p}_3^T = (-3,3)$	$\mathbf{p}_{3,1}^T = (3,3)$	$\mathbf{p}_{3,2}^T = (-20.5,4.5)$
$\mathbf{p}_4^T = (-3,-3)$	$\mathbf{p}_{4,1}^T = (3,-3)$	$\mathbf{p}_{4,2}^T = (-20.5,-4.5)$
$\mathbf{q}_1^T = (6,6)$	$\mathbf{q}_{1,1}^T = (6,-6)$	$\mathbf{q}_{1,2}^T = (-23,-3)$
$\mathbf{q}_2^T = (6,-6)$	$\mathbf{q}_{2,1}^T = (6,6)$	$\mathbf{q}_{2,2}^T = (-23,3)$
$\mathbf{q}_3^T = (-4,4)$	$\mathbf{q}_{3,1}^T = (-4,-4)$	$\mathbf{q}_{3,2}^T = (22,-2)$
$\mathbf{q}_4^T = (-4,-4)$	$\mathbf{q}_{4,1}^T = (-4,4)$	$\mathbf{q}_{4,2}^T = (22,2)$

We summarize:

Theorem 3: *A 1^{st}-order flexible framework $\mathbf{F}_\text{b}$ with non-collinear triples $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of vertices and $\mathbf{p}_1, \ldots, \mathbf{q}_3 \in c$ is infinitesimally flexible of order two if and only if there are second-order curves p' through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and q' through $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ such that the pencil spanned by p' and q' shares a curve d' with the pencil spanned by c and the associated d which is listed in* (28) *together with a possible curve d'.*
As a consequence, a second-order flexible framework $\mathbf{F}_\text{b}$ can in general be extended by two additional vertices $\mathbf{p}_4, \mathbf{q}_4$ without restricting the infinitesimal flexibility.

Remark: Second-order flexibility is related to curvatures (cf. Stachel (1999)). In the sense of Fig. 2 there is also a more kinematic characterization of 2^{nd}-order flexibility of $\mathbf{F}_\text{b}$: A 1^{st}-order flexing framework $\mathbf{F}_\text{b}$ is flexible of order two if and only if the *curvature transformation induced by the coupler motion Σ_3/Σ_1 is among the curvature transformations of one-parameter motions included in the two-parametric motion $\mathcal{B}$ composed from the coupler motions Σ_2/Σ_1 and Σ_3/Σ_2.* According to the results of Stachel (1979) one has to pay attention to the parabolic projectivities π_{ij} induced by the curvature transformations of Σ_i/Σ_j on the instantaneous pole-axis a of $\mathcal{B}$. On this axis a, which is the line through the relative poles 21, 31, 32 (see Fig. 2), there is a projectivity $\sigma : 32 \mapsto 21$ obeying $\sigma^2 = \pi_{21} \circ \pi_{32}$. Now the exact, but not very practicable characterization reads:
The 1^{st}-order flexing framework $\mathbf{F}_\text{b}$ is flexible of order two if and only if the product $\pi_{31}^{-1} \circ \sigma$ is involutory. And this is equivalent to the statement that the parabolic projectivity π_{31} with fixed point 31 maps the preimage $\sigma^{-1}(31)$ of 31 under σ onto the image $\sigma(31)$.

The coefficients of t^3 in (19) and (20) give rise to the equations $f_3(\mathbf{p}_i) = 0$ and $g_3(\mathbf{q}_j) = 0$ for all $i, j = 1, 2, 3$, with quadratic functions

$$\begin{aligned} f_3(\mathbf{x}) := \; & \mathbf{x}^T(A_3^T + A_2^T A_1 + A_1^T A_2 + A_3)\mathbf{x} + 2\left[(\mathbf{a}_1^T - \mathbf{b}_1^T)A_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)A_1 + \right. \\ & \left. +(\mathbf{a}_3^T - \mathbf{b}_3^T)\right]\mathbf{x} + \left[(\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{a}_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\mathbf{a}_1\right] - \gamma_3 \end{aligned} \tag{29}$$

$$\begin{aligned} g_3(\mathbf{x}) := \; & \mathbf{x}^T(A_3 - A_2A_1 - A_1A_2 + A_1^3 - A_2A_1^T + A_1^2A_1^T - A_1A_2^T + A_1A_1^{T2} + \\ & +A_3^T - A_1^TA_2^T - A_2^TA_1^T + A_1^{T3})\mathbf{x} + 2\left[(\mathbf{a}_1^T - \mathbf{b}_1^T)(-A_2^T + A_1^{T2}) - \right. \\ & \left. -(\mathbf{a}_2^T - \mathbf{b}_2^T)A_1^T + (\mathbf{a}_3^T - \mathbf{b}_3^T)\right]\mathbf{x} + \left[(\mathbf{a}_1^T - \mathbf{b}_1^T)\mathbf{b}_2 + (\mathbf{a}_2^T - \mathbf{b}_2^T)\mathbf{b}_1\right] - \gamma_3 \end{aligned} \tag{30}$$

Theorem 4: *A bipartite framework $\mathbf{F}_\text{b}$ with non-collinear triples $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of vertices is infinitesimally flexible of order three if and only if there are vectors $\mathbf{a}_1, \ldots, \mathbf{b}_3$ and matrices A_1, A_2, A_3 and real numbers $\gamma_1, \gamma_2, \gamma_3$ such that the vertices $\mathbf{p}_i$ obey the equations $f_1(\mathbf{x}) = f_2(\mathbf{x}) = f_3(\mathbf{x}) = 0$ and $\mathbf{q}_j$ obey $f_1(\mathbf{x}) = g_2(\mathbf{x}) = g_3(\mathbf{x}) = 0$ according to* (22), (24), (29), (25), *and* (30).

A geometric interpretation of this condition is still open. This together with the investigation of orders ≥ 4 and the discussion of the remaining cases with collinear $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ or $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is left for future publications.

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