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CONFIGURATION THEOREMS ON BIPARTITE FRAMEWORKS

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Dedicated to Peter Gruber at the occasion of his 60th birthday

This paper discusses the relative position between two incongruent configurations of a complete bipartite framework with given edge lengths in the Euclidean n-space. It is proved that there is an appropriate displacement of one configuration such that in the generic case the two locations of each knot become corresponding points of two confocal quadrics. Then the equal lengths are a direct consequence of Ivory's theorem. Modifications of the first configuration theorem cover the special cases where one or both classes of knots are located in a hyperplane.

Introduction

There are many examples showing that generically rigid frameworks can become flexible when the knots are somehow related to second order surfaces (see e.g. [4, 5, 10, 11, 8, 9]). It is well known that a bipartite framework is infinitesimally flexible of first order if and only if the knots are located on a quadric (see e.g. [18, 12, 8]). Also the infinitesimally flexible cross-polytopes are in a certain way linked to quadrics ([13]). The following theorems reveal one reason for this connection between flexibility and quadrics.

The first theorem treats the generic case in the Euclidean n-space $\mathbb{E}^n$: Two incongruent configurations of a complete bipartite framework with sufficiently many knots and given edge lengths are always associated to two confocal quadrics such that the equal distances in both configurations result from Ivory's theorem. In the second theorem the knots of one class are coplanar. The third theorem characterizes two configurations in the particular case where the two classes of knots are located in two perpendicular hyperplanes.

These theorems generalize results presented in [10] for $\mathbb{E}^3$. This time the necessary and sufficient conditions for the configuration theorems are formulated more precisely.

[1] *Parole chiave:* rigidity, flexibility, framework, bipartite graph, Ivory's theorem
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And the new proofs reveal that Ivory's theorem is intimately related to selfadjoint affine transformations.

The results of [10] have originally been used to characterize singular positions in satellite geodesy (compare also [17]). Later it turned out that they were of basic importance for proving the flexibility of structures. E.g., it was a basis for a new proof of the uniqueness of Bricard's flexible octahedra [11]. One can expect that the new theorems enable to prove some new results on the flexibility of geometric structures, e.g. of polytopes (cf. [1]). They might offer a chance to prove or disprove the existence, e.g., of flexible suspensions in $\mathbb{E}^3$ with a pentagonal equator or of flexible cross-polytopes in higher dimensions (see [14] and references there).

These configuration theorems are also true in the spherical n-space. There are good reasons to conjecture that they also hold in the Minkowski-n-space and in hyperbolic spaces.

The first configuration theorem

THEOREM 1: *Let $\mathcal{F}_0$ be a complete bipartite framework of type $\mathcal{K}_{n+1,q+1}$ with given bar lenghts l_{ik}. Suppose this framework admits two incongruent configurations $\mathcal{F}$ and $\mathcal{F}'$ in the Euclidean n-space $\mathbb{E}^n$. This means that the knots $X_0,\dots,X_n,Y_0,\dots,Y_q$ of $\mathcal{F}$ and $X'_0,\dots,X'_n,Y'_0,\dots,Y'_q$ of $\mathcal{F}'$ obey the $(n+1)(q+1)$ quadratic equations*

$$l_{ik}^2 = \overline{X'_iY'_k}^2 = \overline{X_iY_k}^2 \text{ for all } i\in\{0,\dots,n\} \text{ and } k\in\{0,\dots,q\}. \tag{1}$$

The knots $X_0,\dots,X_n$ are supposed to form a simplex.

1. *There is an appropriate displacement $\beta\colon \mathbb{E}^n\to\mathbb{E}^n$ such that for $\mathcal{F}$ and the displaced $\beta(\mathcal{F}')$ the following statement holds:*
   For all i and k the knots $X_i\mapsto\beta(X'_i)$ and $\beta(Y'_k)\mapsto Y_k$ are corresponding points of two confocal[2] surfaces Φ, Ψ of second order (see Fig. 1).
2. *For any $r,s\in\mathbb{N}$ the framework $\mathcal{F}_0$ can be extended to a complete bipartite framework $\widetilde{\mathcal{F}}$ of type $\mathcal{K}_{n+r+1,q+s+1}$ which still admits two incongruent configurations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ with knots $X_0,\dots,X_{n+r}\in\Phi$ and $Y_0,\dots,Y_{q+s}\in\Psi$.*
3. *For knots of the second class this is the only choice for extending $\mathcal{F}_0$, i.e., for any pair of points Y, Y' the equations*
   $$\overline{X'_iY'} = \overline{X_iY} \text{ for all } i\in\{0,\dots,n\}$$
   imply $Y'\in\beta^{-1}(\Phi)$.
4. *The analogous statement that*
   $$\overline{X'Y'_k} = \overline{XY_k} \text{ for all } k\in\{0,\dots,q\}$$
   implies $X\in\Phi$ is not true. It holds for $q\geq n$ under the condition that the point set $Y'_0,\dots,Y'_q$ contains a simplex.

[2]The terms *corresponding points* and *confocal* 2^{nd}*-order surfaces* in $\mathbb{E}^n$ are used according to [16] and they will be explained in detail in the following proof. The surface Ψ can be singular.

5. *Two different knots X_i, X_j of $\mathcal{F}$ share their distance with the corresponding points of X_i', X_j' of $\mathcal{F}'$ if and only if the spanned line $[X_i\, X_j]$ is subset of the second-order surface Φ, i.e.,*
$$\overline{X_iX_j} = \overline{X_i'X_j'} \iff [X_i\, X_j] \subset \Phi\,.$$
Conversely, for any $Y_i', Y_j' \in \mathcal{F}'$ and the corresponding points $Y_i, Y_j \in \mathcal{F}$ we have
$$\overline{Y_i'Y_j'} = \overline{Y_iY_j} \iff [Y_i'\, Y_j'] \subset \beta^{-1}(\Phi).$$

An immediate consequence of Theorem 1 for the flexibility of bipartite frameworks reads:

COROLLARY 1: *A complete bipartite framework of type $\mathcal{K}_{p,q}$ in $\mathbb{E}^n$ with knots $X_1, \ldots, X_p$ spanning $\mathbb{E}^n$ admits a continuous flexion if and only if there is an at least one-parametric set of quadrics Φ passing through $X_1, \ldots, X_p$ such that a confocal quadric Ψ contains simultaneously all $Y_1, \ldots, Y_q$.*

Specify for example the X-knots as the vertices $(\pm c_{i,1}, \ldots, \pm c_{i,n})$, $i = 1, \ldots, n-1$, of $n-1$ boxes symmetric with respect to the axes of a cartesian coordinate system in $\mathbb{E}^n$. And select the Y-knots as the vertices of another such box. Then the framework of type $\mathcal{K}_{(n-1)2^n, 2^n}$ is continuously flexible. It generalizes one of Dixon's mechanisms (compare [5, 8, 10]).

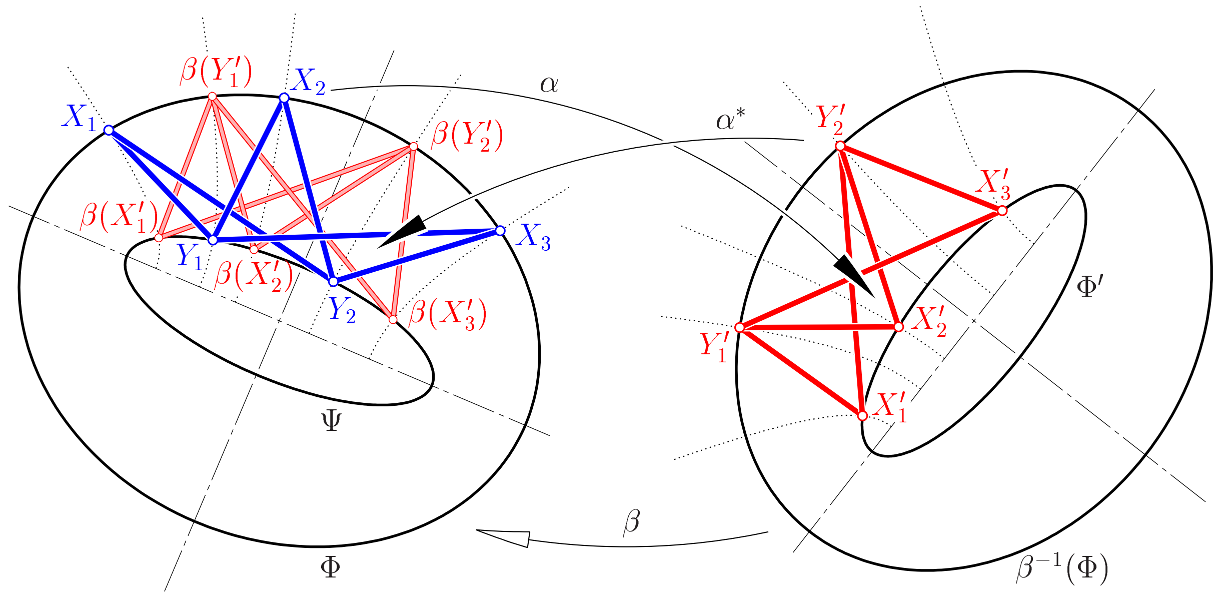

Figure 1: The statement of Theorem 1 for dimension $n = 2$

Proof of Theorem 1

We use two cartesian coordinate systems $\mathcal{S}$ and $\mathcal{S}'$ in $\mathbb{E}^n$, one for each configuration. Let $\mathsf{x}_i, \mathsf{y}_k$ denote the coordinate vectors of the knots X_i, Y_k with respect to $\mathcal{S}$, and $\mathsf{x}_i', \mathsf{y}_k'$ those of X_i', Y_k' with respect to $\mathcal{S}'$:[3] Then the equations (1) imply

$$(\mathsf{x}_i - \mathsf{y}_k)^2 = (\mathsf{x}_i' - \mathsf{y}_k')^2 \text{ for all } i \in \{0, \dots, n\} \text{ and } k \in \{0, \dots, q\}. \tag{2}$$

After subtracting the equations for the indices $(0, k)$ and (i, k) we obtain

$$\mathsf{x}_0^2 - \mathsf{x}_i^2 + 2(\mathsf{x}_i - \mathsf{x}_0)\cdot\mathsf{y}_k = \mathsf{x}_0'^2 - \mathsf{x}_i'^2 + 2(\mathsf{x}_i' - \mathsf{x}_0')\cdot\mathsf{y}_k' . \tag{3}$$

For each k this can be seen as a fixed system of n linear equations for the unknown vector y_k. As $X_0, \dots, X_n$ is supposed to form a simplex, this system has a unique solution for any given y_k'. In order to express this solution in an appropriate form we take into account that there is a unique affine transformation

$$\alpha\colon\ \mathbb{E}^n \to \mathbb{E}^n,\ \mathsf{x} \mapsto \alpha(\mathsf{x}) = \mathsf{a}' + l(\mathsf{x}) \quad \text{with } \mathsf{x}_i \mapsto \mathsf{x}_i' \text{ for all } i \in \{0, \dots, n\}. \tag{4}$$

Here l denotes the induced linear map defined by $l(\mathsf{x}_i - \mathsf{x}_0) = \mathsf{x}_i' - \mathsf{x}_0'$, $i = 1, \dots, n$. Let $l^*\colon \mathbb{R}^n \to \mathbb{R}^n$ be the adjoint map obeying

$$l(\mathsf{u})\cdot\mathsf{v}' = \mathsf{u}\cdot l^*(\mathsf{v}') \quad \text{for all} \quad \mathsf{u}, \mathsf{v}' \in \mathbb{R}^n. \tag{5}$$

In the case $q \geq 1$ we subtract from (3) the equation for $k = 0$ and get

$$(\mathsf{x}_i - \mathsf{x}_0)\cdot(\mathsf{y}_k - \mathsf{y}_0) = l(\mathsf{x}_i - \mathsf{x}_0)\cdot(\mathsf{y}_k' - \mathsf{y}_0') \text{ for all } i \in \{1, \dots, n\} \text{ and } k \in \{1, \dots, q\}. \tag{6}$$

Then we fulfill all equations (6) by setting

$$\mathsf{y}_k - \mathsf{y}_0 = l^*(\mathsf{y}_k' - \mathsf{y}_0') \text{ for all } k = 1, \dots, q.$$

This adjoint map l^* together with one pair $\mathsf{y}_0' \mapsto \mathsf{y}_0$ of points defines the affine transformation

$$\alpha^*\colon\ \mathbb{E}^n \to \mathbb{E}^n,\ \mathsf{y}' \mapsto \alpha^*(\mathsf{y}') = \mathsf{b} + l^*(\mathsf{y}') \quad \text{with } \mathsf{y}_k' \mapsto \mathsf{y}_k \text{ for all } k \in \{0, \dots, q\}. \tag{7}$$

We call α and α^* *"adjoint"* affine transformations. Thus we can formulate the following necessary condition:

LEMMA 1: *Let $\mathcal{F}$, $\mathcal{F}'$ in $\mathbb{E}^n$ be two incongruent configurations of $\mathcal{F}_0$ as defined in Theorem 1. Then there are two adjoint affine transformations α, α^* with*

$$\alpha\colon X_i \mapsto X_i' \textit{ for all } i = 0, \dots, n, \quad \textit{and } \alpha^*\colon Y_k \mapsto Y_k' \textit{ for all } k = 0, \dots, q.$$

Neither α nor α^* is an isometry.

Proof of the last statement in Lemma 1: Suppose α is an isometry. Then we specify the coordinate system $\mathcal{S}'$ such that $\mathsf{x}_i' = \mathsf{x}_i$ holds for each $i = 0, \dots, n$. Eq. (3) gives rise to a homogeneous system of linear equations

$$(\mathsf{x}_i - \mathsf{x}_0)\cdot(\mathsf{y}_k - \mathsf{y}_k') = 0 \text{ for each } k \in \{0, \dots, q\}.$$

which admits only the trivial solution $\mathsf{y}_k' = \mathsf{y}_k$. The two configurations $\mathcal{F}$, $\mathcal{F}'$ are congruent — thus contradicting the initial assumption. □

[3]For the sake of brevity we identify points with their coordinate vectors.

Now we replace in (2) x_i' by $\alpha(\mathsf{x}_i)$ and y_k by $\alpha^*(\mathsf{y}_k')$ and obtain

$$\mathsf{x}_i^2 - 2\mathsf{x}_i\cdot[\mathsf{b} + l^*(\mathsf{y}_k')] + [\mathsf{b} + l^*(\mathsf{y}_k')]^2 = [\mathsf{a}' + l(\mathsf{x}_i)]^2 - 2[\mathsf{a}' + l(\mathsf{x}_i)]\cdot\mathsf{y}_k' + \mathsf{y}_k'^2$$

or — because of $\mathsf{x}_i\cdot l^*(\mathsf{y}_k') = l(\mathsf{x}_i)\cdot\mathsf{y}_k'$ due to (5)

$$\mathsf{x}_i^2 - l(\mathsf{x}_i)^2 - 2\mathsf{x}_i\cdot\mathsf{b} - 2l(\mathsf{x}_i)\cdot\mathsf{a}' + \mathsf{b}^2 = \mathsf{y}_k'^2 - l^*(\mathsf{y}_k')^2 - 2\mathsf{y}_k'\cdot\mathsf{a}' - 2l^*(\mathsf{y}_k')\cdot\mathsf{b} + \mathsf{a}'^2.$$

This equation holds for all $i \in \{0,\dots,n\}$ and all $k \in \{0,\dots,q\}$. As the left side depends on i, the right side on k only, both sides must equal any constant C. This results in two quadratic functions

$$(8)\qquad \begin{aligned} f(\mathsf{x}) &:= \mathsf{x}^2 - l(\mathsf{x})^2 - 2\mathsf{x}\cdot[\mathsf{b} + l^*(\mathsf{a}')] + \mathsf{b}^2 - C, \\ g'(\mathsf{y}') &:= \mathsf{y}'^2 - l^*(\mathsf{y}')^2 - 2\mathsf{y}'\cdot[\mathsf{a}' + l(\mathsf{b})] + \mathsf{a}'^2 - C \end{aligned} \quad \text{and} \quad \begin{aligned} f(\mathsf{x}_i) &= 0 \ \ \forall i = 0,\dots,n\,, \\ g'(\mathsf{y}_k') &= 0 \ \ \forall k = 0,\dots,q. \end{aligned}$$

Conversely, for all $\mathsf{x},\mathsf{y}' \in \mathbb{R}^n$ the equations $f(\mathsf{x}) = g'(\mathsf{y}') = 0$ imply

$$\|\mathsf{x} - \alpha^*(\mathsf{y}')\| = \|\alpha(\mathsf{x}) - \mathsf{y}'\|.$$

We will see in the sequel that this is exactly Ivory's theorem for corresponding points of confocal surfaces in $\mathbb{E}^n$ (see [16]). We summarize in

LEMMA 2: *Let α and α^* be adjoint affine transformations according to* (4) *and* (7). *For any $\mathsf{x}_0,\dots,\mathsf{x}_p,\mathsf{y}_0',\dots,\mathsf{y}_q' \in \mathbb{R}^n$ the points*

$$\mathsf{x}_0,\dots,\mathsf{x}_p,\alpha^*(\mathsf{y}_0'),\dots,\alpha^*(\mathsf{y}_q') \ \text{ and } \ \alpha(\mathsf{x}_0),\dots,\alpha(\mathsf{x}_p),\mathsf{y}_0',\dots,\mathsf{y}_q' \ \text{ in } \ \mathbb{E}^n$$

constitute two configurations of a complete bipartite framework of type $\mathcal{K}_{p+1,\,q+1}$, i.e. with

$$\|\mathsf{x}_i - \alpha^*(\mathsf{y}_k')\| = \|\alpha(\mathsf{x}_i) - \mathsf{y}_k'\|,$$

if and only if there is a constant C such that for f and g' according to (8) *the following equations hold true:*

$$f(\mathsf{x}_i) = 0 \ \ \forall i = 0,\dots,p \ \text{ and } \ g'(\mathsf{y}_k') = 0 \ \ \forall k = 0,\dots,q.$$

We are now going to specify the coordinate systems $\mathcal{S}$ and $\mathcal{S}'$: It is well known that the composite mapping $l^*\circ l\colon \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint as

$$\mathsf{u}\cdot[l^*\circ l(\mathsf{v})] = l(\mathsf{u})\cdot l(\mathsf{v}) = [l^*\circ l(\mathsf{u})]\cdot\mathsf{v} \ \text{ for all } \ \mathsf{u},\mathsf{v} \in \mathbb{R}^n.$$

There is an orthonormal basis of eigenvectors $\mathsf{e}_1,\dots,\mathsf{e}_n$ obeying

$$l^*\circ l(\mathsf{e}_i) = \lambda_i^2\mathsf{e}_i \ \text{ with } \ \lambda_i^2 = l(\mathsf{e}_i)\cdot l(\mathsf{e}_i) \ge 0,$$

because of

$$(9)\qquad l(\mathsf{e}_i)\cdot l(\mathsf{e}_j) = [l^*\circ l(\mathsf{e}_i)]\cdot\mathsf{e}_j = \lambda_i^2(\mathsf{e}_i\cdot\mathsf{e}_j) = \lambda_i^2\delta_{ij}.$$

We assume

(10) $$\lambda_1 = \cdots = \lambda_s = 0, \quad \lambda_{s+1}, \ldots, \lambda_r \neq 0, 1, \quad \lambda_{r+1} = \cdots = \lambda_n = 1 \\ \text{with } 0 \le s \le r \le n \text{ and } r \ge 1$$

because of $l^* \circ l \neq \mathrm{id}_{\mathbb{R}^n}$ according to Lemma 1.

The eigenvectors $\mathsf{e}_1, \ldots, \mathsf{e}_n$ have pairwise orthogonal images $l(\mathsf{e}_1), \ldots, l(\mathsf{e}_n)$ due to (9). Hence there is also an orthonormal basis $\mathsf{e}'_1, \ldots, \mathsf{e}'_n$ such that

(11) $$l(\mathsf{e}_i) = \lambda_i \mathsf{e}'_i \text{ and } l^*(\mathsf{e}'_k) = \lambda_k \mathsf{e}_k \text{ for all } i, k \in \{1, \ldots, n\}.$$

We use $\mathsf{e}_1, \ldots, \mathsf{e}_n$ and $\mathsf{e}'_1, \ldots, \mathsf{e}'_n$ as the basis vectors of $\mathcal{S}$ and $\mathcal{S}'$, respectively. For an appropriate choice of the origin there are two cases to distinguish:

CASE 1: The composite affine transformation $\alpha^* \circ \alpha$ keeps a point f fixed: Then we choose f as the origin of $\mathcal{S}$ and $\alpha(\mathsf{f})$ as the origin of $\mathcal{S}'$. Due to (11) we obtain

$$\mathsf{x} = \mathsf{f} + \sum_{i=1}^n x_i \mathsf{e}_i \implies \alpha(\mathsf{x}) = \alpha(\mathsf{f}) + \sum_{i=1}^n \lambda_i x_i \mathsf{e}'_i,$$
$$\mathsf{y}' = \alpha(\mathsf{f}) + \sum_{k=1}^n y'_k \mathsf{e}'_k \implies \alpha^*(\mathsf{y}') = \mathsf{f} + \sum_{k=1}^n \lambda_k y'_k \mathsf{e}_k,$$

and therefore $\mathsf{a}' = \mathsf{o}$ in (4) and $\mathsf{b} = \mathsf{o}$ in (7). Hence because of (10) the coordinate representations of the quadratic functions in (8) read

(12) $$f(\mathsf{x}) = x_1^2 + \cdots + x_s^2 + (1 - \lambda_{s+1}^2) x_{s+1}^2 + \cdots + (1 - \lambda_r^2) x_r^2 - C = g'(\mathsf{x}).$$

The equation $f(\mathsf{x}) = 0$ defines a surface Φ of second order. Its affine image $\Phi' := \alpha(\Phi)$ obeys

(13) $$\begin{aligned} &\text{for } s = 0: && f'(\mathsf{x}') := \tfrac{1-\lambda_1^2}{\lambda_1^2} x_1'^2 + \cdots + \tfrac{1-\lambda_r^2}{\lambda_r^2} x_r'^2 - C = 0, \\ &\text{for } s > 0: && x'_1 = \cdots = x'_s = 0 \text{ and} \\ & && f'(\mathsf{x}') := \tfrac{1-\lambda_{s+1}^2}{\lambda_{s+1}^2} x_{s+1}'^2 + \cdots + \tfrac{1-\lambda_r^2}{\lambda_r^2} x_r'^2 - C \le 0. \end{aligned}$$

In the "singular" case $s > 0$ we use the symbol $\partial\Phi'$ for the boundary of $\Phi' = \alpha(\Phi)$, i.e., for the set

$$\partial\Phi' = \{\mathsf{x}' = (0, \ldots, 0, x'_{s+1}, \ldots, x'_n) \mid f'(\mathsf{x}') = 0\}.$$

Now we displace the second configuration such that the associated coordinate system $\mathcal{S}'$ coincides with $\mathcal{S}$. This isometry is denoted by β (see Fig. 1). Then because of

$$\frac{\lambda_i^2}{1 - \lambda_i^2} = \frac{1}{1 - \lambda_i^2} - 1$$

the second-order surface Φ and the displaced affine image $\Psi := \beta \circ \alpha(\Phi)$ or (in case $s > 0$) $\partial\Psi$ are *confocal* in $\mathbb{E}^n$ (compare [16])[4].

These confocal surfaces have e.g. the following properties:

[4] As an example for the singular case in $\mathbb{E}^3$ take an ellipsoid Φ and its focal ellipse $\partial\Psi$. The affine image $\beta \circ \alpha(\Phi)$ is the elliptic disc bounded by $\partial\Psi$.

- Under $C \neq 0$ the sections of Φ and Ψ with any principal 2-plane spanned by e_i and e_j for $s < i < j \leq r$ are confocal conics

$$\frac{x_i^2}{\frac{1}{1-\lambda_i^2}} + \frac{x_j^2}{\frac{1}{1-\lambda_j^2}} = C \;\text{ and }\; \frac{x_i^2}{\frac{1}{1-\lambda_i^2} - 1} + \frac{x_j^2}{\frac{1}{1-\lambda_j^2} - 1} = C$$

of the same type. For $i \leq s$ the second conic degenerates.

- For $s = 0$ the intersections of Φ, Ψ with the space spanned by the coordinate axes $x_1, \dots, x_r$ are regular second-order surfaces. If they are seen as quadratic sets of tangential hyperplanes, they span a one-parametric linear system together with the set of isotropic hyperplanes. This results from the fact that the tangential equations of these quadrics in homogeneous hyperplane coordinates $(\xi_0, \dots, \xi_r)$ obey

$$\left(\tfrac{1}{1-\lambda_1^2}\xi_1^2 + \cdots + \tfrac{1}{1-\lambda_r^2}\xi_r^2 - \tfrac{1}{C}\xi_0^2\right) - \left(\xi_1^2 + \cdots + \xi_r^2\right) = \left(\tfrac{\lambda_1^2}{1-\lambda_1^2}\xi_1^2 + \cdots + \tfrac{\lambda_r^2}{1-\lambda_r^2}\xi_r^2 - \tfrac{1}{C}\xi_0^2\right).$$

Hence Φ and Ψ share the isotropic tangential hyperplanes. This holds also in the singular case $s \geq 1$ for Φ and $\partial\Psi$ in the space spanned by $x_{s+1}, \dots, x_r$.

- Points $\mathsf{x} \in \Phi$ and their images $\beta\circ\alpha(\mathsf{x}) \in \Psi$ are called *corresponding* in the sense of [16] (see $X_i, \beta(X_i')$ in Fig. 1). Whenever another surface confocal to Φ passes through x, then it passes also through the image $\beta\circ\alpha(\mathsf{x})$.[5] This can be proved as follows: The confocal set is given by

$$F(\mathsf{x}, t) := \frac{x_1^2}{\frac{1}{1-\lambda_1^2} + t} + \cdots + \frac{x_r^2}{\frac{1}{1-\lambda_r^2} + t} - C = 0, \quad t \in \mathbb{R} \setminus \left\{\tfrac{-1}{1-\lambda_1^2}, \dots, \tfrac{-1}{1-\lambda_r^2}\right\}.$$

From $F(\mathsf{x}, 0) = f(\mathsf{x}) = 0$ and $F(\mathsf{x}, t) = 0$ with $t \neq 0$ we conclude $F(\alpha(\mathsf{x}), t) = 0$ as the latter can be expressed as the affine combination

$$F(\alpha(\mathsf{x}), t) = -\tfrac{1}{t}\, F(\mathsf{x}, 0) + \left(1 + \tfrac{1}{t}\right) F(\mathsf{x}, t).$$

This is a consequence of

$$\frac{\lambda^2}{\frac{1}{1-\lambda^2} + t} = -\frac{1}{t}(1 - \lambda^2) + \left(1 + \frac{1}{t}\right) \frac{1}{\frac{1}{1-\lambda^2} + t}.$$

CASE 2: There is *no fixed point* of

$$\alpha^*\circ\alpha\colon\ \mathsf{x} \;\mapsto\; \mathsf{b} + l^*(\mathsf{a}' + l(\mathsf{x})) = \mathsf{b} + l^*(\mathsf{a}') + l^*\circ l(\mathsf{x}).$$

This means that the system of linear equations

$$(l^*\circ l - \mathrm{id}_{\mathbb{R}^n})(\mathsf{x}) = -\mathsf{b} - l^*(\mathsf{a}')$$

has no solution, i.e., the rank $\mathrm{rk}(l^*\circ l - \mathrm{id}_{\mathbb{R}^n}) = r$ is smaller than n, hence 1 is an eigenvalue of $l^*\circ l$, and

$$\mathsf{d} := -\mathsf{b} - l^*(\mathsf{a}') \notin (l^*\circ l - \mathrm{id}_{\mathbb{R}^n})(\mathbb{R}^n) = [\mathsf{e}_1, \dots, \mathsf{e}_r].$$

[5]See in Fig. 1 the dotted hyperbolas passing simultaneously through X_i and $\beta(X_i')$.

We decompose

$$\mathsf{d} = \mathsf{d}_0 + \mathsf{d}_1 \quad \text{such that} \quad \mathsf{d}_0 \in [\mathsf{e}_1, \dots, \mathsf{e}_r] \ \text{and} \ \mathsf{d}_1 \in [\mathsf{e}_{r+1}, \dots, \mathsf{e}_n]$$

with $\mathsf{d}_1 \neq \mathsf{o}$. Now we specify the origin f of $\mathcal{S}$ as a solution of

$$(l^* \circ l - \mathrm{id}_{\mathbb{R}^n})(\mathsf{f}) = \mathsf{d}_0\,.$$

And we modify the basis vectors $\mathsf{e}_{r+1}, \dots, \mathsf{e}_n$ in the eigenspace of 1 such that $\mathsf{d}_1 = -2a\mathsf{e}_n$ with $a \neq 0$. This implies

$$\alpha^* \circ \alpha(\mathsf{f}) = -\mathsf{d}_0 - \mathsf{d}_1 + l^* \circ l(\mathsf{f}) = -\mathsf{d}_1 + \mathsf{f} = 2a\mathsf{e}_n + \mathsf{f}.$$

Let $\mathsf{f}' := \alpha(\mathsf{f}) - a\mathsf{e}'_n$ be the origin of $\mathcal{S}'$. Then we get with (11) and $\lambda_n = 1$

$$\alpha(\mathsf{f}) = \mathsf{f}' + a\mathsf{e}'_n \ \text{ and } \ \alpha^*(\mathsf{f}') = \alpha^* \circ \alpha(\mathsf{f}) - a l^*(\mathsf{e}'_n) = \mathsf{f} + a\mathsf{e}_n$$

and therefore $\mathsf{a}' = a\mathsf{e}'_n$ in (4) and $\mathsf{b} = a\mathsf{e}_n$ in (7). From (8) we obtain

$$(14) \qquad f(\mathsf{x}) = x_1^2 + \dots + x_s^2 + (1 - \lambda_{s+1}^2)x_{s+1}^2 + \dots + (1 - \lambda_r^2)x_r^2 - 4ax_n + D = g'(\mathsf{x})$$

with $D := a^2 - C$. The image Φ' of the quadratic surface $\Phi\colon f(\mathsf{x}) = 0$ under α obeys

$$(15) \qquad x'_1 = \dots = x'_s = 0,\ f'(\mathsf{x}') := \tfrac{1-\lambda_{s+1}^2}{\lambda_{s+1}^2}\, x'^2_{s+1} + \dots + \tfrac{1-\lambda_r^2}{\lambda_r^2}\, x'^2_r - 4a(x'_n - a) + D \leq 0$$

where equality holds in the regular case $s = 0$ only.

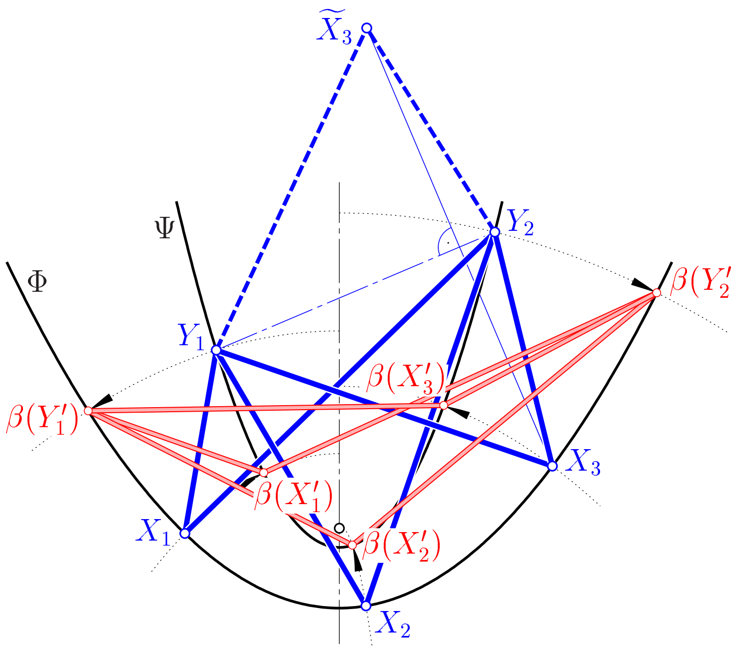

Figure 2: Theorem 1 in the parabolic case (Case 2)

Again an isometry β brings $\mathcal{S}'$ in coincidence with $\mathcal{S}$, and this gives *confocal* second-order surfaces with analogous properties as listed above. The sections of Φ and Ψ or $\partial\Psi$

with principal 2-planes spanned by e_i and e_n for $s < i \leq r$ are confocal parabolas with equations

$$(1-\lambda_i^2)x_i^2 - 4ax_n + D = 0, \quad \frac{1-\lambda_i^2}{\lambda_i^2}\,x_i^2 - 4a(x_n - a) + D = 0$$

as they share the axis of symmetry ($=x_n$-axis) and the focal point

$$(x_i, x_n) = \left(0,\ a + \frac{D}{4a} + \frac{a\lambda_i^2}{1-\lambda_i^2}\right) = \left(0,\ \frac{D}{4a} + \frac{a}{1-\lambda_i^2}\right).$$

This proves

LEMMA 3: *Let $\mathcal{F}$, $\mathcal{F}'$ in $\mathbb{E}^n$ be two incongruent configurations as defined in Theorem 1. There is an isometry $\beta : \mathbb{E}^n \to \mathbb{E}^n$ such that the knots of $\mathcal{F}$ and $\beta(\mathcal{F}')$ are corresponding points of two confocal surfaces Φ, Ψ, i.e., with*

$$X_0, \dots, X_n, \beta(Y_0'), \dots, \beta(Y_q') \in \Phi \text{ and } \beta(X_0'), \dots, \beta(X_n'), Y_0, \dots, Y_q \in \Psi.$$

Remark 1: After applying β the affine transformations α and α^* coincide, or more precisely

$$\beta\circ\alpha = \alpha^*\circ\beta^{-1} = (\beta\circ\alpha)^*.$$

The last equation results from $\beta^{-1} = \beta^*$ for isometries and $(\beta\circ\alpha)^* = \alpha^*\circ\beta^*$. Hence the affine transformation $\beta\circ\alpha$ is self-adjoint (compare [15]).

Proof of Theorem 1: The items 1, 2 and 3 are a consequence of Lemmas 1, 2 and 3. For proving item 4, take any point $X \in \Phi$ and $X' = \alpha(X)$. Then according to 2. we have $\overline{X'Y_k'} = \overline{XY_k}$ for all $k \in \{0, \dots, q\}$. Suppose that the affine span $\mathrm{Y} := [Y_0 \dots Y_q]$ has a dimension $< n$, and let $\widetilde{X}$ be the image of X under the reflection in Y (or any other motion which keeps Y fixed). Then we obtain again

$$\overline{\widetilde{X}Y_k} = \overline{XY_k} = \overline{X'Y_k'},$$

though $\widetilde{X}$ needs not be a point of Φ (note e.g. $\widetilde{X}_3$ in Fig. 2).

In order to prove item 5 in Theorem 1, we notice that $[X_i\,X_j] \subset \Phi$ is equivalent to the statement that $f\,(\lambda\mathsf{x}_i + (1-\lambda)\mathsf{x}_j) = 0$ for all $\lambda \in \mathbb{R}$. From (8) and $f(\mathsf{x}_i) = f(\mathsf{x}_j) = 0$ we get

$$\begin{aligned}
&f\,(\lambda\mathsf{x}_i + (1-\lambda)\mathsf{x}_j) = \\
&= [\lambda\mathsf{x}_i + (1-\lambda)\mathsf{x}_j]^2 - l\,(\lambda\mathsf{x}_i + (1-\lambda)\mathsf{x}_j)^2 - 2\,[\lambda\mathsf{x}_i + (1-\lambda)\mathsf{x}_j]\cdot[\mathsf{b} + l^*(\mathsf{a}')] + \mathsf{b}^2 - C = \\
&= \lambda^2[\mathsf{x}_i^2 - l(\mathsf{x}_i)^2] + (1-\lambda)^2[\mathsf{x}_j^2 - l(\mathsf{x}_j)^2] + 2\lambda(1-\lambda)[\mathsf{x}_i\cdot\mathsf{x}_j - l(\mathsf{x}_i)\cdot l(\mathsf{x}_j)] - \\
&\quad - \lambda[\mathsf{x}_i^2 - l(\mathsf{x}_i)^2] - (1-\lambda)[\mathsf{x}_j^2 - l(\mathsf{x}_j)^2] = \\
&= \lambda(1-\lambda)\left[-\mathsf{x}_i^2 - \mathsf{x}_j^2 + l(\mathsf{x}_i)^2 + l(\mathsf{x}_j)^2 + 2\mathsf{x}_i\cdot\mathsf{x}_j - 2l(\mathsf{x}_i)\cdot l(\mathsf{x}_j)\right] = \\
&= \lambda(\lambda-1)\left[(\mathsf{x}_i - \mathsf{x}_j)^2 - (l(\mathsf{x}_i) - l(\mathsf{x}_j))^2\right] = \lambda(\lambda-1)\left[(\mathsf{x}_i - \mathsf{x}_j)^2 - (\alpha(\mathsf{x}_i) - \alpha(\mathsf{x}_j))^2\right],
\end{aligned}$$

which gives the equivalence

$$[X_i\,X_j] \subset \Phi \iff \|\mathsf{x}_i - \mathsf{x}_j\| = \|\alpha(\mathsf{x}_i) - \alpha(\mathsf{x}_j)\|.$$

The analogous computation shows that $[Y_i'\,Y_j'] \subset \beta^{-1}(\Phi)$ is equivalent to $\overline{Y_iY_j} = \overline{Y_i'Y_j'}$. This is also a consequence of statements 2 and 3 in Theorem 1 as $\overline{Y_iY_j} = \overline{Y_i'Y_j'}$ implies congruent triangles $Y_iY_jX_k = Y_i'Y_j'X_k'$ for each $k \in \{0, \dots, n\}$. Hence for any point Y of the line $[Y_i\,Y_j]$ and its image $Y' \in [Y_i'\,Y_j']$ under the isometry $[Y_i\,Y_j] \to [Y_i'\,Y_j']$ with $Y_i \mapsto Y_i'$ and $Y_j \mapsto Y_j'$ we have $\overline{Y X_k} = \overline{Y' X_k'}$ and therefore $Y' \in \beta^{-1}(\Phi)$. □

The other configuration theorems

We now suppose that at the bipartite framework the knots of one class, say the X-knots, are located in a hyperplane H_x. In this case incongruent configurations with the same edge lengths arise by reflecting single Y-knots in H_x. We exclude these in the following theorem by saying that we treat pairs of *"essentially different"* configurations only.

THEOREM 2: *Let $X_0, \dots, X_{n-1}, Y_0, \dots, Y_q$ and $X_0', \dots, X_{n-1}', Y_0', \dots, Y_q'$ be the knots of two essentially different configurations $\mathcal{F}$, $\mathcal{F}'$ of a complete bipartite framework $\mathcal{F}_0$ of type $\mathcal{K}_{n,q+1}$ with given bar lenghts l_{ik} in $\mathbb{E}^n$. Suppose that $X_0, \dots, X_{n-1}$ span a hyperplane H_x which also contains the knots $X_0', \dots, X_{n-1}'$.*

1. *There is an appropriate displacement $\beta\colon \mathbb{E}^n \to \mathbb{E}^n$ with $H_x \mapsto H_x$ such that for all $i \in \{0, \dots, n-1\}$ the knots $X_i \mapsto \beta(X_i')$ are corresponding points of two confocal quadrics Φ, $\beta(\Phi')$ in H_x.*
   For all $k \in \{0, \dots, q\}$ the knots $\beta(Y_k') \mapsto Y_k$ are corresponding points of two confocal quadrics $\beta(\Psi_k')$, Ψ_k which are symmetric with respect to H_x and intersect H_x along Φ and $\beta(\Phi')$, respectively (see Fig. 3).
   Thus the statement $\overline{\beta(X_i')\,\beta(Y_k')} = \overline{X_iY_k}$ is a direct consequence of Ivory's theorem.
2. *Two different knots Y_j, Y_k of $\mathcal{F}$ share their distance with the corresponding points of Y_j', Y_k' of $\mathcal{F}'$ if and only if the spanned line $[Y_j'\,Y_k']$ is tangent to the second-order surface $\widehat{\Psi}'$ whose corresponding quadric $\widehat{\Psi}$ through $\beta(\Phi')$ is flat (see Fig. 4). Conversely — like in Theorem 1 — the equivalence holds:*

$$\overline{X_iX_j} = \overline{X_i'X_j'} \iff [X_i\,X_j] \subset \Phi\,.$$

3. *For any $r, s \in \mathbb{N}$ the framework $\mathcal{F}_0$ can be extended to a complete bipartite framework $\widetilde{\mathcal{F}}$ of type $\mathcal{K}_{n+r,\,q+1+s}$ which still admits two essentially different configurations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ with knots $X_0, \dots, X_{n-1+r} \in \Phi$ and $Y_0', \dots, Y_{q+s}' \in \mathbb{E}^n \setminus \mathrm{int}(\widehat{\Psi}')$.*

Proof: Again, each configuration $\mathcal{F}, \mathcal{F}'$ gets its own cartesian coordinate system $\mathcal{S}, \mathcal{S}'$, respectively. We suppose that in both systems the hyperplane H_x is spanned by the first $n-1$ coordinate axes. If $\chi\colon \mathbb{E}^n \to H_x$ denotes the orthogonal projection onto H_x, i.e., parallel to the last basis vector e_n of $\mathcal{S}$ and e_n' of $\mathcal{S}'$, then we get

$$\mathsf{y}_k = \chi(\mathsf{y}_k) + y_{k,n}\mathsf{e}_n, \quad \mathsf{y}_k' = \chi(\mathsf{y}_k') + y_{k,n}'\mathsf{e}_n'.$$

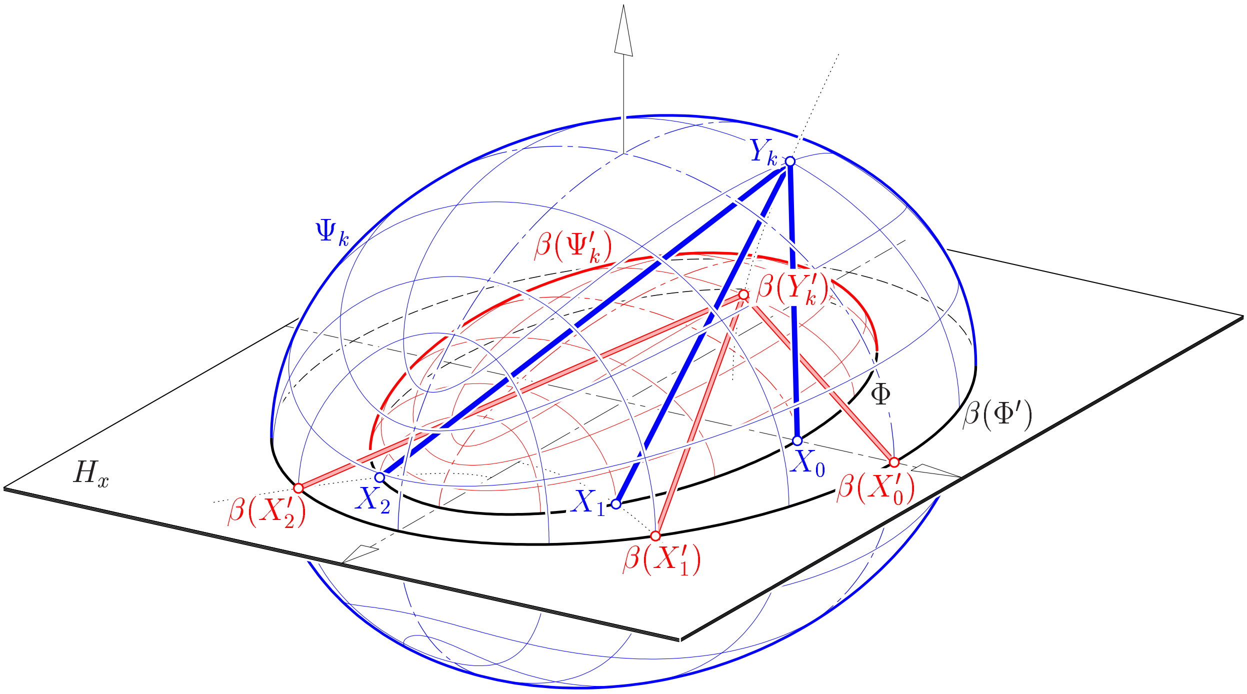

Figure 3: The statement of Theorem 2 for dimension $n = 3$

We subtract the quadratic equations (2) and notice that in (3) only the first $n-1$ coordinates of y_k are involved. In analogy to Lemma 1 we obtain the necessary conditions: There are two adjoint affine transformations

$$\begin{array}{llll} \alpha\colon & H_x \to H_x, & \mathsf{x}_i \mapsto \mathsf{x}'_i, & i = 0, \dots, n-1 \\ \alpha^*\colon & H_x \to H_x, & \chi(\mathsf{y}'_k) \mapsto \chi(\mathsf{y}_k), & k = 0, \dots, q. \end{array}$$

As the two configurations $\mathcal{F}$ and $\mathcal{F}'$ are supposed to be essentially different, neither α nor α^* is an isometry.

Now we substitute the representations (4) of α and (7) of α^* in (2) and obtain that

$$\|\mathsf{x}_i - \mathsf{y}_k\| = \|\mathsf{x}_i - \alpha^*\circ\chi(\mathsf{y}'_k) - y_{k,n}\mathsf{e}_n\| = \|\mathsf{x}'_i - \mathsf{y}'_k\| = \|\alpha(\mathsf{x}_i) - \chi(\mathsf{y}'_k) - y'_{k,n}\mathsf{e}'_n\|$$

is equivalent to

$$\begin{aligned} \mathsf{x}_i^2 - l(\mathsf{x}_i)^2 - 2\mathsf{x}_i\cdot\mathsf{b} - 2l(\mathsf{x}_i)\cdot\mathsf{a}' + \mathsf{b}^2 = \\ = \chi(\mathsf{y}'_k)^2 - l^*\circ\chi(\mathsf{y}'_k)^2 - 2\chi(\mathsf{y}'_k)\cdot\mathsf{a}' - 2l^*\circ\chi(\mathsf{y}'_k)\cdot\mathsf{b} + \mathsf{a}'^2 + y'^2_{k,n} - y^2_{k,n}. \end{aligned} \tag{16}$$

This equation holds for all $i \in \{0, \dots, n-1\}$ and all $k \in \{0, \dots, q\}$. As the left side depends on i, the right side on k only, both sides must equal any constant C. This results in two quadratic functions

(17)

$$\begin{aligned} f(\mathsf{x}) :=\ & \mathsf{x}^2 - l(\mathsf{x})^2 - 2\mathsf{x}\cdot[\mathsf{b} + l^*(\mathsf{a}')] + \mathsf{b}^2 - C, \\ g(\mathsf{y}, \mathsf{y}') :=\ & \chi(\mathsf{y}')^2 - l^*\circ\chi(\mathsf{y}')^2 - 2\chi(\mathsf{y}')\cdot[\mathsf{a}' + l(\mathsf{b})] + \mathsf{a}'^2 - C + (\mathsf{y}'\cdot\mathsf{e}'_n)^2 - (\mathsf{y}\cdot\mathsf{e}_n)^2, \end{aligned}$$

and

$$f(\mathsf{x}_i) = 0 \ \ \forall i = 0, \dots, n-1\,, \quad g(\mathsf{y}_k, \mathsf{y}'_k) = 0 \ \ \forall k = 0, \dots, q.$$

We now specify the coordinate systems $\mathcal{S}, \mathcal{S}'$ appropriate to α and α^* like previously in the proof of Theorem 1 for the Cases 1 and 2:
In analogy to (12) and (13) we obtain in Case 1 the coordinate representations

(18)
$$\begin{aligned} f(\mathsf{x}) &= x_1^2 + \cdots + x_s^2 + (1-\lambda_{s+1}^2)x_{s+1}^2 + \cdots + (1-\lambda_r^2)x_r^2 - C, \\ g(\mathsf{y}, \mathsf{y}') &= y_1'^2 + \cdots + y_s'^2 + (1-\lambda_{s+1}^2)y_{s+1}'^2 + \cdots + (1-\lambda_r^2)y_r'^2 - C + y_n'^2 - y_n^2 \end{aligned}$$

with $0 \le s \le r \le n-1$. The points x_i are located on the quadric

$$\Phi:\ f(\mathsf{x}) = x_n = 0 \ \text{ in } \ H_x.$$

The knots x_i' belong to the affine image $\Phi' := \alpha(\Phi) \subset H_x$ obeying

$$x_1' = \cdots = x_s' = x_n' = 0 \ \text{ and } \ \frac{1-\lambda_{s+1}^2}{\lambda_{s+1}^2}\, x_{s+1}'^2 + \cdots + \frac{1-\lambda_r^2}{\lambda_r^2}\, x_r'^2 - C \le 0.$$

If the isometry β brings $\mathcal{S}'$ in coincidence with $\mathcal{S}$ then $\beta(\Phi')$ becomes confocal to Φ with the pairs $(\mathsf{x}_i\,,\ \beta(\mathsf{x}_i'))$ of corresponding points.

In which way are for any $k \in \{0, \ldots, q\}$ the points y_k' and y_k related under the conditions $\chi(\mathsf{y}_k) = \alpha^* \circ \chi(\mathsf{y}_k')$ and $g(\mathsf{y}_k, \mathsf{y}_k') = 0$?
Due to (18) and $y_{k,n}^2 \ge 0$ there is a $\lambda_{k,n} \in \mathbb{R}$ such that for the quadratic function

(19)
$$G_k'(\mathsf{y}') := y_1'^2 + \cdots + y_s'^2 + (1-\lambda_{s+1}^2)y_{s+1}'^2 + \cdots + (1-\lambda_r^2)y_r'^2 - C + (1-\lambda_{k,n}^2)y_n'^2$$

$G_k'(\mathsf{y}_k') = 0$ holds. Hence $\Psi_k':\ G_k'(\mathsf{y}') = 0$ is a quadric passing through y_k'. From

$$g(\mathsf{y}, \mathsf{y}') = G_k'(\mathsf{y}') + \lambda_{k,n}^2 y_n'^2 - y_n^2 \ \text{ and } \ g(\mathsf{y}_k, \mathsf{y}_k') = G_k'(\mathsf{y}_k') = 0$$

we deduce $y_{k,n} = \lambda_{k,n} y_{k,n}'$ which specifies the sign of $\lambda_{k,n}$ uniquely. This implies together with (11) for $i = 0, \ldots, n-1$ that we can extend $\alpha^*\colon H_x \to H_x$ to an affine transformation $\alpha_k^*\colon \mathbb{E}^n \to \mathbb{E}^n$ with $\mathsf{y}_k' \mapsto \mathsf{y}_k$. The affine image $\Psi_k := \alpha_k^*(\Psi_k')$ obeys under $\lambda_{k,n} \ne 0$

$$y_1 = \cdots = y_s = 0 \ \text{ and } \ \frac{1-\lambda_{s+1}^2}{\lambda_{s+1}^2}\, y_{s+1}^2 + \cdots + \frac{1-\lambda_r^2}{\lambda_r^2}\, y_r^2 - C + \frac{1-\lambda_{k,n}^2}{\lambda_{k,n}^2}\, y_n^2 \le 0.$$

Let $\widehat{\Psi}'$ denote the quadric Ψ_k' in the special case $\lambda_{k,n} = 0$. Then $\widehat{\Psi} = \alpha_k^*(\widehat{\Psi}')$ obeys

$$y_1 = \cdots = y_s = y_n = 0 \ \text{ and } \ \frac{1-\lambda_{s+1}^2}{\lambda_{s+1}^2}\, y_{s+1}^2 + \cdots + \frac{1-\lambda_r^2}{\lambda_r^2}\, y_r^2 - C \le 0.$$

In all cases Ψ_k is confocal to the displaced $\beta(\Psi_k')$, and these two quadrics have the properties listed in Theorem 2 (compare Fig. 3). $\beta(\mathsf{y}_k')$ and y_k are corresponding points. So, each knot y_k' defines its own pair $(\beta(\Psi_k'),\ \Psi_k)$ of confocal quadrics.

The discussion of Case 2 leads to an analogous result which proves the statement 1 of Theorem 2 completely.

Ad 2. Suppose $\|\mathsf{y}_j - \mathsf{y}_k\| = \|\mathsf{y}_j' - \mathsf{y}_k'\|$. As explained in the proof of Theorem 1, this is equivalent to the statement that all pairs of points $\big(\mathsf{y} := \mu\mathsf{y}_j + (1-\mu)\mathsf{y}_k\,,$

$\mathsf{y}' := \mu\mathsf{y}'_j + (1-\mu)\mathsf{y}'_k\big)$ of the spanned lines $h = [\mathsf{y}_j\,\mathsf{y}_k]$, $h' = [\mathsf{y}'_j\,\mathsf{y}'_k]$ share the property $\|\mathsf{x}_i - \mathsf{y}\| = \|\mathsf{x}'_i - \mathsf{y}'\|$ for all $i = 0,\dots,n-1$.[6] The quadrics Ψ'_k with equations $G'_k(\mathsf{y}') = 0$, $k = 0,\dots,q$, according to (19) belong to a pencil spanned e.g. by the double hyperplane $H_x\colon y_n'^2 = 0$ and the quadric

(20)
$$\widehat{\Psi}'\colon\ \widehat{G}'(\mathsf{y}') := y_1'^2 + \dots + y_s'^2 + (1-\lambda_{s+1}^2)y_{s+1}'^2 + \dots + (1-\lambda_r^2)y_r'^2 - C + y_n'^2 = 0.$$

Therefore all quadrics Ψ'_k share the tangential hyperplanes at the points of the intersection $\widehat{\Psi}' \cap H_x$.

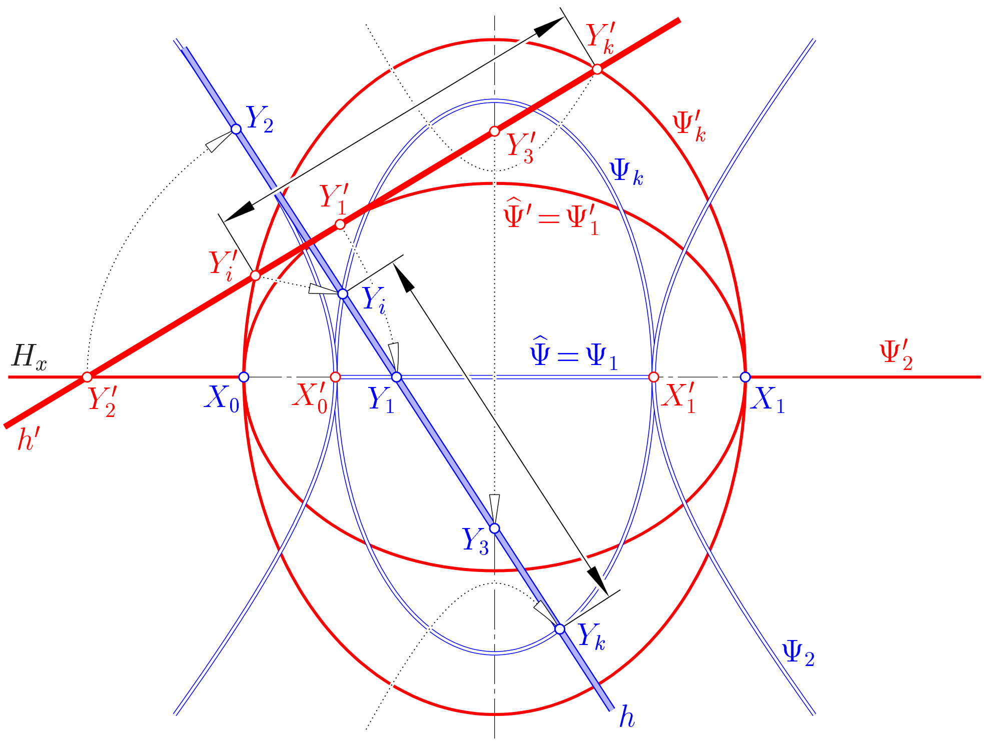

Figure 4: The second statement of Theorem 2 in the special case $n = 2$, $\beta = \mathrm{id}_{\mathbb{E}^2}$, $\Phi = \{X_0, X_1\}$, $\Phi' = \{X'_0, X'_1\}$

Without loss of generality we may suppose that the two points $\mathsf{y}'_j, \mathsf{y}'_k$ belong to the same Ψ'_k, i.e., $G'_k(\mathsf{y}'_j) = G'_k(\mathsf{y}'_k) = 0$ (see Fig. 4). Then in Case 1 we have

$$\begin{aligned} \mathsf{y}'_j &= (y'_{j,1},\dots,y'_{j,n-1},\ y'_{j,n}), & \mathsf{y}_j &= (\lambda_1 y'_{j,1},\dots,\lambda_{n-1}y'_{j,n-1},\ \pm\lambda_{k,n}y'_{j,n}),\\ \mathsf{y}'_k &= (y'_{k,1},\dots,y'_{k,n-1},\ y'_{k,n}), & \mathsf{y}_k &= (\lambda_1 y'_{k,1},\dots,\lambda_{n-1}y'_{k,n-1},\ \lambda_{k,n}y'_{k,n}),\end{aligned}$$

and the distance equation

$$\mathsf{y}_j^2 - 2\mathsf{y}_j\!\cdot\!\mathsf{y}_k + \mathsf{y}_k^2 = \mathsf{y}_j'^2 - 2\mathsf{y}'_j\!\cdot\!\mathsf{y}'_k + \mathsf{y}_k'^2$$

[6] It results also from the identity

$$\big(\mathsf{x}_i - \mu\mathsf{y}_j - (1-\mu)\mathsf{y}_k\big)^2 = \mu(\mathsf{x}_i - \mathsf{y}_j)^2 + (1-\mu)(\mathsf{x}_i - \mathsf{y}_k)^2 - \mu(1-\mu)(\mathsf{y}_j - \mathsf{y}_k)^2.$$

is equivalent to

$$(1-\lambda_1^2)y'_{j,1}y'_{k,1}+\cdots+(1-\lambda_r^2)y'_{j,r}y'_{k,r}+(1\mp\lambda_{k,n}^2)y'_{j,n}y'_{k,n}-C=0. \tag{21}$$

The upper sign characterizes conjugate position of y'_j and y'_k with respect to Ψ'_k with equation $G'_k(\mathsf{y}')=0$ due to (19), i.e., the connecting line $[\mathsf{y}'_j\,\mathsf{y}'_k]$ is a generator of Ψ'_k and therefore tangent to $\widehat{\Psi}$ at a point in H_x.
In order to figure out the meaning of the condition with the lower sign, we compute the discriminant of the polynomial $\widehat{G}'(\mu\mathsf{y}'_j+(1-\mu)\mathsf{y}'_k)$ which is quadratic in μ according to (20). Straight forward computation reveals that this discriminant equals the product of the left-side terms in (21) with different signs. So eq. (21) holds if and only if the line $[\mathsf{y}'_j\,\mathsf{y}'_k]$ is tangent to the quadric $\widehat{\Psi}'$: $\widehat{G}'(\mathsf{y}')=0$ or subset of this quadric.
An analogous discussion of Case 2 concludes the proof of statement 2 in Theorem 2, as the second part has already been proved with Theorem 1.

Remark 2: The equation $g(\mathsf{y},\mathsf{y}')=0$ in (17) or (18) defines a 2-2-correspondence between the points $\mathsf{y}'=\chi(\mathsf{y}')+y'_n\mathsf{e}'_n$ and $\mathsf{y}=\alpha^*\circ\chi(\mathsf{y}')+y_n\mathsf{e}_n$. The quadric $\widehat{\Psi}'$ is mapped into the hyperplane H_x. This gives exactly Jacobi's generation of a quadric $\widehat{\Psi}'$ in $\mathbb{E}^n$ as the set of points y' with $\|\mathsf{y}'-\mathsf{x}'_i\|=\|\mathsf{y}-\mathsf{x}_i\|$, $i=0,\dots,n-1$, for any $\mathsf{y}\in H_x$ (see [7]).
In the case of equal distances $\|\mathsf{y}_j-\mathsf{y}_k\|=\|\mathsf{y}'_j-\mathsf{y}'_k\|$ the 2-2-correspondence must map the line $h'=[\mathsf{y}'_j\,\mathsf{y}'_k]$ onto the line $h=[\mathsf{y}_j\,\mathsf{y}_k]$. Therefore $[\mathsf{y}'_j\,\mathsf{y}'_k]$ can intersect the quadric $\widehat{\Psi}'$ in one point only (see Fig. 4) or it totally lies on $\widehat{\Psi}'$, as the common points correspond to $[\mathsf{y}_j\,\mathsf{y}_k]\cap H_x$.

Ad 3. This follows from the fact that $\|\mathsf{y}-\mathsf{x}_i\|=\|\mathsf{y}'-\mathsf{x}'_i\|$ for all $i=0,\dots,n-1$ is equivalent to eq. (16) or in Case 1 to $f(\mathsf{x}_i)=0$ and $g(\mathsf{y},\mathsf{y}')=0$ in (18). For any point y' there is a corresponding $\mathsf{y}=\alpha^*\circ\chi(\mathsf{y}')+y_n\mathsf{e}_n$ with $g(\mathsf{y},\mathsf{y}')=0$ if and only if the inequality

$$y_n^2=y_1'^2+\cdots+y_s'^2+(1-\lambda_{s+1}^2)y_{s+1}'^2+\cdots+(1-\lambda_r^2)y_r'^2-C+y_n'^2\geq 0.$$

is true. This inequality excludes points in the "interior" of $\widehat{\Psi}'$ with equation (20). The same holds in Case 2. □

THEOREM 3: *Let $X_0,\dots,X_{n-2},Y_0,\dots,Y_q$ and $X'_0,\dots,X'_{n-2},Y'_0,\dots,Y'_q$ in $\mathbb{E}^n$ be the knots of two essentially different configurations $\mathcal{F}$, $\mathcal{F}'$ of a complete bipartite framework $\mathcal{F}_0$ of type $\mathcal{K}_{n-1,q+1}$ with given bar lenghts l_{ik}. The knots $X_0,\dots,X_{n-2},X'_0,\dots,X'_{n-2}$ are located in a hyperplane H_x. The knots $Y_0,\dots,Y_q,Y'_0,\dots,Y'_q$ of the other class belong to a perpendicular hyperplane H_y.*
The orthogonal projections of $X_0,\dots,X_{n-2}$ onto H_y are supposed to span the $(n-2)$-dimensional intersection space $S:=H_x\cap H_y$.

1. *There is an appropriate displacement $\beta\colon\mathbb{E}^n\to\mathbb{E}^n$ with $H_x\mapsto H_x$ and $H_y\mapsto H_y$ such that for all $i\in\{0,\dots,n-2\}$ the knots $X_i\mapsto\beta(X'_i)$ are corresponding points of two confocal quadrics Φ_i, $\beta(\Phi'_i)$ in H_x, symmetric with respect to S.*
   For all $k\in\{0,\dots,q\}$ the knots $\beta(Y'_k)\mapsto Y_k$ are corresponding points of two confocal quadrics $\beta(\Psi'_k),\Psi_k\subset H_y$ which are symmetric with respect to S.
   All Φ_i and $\beta(\Psi'_k)$ intersect S along the same quadric Σ which is confocal to the common intersection $\Sigma'=\beta(\Phi'_i)\cap S=\Psi_k\cap S$ (see Fig. 5).
   Hence the statement $\overline{\beta(X'_i)\,\beta(Y'_k)}=\overline{X_iY_k}$ is again a direct consequence of Ivory's theorem.

2. *Two different knots* X_i, X_j *of* $\mathcal{F}$ *share their distance with the corresponding points of* X'_i, X'_j *of* $\mathcal{F}'$ *if and only if the spanned line* $[X'_i\, X'_j]$ *is tangent to the second-order surface* $\widehat{\Phi}$ *whose corresponding quadric* $\beta(\widehat{\Phi}')$ *through* Σ' *is flat in* S.
   Conversely — like in Theorem 2 — we have $\overline{Y_jY_k} = \overline{Y'_jY'_k}$ *if and only if the spanned line* $[Y'_j\, Y'_k]$ *is tangent to the quadric* $\widehat{\Psi}'$ *with a flat corresponding quadric* Ψ *through* Σ'.
3. *For any* $r, s \in \mathbb{N}$ *the framework* $\mathcal{F}_0$ *can be extended to a complete bipartite framework* $\widetilde{\mathcal{F}}$ *of type* $\mathcal{K}_{n-1+r,\,q+1+s}$ *which still admits two essentially different configurations* $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ *with arbitrary knots* $X_0, \ldots, X_{n-2+r} \in H_x \backslash \mathrm{int}(\widehat{\Phi})$ *and* $Y'_0, \ldots, Y'_{q+s} \in H_y \backslash \mathrm{int}(\widehat{\Psi}')$.

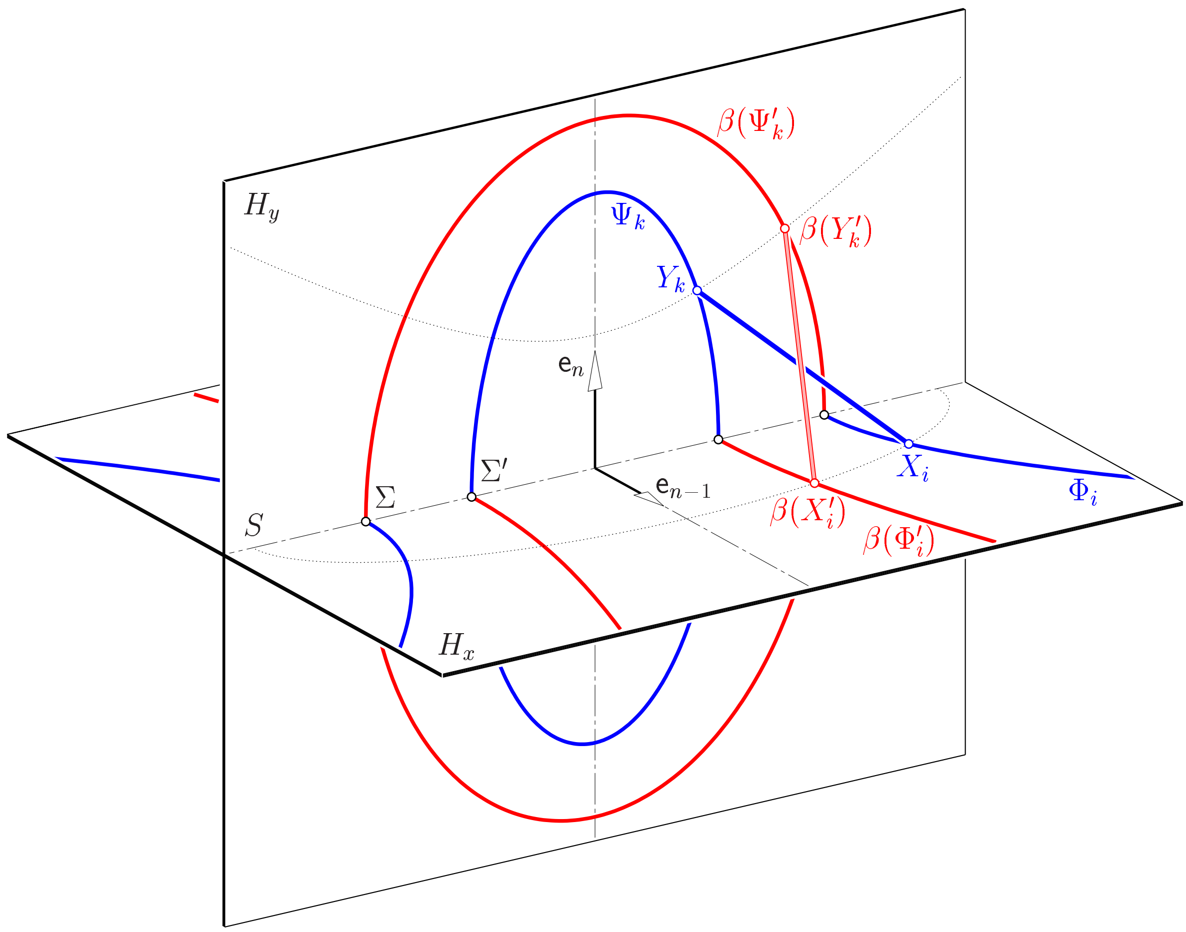

Figure 5: The statement of Theorem 3 for dimension $n = 3$

Proof: We again start with two different coordinate systems $\mathcal{S}, \mathcal{S}'$. In both cases the given hyperplanes H_x, H_y are supposed as coordinate planes: H_x is perpendicular to the last basis vectors e_n and e'_n. H_y is perpendicular to e_{n-1} and e'_{n-1}. Let $\sigma : \mathbb{E}^n \to S$ denote the orthogonal projection onto S. Thus we get

$$\mathsf{x}_i = \sigma(\mathsf{x}_i) + x_{i,n-1}\mathsf{e}_{n-1}, \qquad \mathsf{x}'_i = \sigma(\mathsf{x}'_i) + x'_{i,n-1}\mathsf{e}'_{n-1},$$
$$\mathsf{y}_k = \sigma(\mathsf{y}_k) + y_{k,n}\mathsf{e}_n, \qquad \mathsf{y}'_k = \sigma(\mathsf{y}'_k) + y'_{k,n}\mathsf{e}'_n.$$

After subtracting the quadratic equations (2), we obtain (3) where only the first $n-2$ coordinates of x_i and y'_k are involved. Since $\sigma(\mathsf{x}_0),\dots,\sigma(\mathsf{x}_{n-2})$ are supposed to constitute a simplex in S, we derive in analogy to Lemma 1 the following necessary conditions: There are two adjoint affine transformations

$$\begin{array}{llll}\alpha: & S\to S, & \sigma(\mathsf{x}_i)\ \mapsto\ \sigma(\mathsf{x}'_i), & i=0,\dots,n-2,\\ \alpha^*: & S\to S, & \sigma(\mathsf{y}'_k)\mapsto\sigma(\mathsf{y}_k), & k=0,\dots,q.\end{array}$$

Now we substitute the representations (4) of α and (7) of α^* in (2) and obtain that

$$\|\sigma(\mathsf{x}_i)+x_{i,n-1}\mathsf{e}_{n-1}-\alpha^*\circ\sigma(\mathsf{y}'_k)-y_{k,n}\mathsf{e}_n\| = \|\alpha\circ\sigma(\mathsf{x}_i)+x'_{i,n-1}\mathsf{e}'_{n-1}-\sigma(\mathsf{y}'_k)-y'_{k,n}\mathsf{e}'_n\|$$

is equivalent to

$$\begin{aligned}&\sigma(\mathsf{x}_i)^2-l\circ\sigma(\mathsf{x}_i)^2-2\sigma(\mathsf{x}_i)\cdot\mathsf{b}-2l\circ\sigma(\mathsf{x}_i)\cdot\mathsf{a}'+\mathsf{b}^2+x^2_{i,n-1}-x'^2_{i,n-1}=\\ &=\sigma(\mathsf{y}'_k)^2-l^*\circ\sigma(\mathsf{y}'_k)^2-2\sigma(\mathsf{y}'_k)\cdot\mathsf{a}'-2l^*\circ\sigma(\mathsf{y}'_k)\cdot\mathsf{b}+\mathsf{a}'^2+y'^2_{k,n}-y^2_{k,n}.\end{aligned}\tag{22}$$

Again we conclude two quadratic functions

(23)
$$\begin{aligned}f(\mathsf{x},\mathsf{x}') &:= \sigma(\mathsf{x})^2-l\circ\sigma(\mathsf{x})^2-2\sigma(\mathsf{x})\cdot[\mathsf{b}+l^*(\mathsf{a}')]+\mathsf{b}^2-C+(\mathsf{x}\cdot\mathsf{e}_{n-1})^2-(\mathsf{x}'\cdot\mathsf{e}'_{n-1})^2,\\ g(\mathsf{y},\mathsf{y}') &:= \sigma(\mathsf{y}')^2-l^*\circ\sigma(\mathsf{y}')^2-2\sigma(\mathsf{y}')\cdot[\mathsf{a}'+l(\mathsf{b})]+\mathsf{a}'^2-C+(\mathsf{y}'\cdot\mathsf{e}'_n)^2-(\mathsf{y}\cdot\mathsf{e}_n)^2,\end{aligned}$$

and

$$f(\mathsf{x}_i,\mathsf{x}'_i)=0\quad\forall i=0,\dots,n-2,\quad\text{and}\quad g(\mathsf{y}_k,\mathsf{y}'_k)=0\quad\forall k=0,\dots,q.$$

Now a discussion like in the proof of Theorem 2 leads to the stated properties. □

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