Gears and Belt Drives for Non-Uniform Transmission

Hellmuth Stachel

Abstract Ordinarily, gears and belt drives are used for uniform transmission of rotations between parallel axes. Here we focus on the nonuniform case, i.e., with non-constant transmission ratio. We treat the geometry of tooth profiles and pulleys and their algorithmic computation. Concerning gears, we recall a method due to S. Finsterwalder. Concerning belt drives, we study their relation to tooth profiles and focus on 'strict' cases which work without tightener.

Keywords Nonuniform Transmission, Planar Gearing, Belt Drives, Geometric Kinematics

Introduction

The problem of designing *noncircular gears* has often been addressed in publications (see [1, 2, 3, 4] and in particular Litvin's monograph [5], pp. 346–381). Such gears are used for nonuniform transmission between parallel axes. Similarly, certain cam-like mechanisms have rolling centrode contact surfaces. Here we recall an applicable, classical design method formulated by S. Finsterwalder (1862–1951, see [6], [7, p. 284] or [8, p. 205]) which is also useful for an algorithmic computation.

J. Hoschek [9] and F. Freudenstein [10] (see also [11]) created methods to design belt drives for given nonuniform transmission such that the length of the surrounding belt remains constant. Such belt drives with belt slack zero will briefly be called *strict*. After discussing geometric properties of belt drives we present a modification of Hoschek's method for computing conjugate pulleys for strict belt drives that is not confined to discrete (i.e., polygonal) models, like those treated in [12] or [13], but nevertheless produces satisfactory results. However implementation of large transmission ratio variations results in pulley profiles with singularities analogous to

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M. Ceccarelli (ed.), *Proceedings of EUCOMES 08*,

undercut gear teeth. Hoschek's method as well as ours fail in the case of global 1:1 transmissions where a full rotation of the input wheel corresponds to a full rotation of the output wheel.

Gearing for Nonuniform Transmission

Let the driving wheel Σ_1 rotate about the center $O_1 = 01$ through the angle φ_1 with respect to the frame of the fixed housing Σ_0 while the output wheel Σ_2 rotates about $O_2 = 02$ through φ_2 . Then the pole 12 of the relative motion Σ_2/Σ_1 divides the segment O_1O_2 in the ratio of instantaneous angular velocities, i.e., $O_1\,12 : O_2\,12 = \dot\varphi_2 : \dot\varphi_1 = \omega_2 : \omega_1$.

In the sequel we seek gears and belt drives that transmit rotary motion according to some *transmission function*

$$\varphi_2 = f(\varphi_1) \quad \text{for} \quad 0 \le \varphi_1 \le 2\pi$$

Function $f(\varphi_1)$ is assumed to be strictly monotonic, quite often differentiable, and obeys $f(\varphi_1 + 2\pi) = f(\varphi_1) + 2\pi/n$ for , $n \in \mathbb{Z}$, $n \neq 0$. The integer n is called *global transmission ratio*. The transmission function $f(\varphi_1)$ defines the associated *perturbation function* $g(\varphi_1)$ by

$$g(\varphi_1) = n.f(\varphi_1) - \varphi_1 \quad \text{or} \quad f(\varphi_1) = [\varphi_1 + g(\varphi_1)]/n$$

Because of $g(\varphi_1 + 2\pi) = g(\varphi_1)$ function $g(\varphi_1)$ is periodic and can be set up as a Fourier series. In the gear box Σ_0 we use a coordinate frame (see Fig. 3) with $O_1 = (0,0)$, $O_2 = (e,0)$, and $12 = (r_1(\varphi_1), 0)$. Then the coordinate $r_1(\varphi_1)$ of 12 obeys $\omega_2 : \omega_1 = r_1 : (r_1 - e)$, hence

$$r_1(\varphi_1) = \frac{e\,f'(\varphi_1)}{f'(\varphi_1) - 1} = \frac{e\,(1 + g'(\varphi_1))}{1 - n + g'(\varphi_1)} \tag{1}$$

when the prime indicates differentiation by φ_1.
$c_1 \subset \Sigma_1$ and $c_2 \subset \Sigma_2$ are conjugate tooth profiles if and only if they are an *enveloping pair* of the relative motion Σ_2/Σ_1. Due to the 'Law of Gearing' the common normal line at the meshing point C passes always through the relative pole 12 (Fig. 1). We express the position of C in polar coordinates $(\rho, \ \psi)$ with respect to the relative pole 12 and choose the negative vector of the pole velocity as zero-axis for measuring the polar angle ψ. We may suppose $0 \le \psi \le \pi$ for $\rho \in \mathbb{R}$. Angle ψ with the meshing normal is unique even for $\rho = 0$. As the two polodes p_1, p_2 are in contact at 12, these coordinates of C are the same with respect to Σ_1 and Σ_2.

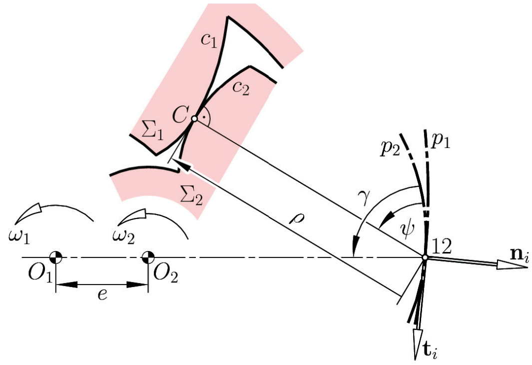

Fig. 1 Conjugate tooth profiles c_1, c_2 for the relative polodes p_1, p_2

Let $(\mathbf{t}_i, \mathbf{n}_i)$ denote the Frenet frame of the polode p_i , $i = 1, 2$, with position vector $\mathbf{p}_i(t)$ and arc length s. The derivatives by time t are

$$\dot{\mathbf{t}}_i = v\,\kappa_i \mathbf{n}_i,\ \dot{\mathbf{n}}_i = -v\,\kappa_i\,\dot{\mathbf{t}}_i \quad \text{and} \quad \dot{\mathbf{p}}_i = v\mathbf{t}_i$$

with pole velocity $v = \mathrm{d}s/\mathrm{d}t$ and curvature κ_i of p_i. The position vector $\mathbf{c}_i = \mathbf{p}_i + \rho(\cos\psi\ \mathbf{t}_i + \sin\psi\ \mathbf{n}_i)$ of the tooth profile has the velocity

$$\dot{\mathbf{c}}_i = v\,\mathbf{t}_i + \dot{\rho}(\cos\psi\ \mathbf{t}_i + \sin\psi\ \mathbf{n}_i) + \rho\dot{\psi}(-\sin\psi\ \mathbf{t}_i + \cos\psi\ \mathbf{n}_i) + \rho\, v\,\kappa_i(\cos\psi\ \mathbf{n}_i - \sin\psi\ \mathbf{t}_i)$$

orthogonal to $\mathbf{c}_i - \mathbf{p}_i$. This implies

$$\dot{\rho} = -v\cos\psi \tag{2}$$

This differential equation for $\mathbf{c}_i(t)$ is independent of the curvature κ_i of the polodes p_i and is therefore the basis for

S. Finsterwalder's principle of gearing [6, p. 243]:[1] *We imagine p_1 as a flexible metal band and replace c_1 by a discrete set of line elements, each attached to the polode p_1 by fixing the angle ψ and the distance ρ (Fig. 2). Then for any flex p_2 of p_1 the curve c_2 formed by the attached line elements is conjugate to c_1 if the relative motion Σ_2/Σ_1 is defined by p_2 rolling along p_1 . This principle works not only for wheels rotating about fixed centers, but for any planar motion given by a pair of polodes.*

In the sense of Differential Geometry the functions and obeying (2) can be called *natural functions of the specified gearing.*

[1] This is the discretized version of the general Reuleaux (or Camus) principle saying that conjugate profiles are envelopes of any curve c_0 while an auxiliary curve atached to c_0 is rolling on the polodes.

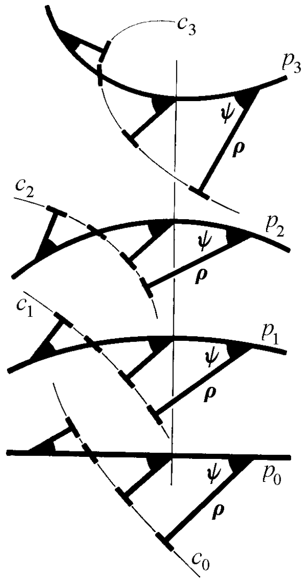

Fig. 2 S. Finsterwalder's principle of gearing (extracted from [7], p. 284, Fig. 446, compare [8], Abb. 151)

If the metal band is stretched like p_0 in Fig. 2 then is the tooth profile of the conjugate broaching rack. A straight line segment defines a constant angle ψ and therefore the nonuniform *involute gearing*.

Theorem 1. *The following conditions are necessary to avoid local undercuts of conjugate tooth profiles given by their natural functions:*

$(1 + v\,\kappa_i)\rho - v\,\sin\psi \neq 0$ for
$i = 1, 2$ and $\dot{\psi} \neq \rho - v\,\sin\psi$.

The proof is based on Frenet equations of the tooth profile c_i . The first condition excludes singularities, the second avoids intersections between c_1 and c_2 because of 3-point contact.

For computing conjugate tooth profiles c_1 , c_2 of a given transmission one has to take the following steps:

(1) Compute the polode p_1 in Σ_1 with polar coordinates $(r_1,\ -\varphi_1)$ and r_1 by (1) as well as p_2 in Σ_2 with coordinates $p_2 = (r_1\ -\ e,\ -\varphi_2)$.
(2) Rectify p_1, i.e., bend it into the straight line p_0. Freely choose the rack tooth profile c_0 as long as it is nowhere orthogonal to p_0 and compute the polar coordinates $(\rho(s),\ \ \psi(s))$ with respect to p_0.
(3) Bend p_0 back into p_1 and p_2 and use $(\rho,\ \ \psi)$ with respect to the Frenet frames of p_1 and p_2 for obtaining c_1 and c_2.

(4) The applicable segments of c_1 and c_2 are arrived at by inspecting their relative movement in view of local and global undercuts.

Geometry of Belt Drives

Let b_1, b_2 be conjugate pulley profiles (Fig. 3). The upper belt span between the contact points $B_1 \in b_1$ and $B_2 \in b_2$ defines a new system Σ_3. As line $b_3 = B_1B_2$ rolls on b_1 and b_2, the points B_1, B_2 are the relative poles 13 and 23, resp., and this implies the necessary condition [10, 9, 14, 8].

Theorem 2. *label2 At each instant the upper belt span must be aligned with the relative pole 12.*

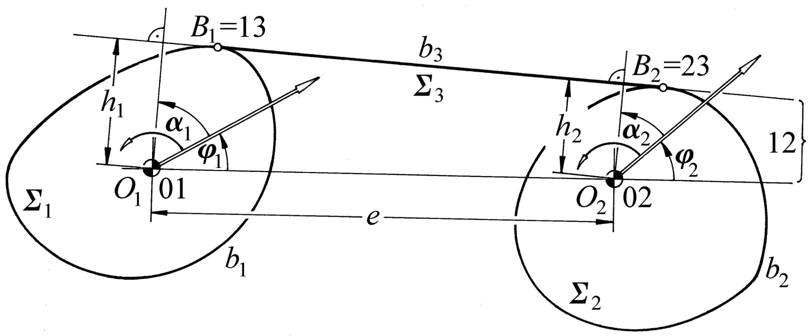

Fig. 3 The upper belt span connects the relative poles 13 and 23

Under the relative motion Σ_3/Σ_1 an arbitrary point C attached to the line B_1B_2 traces an involute c_1 of b_1 (see Fig. 4). The path of $C \in \Sigma_3$ under Σ_3/Σ_2 is an involute c_2 in Σ_2, which contacts c_1 under Σ_2/Σ_1 at C.

Theorem 3. *Conjugate pulley profiles $b_1 \subset \Sigma_1$ and $b_2 \subset \Sigma_2$produce the given transmission from Σ_1 to $\Sigma_2$if and only if b_2 and b_1 are evolutes of an enveloping pair (c_2, c_1) of the relative motion Σ_2/Σ_1. At each instant the endpoints B_2 and B_1 of the upper belt span are corresponding under the curvature transformation of the relative motion Σ_2/Σ_1 (see Fig. 4).*

Of course, b_1 and b_2 must be closed curves – contrary to the locally acting tooth profiles c_1 and c_2.

Now we recall Finsterwalder's method with the polodes p_1, p_2 as metal bands. This time we focus on the envelopes of the attached normal lines making the oriented angle ψ with the polodes. These envelopes b_1, b_2 are conjugate pulley profiles. Hence we obtain

Theorem 4. *For any transmission function and any driving pulley b_1 there is a unique conjugate profile b_2. However, b_2 needs not be convex.*

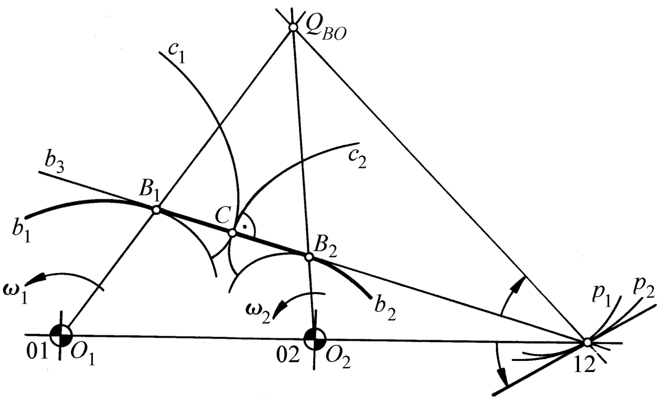

Fig. 4 Points $B_2 \mapsto B_1$ and $O_2 \mapsto O_1$ are corresponding under the curvature transformation of Σ_2/Σ_1 (Bobillier's construction, see [14], Abb. 152)

We call $\psi(s)$ the *natural angle function* of this design. The important case $\psi =$ const. of involute gearing corresponds to *evolutoides*[2] of the polodes as conjugate pulley profiles $b_1, \ b_2$.

Strict Belt Drives

When in each position the length of the surrounding belt with taut spans remains constant, then the lower belt span rolls on b_1 and b_2, too.

Theorem 5. *[9]: Conjugate pulley profiles* b_1 *and* b_2 *operate without needing a tightener if and only if at each instant both the upper and the lower belt span are aligned with the relative pole* 12 *(Fig. 5)*

Wunderlich and Zenow [15] discovered 1975 the following nontrivial example of a nonuniformly transmitting strict belt drive for $n = -1$: Σ_2/Σ_1 is a line-symmetric motion with ellipses as polodes p_1 and p_2. The pulleys are ellipses b_1, b_2 confocal with p_1, p_2 (Theorem of Graves). It should be noted that in the uniform case with $n = 1$ any convex disk together with a translated copy yield a strict belt drive.

Theorem 5 implies an algorithm for computing strict belt drives:

(1) In an arbitrary initial position ($\varphi_1 = \varphi_1^{(0)}$) we specify an upper belt span ($\psi = \psi^{(0)}$) passing through 12. We attach this line to Σ_1.
(2) The next point of intersection of this line with the polode p_1 defines a position ($\varphi_1 = \varphi_1^{(1)}$) where this line becomes a lower belt span. This must be tangent to the conjugate profile b_2, and we attach it to Σ_2.

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[2] An evolutoide of curve p is the envelope of lines meeting p at a constant angle ψ.

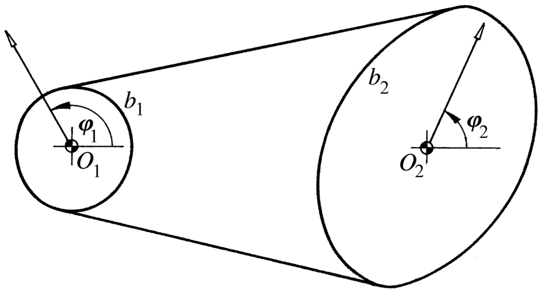

Fig. 5 A strict belt drive for the transmission function $\varphi_2 = 0.5\,[\varphi_1 + 0.2\sin\varphi_1 + 0.16\,\sin 2\varphi_1 + 0.008\,\cos 2\varphi_1]$. The length of the surrounding belt varies within about 0.002 %

(3) When its next point of intersection with p_2 becomes the relative pole ($\varphi_1 = \varphi_1^{(2)}$), the line again covers an upper belt span. In general, this gives a new tangent line of b_1 ($\psi = \psi^{(2)}$).

Iteration gives a finite set of lines tangent to b_1 . In contrast to Hoschek's method using Bézier curves we represent the tangent lines by their support function $h_1(\alpha_1)$ (see Fig. 3). Then we use a least square method for finding the Fourier series of fixed order which approximates the computed tangent lines best. By Theorem 4 we obtain the conjugate c_2.

This algorithm works well (see Fig. 5) in all examples with global transmission ratio $n \neq 1$. The computed lines obviously envelope a unique curve $_1$. This observation together with some arguments give rise to

Conjecture 1: *For a given nonuniform transmission function with global transmission ratio $n \neq 1$ there is a one-parametric set of conjugate pairs (b_1, b_2) of profiles for a strict belt drive. However, these profiles are convex ($h_i + h_i'' > 0$ by [16]) only if the given transmission lies sufficiently close to the uniform transmission with the same global ratio n.*

A rigorous mathematical proof is open but there is some supporting evidence: For convex relative polodes and an analytic transmission function the mapping $(\varphi_1^{(0)}, \psi^{(0)}) \mapsto (\varphi_1^{(2)}, \psi^{(2)})$ of lines is analytic, too. Lines tangent to p_1 ($\psi = 0$) are fixed. Passing through O_1 ($\psi = \gamma$, see Fig. 1) is preserved. And the support function obeys

$$h_1(\varphi_1^{(2)}) : h_1(\varphi_1^{(0)}) = \Omega(\varphi_1^{(2)}) : \Omega(\varphi_1^{(1)}) \qquad \text{with}$$
$$\Omega = \omega_2 / \omega_1$$

The excluded case $n = 1$ shows a strange behavior that was observed –but not reported – by J. Hoschek: The lines obtained by the above algorithm do not envelope any curve. It is proved in [17] that in this case a starting line passing through O_1

after iteration does not rotate fully about O_1 but approaches a limiting position. By continuity, this seems to contradict the required convexity of b_1 and leads to

Conjecture 2: *There is no pure belt drive for nonuniform transmission with global transmission ratio* $n = 1$.

Conclusion

Tooth profiles for gears and pulleys of belt drives are closely related. However, the determination of strict belt drives for a given non-uniform transmission leads to deeper mathematical problems. It is to hope that in the near future the conjectures stated above can be proved rigorously.

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