PROCEEDINGS
13th INTERNATIONAL CONFERENCE ON GEOMETRY AND GRAPHICS
August 4-8, 2008, Dresden (Germany)
ISBN: 978-3-86780-042-6

COMMENTS ON HELICAL DEVELOPABLES

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ABSTRACT: All normal lines of a helical developable Φ have the same distance to its axis a and make the same angle with a. This implies that on coaxial one-sheet hyperboloids of revolution or skew ruled helical surfaces with generators included in the normal-congruence of Φ any orthogonal trajectory of the rulings is located on a translate of Φ.
Let line n be the normal of the helical developable Φ at C. When Φ performs a helical motion or rotation about its axis a with constant velocity, then the point C of intersection with the fixed normal line n runs along n with constant velocity, too. The same holds for circular involutes in the plane. This property offers the most simple way to grasp why helical developables serve as tooth flanks for skew gearing according to Jack Phillips (2003) and why the gear ratio is independent from the relative position of their axes.
The generators of two conjugate tooth flanks Φ_1 and Φ_2 passing through the instantaneous point C of contact remain parallel to their respective initial position. The corresponding osculating cones of the tooth flanks have axes fixed in the gear box and parallel to the related helical axis.

Keywords: helical developable, ruled surfaces, orthogonal trajectory, spatial involute gearing

1. INTRODUCTION

We start with a lemma from plane kinematics:

Lemma 1: *Let c be an involute of the circle c^* with radius r. Suppose, n is the normal line of c at any point $C \in c$ (Figure 1).*
When c rotates about its center O with constant angular velocity ω and n is kept fixed, then the point C of intersection with n runs with constant velocity $v_C = r\omega$ along n while c remains orthogonal to n. The point C^* of contact between n and c^* is the common curvature center of C at each posture of c.

Proof: All this follows from the fact, that the involute c is traced by $C \in n$ while n is rolling on the fixed evolute c^*. The orientation of v_C is induced by the rotation about O. Another explanation results from seeing c^* as a spool and C as the endpoint of a tightened thread which is rolled up on c^*. Now imagine that you pull the thread at its end C in direction of n, while center O is fixed. □

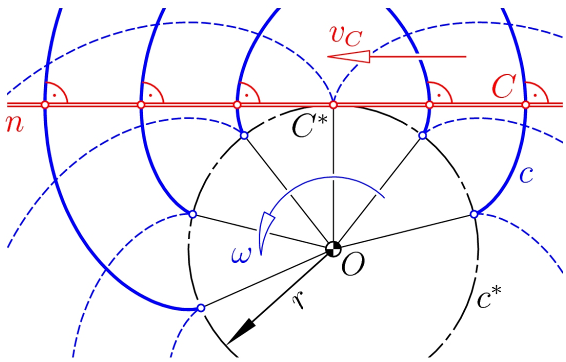

Figure 1: Rotation of the involute c.

As a consequence of Lemma 1, take two circular involutes c_1, c_2 (radii r_1, r_2), which are in contact at C and which therefore share the normal line n at C. If both involutes rotate about their centers O_i, $i=1,2$, with velocities ω_i such that

$$r_1\,\omega_1 = r_2\,\omega_2 \tag{1}$$

then the contact at C will be preserved during this simultaneous movement (Figure 2). This results because both curves remain orthogonal to n and their point of intersection with n runs with the same velocity v_C with respect to both curves.

This is the geometric background of *involute gearing*, invented 1765 by L. Euler (see [11, 3, 4]). c_1, c_2 are conjugate tooth profiles. n is the fixed contact normal and meshing line. The curvature centers C_1^*, C_2^* of c_1 and c_2, resp., at each meshing point $C \in n$ are fixed in the gear box (Figure 2).

We immediately conclude from (1) that involute gearing is insensitive against errors of assembly. The corresponding transmission ratio $\omega_2/\omega_2 = r_1/r_2$ does not depend on the distance $\overline{O_1 O_2}$ but only on the dimensions of the two involutes.

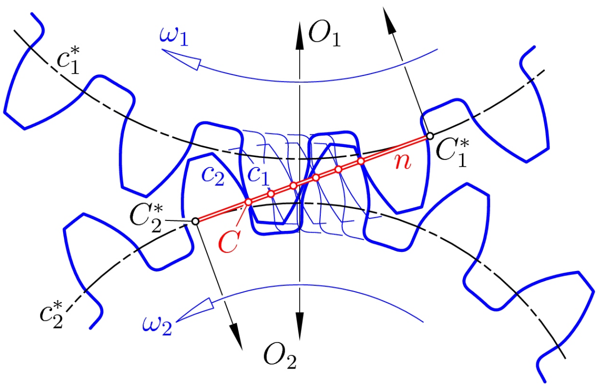

Figure 2: Planar involute gearing.

In theory, the transmission would be preserved even after C passes the cusp on c_i and the convex side of c_i is replaced by the concave one (note Figure 1). Of course, a real world tooth profile cannot have any cusp. Hence only segments of the involutes can be used in practice.

In the sequel we study the 3D-analogue: Instead of the circular involute we use a helical developable Φ – sometimes called *helical involute* – and let it perform a helical motion along its axis a. Then an analogous lemma holds true (Lemma 3), which offers a most simple approach to all the surprising properties of *spatial involute gearing*. This is a new type of skew gearing invented by Jack Phillips and first presented 2003 in the monograpy [5] (note also [8, 2]).

The slip tracks of this gearing are orthogonal trajectories of the rulings on a ruled helical surface or on a one-sheet hyperboloid of revolution. This implies that such trajectories are always located on certain coaxial helical developables.

2. HELICAL DEVELOPABLES

A *helical developable, helical torse* or *helical involute* Φ is a ruled surface with generators g tangent to a helix s (Figure 3). This helix is called *cuspidal edge* of Φ or *line of regression.* Let $r > 0$ be the radius of s and $p \neq 0$ be its pitch.

In analogy to circular involutes, the helical developable Φ is swept out by a line g attached to a tape when this is unrolled from the cylinder of revolution through s (see [11], Abb. 151).

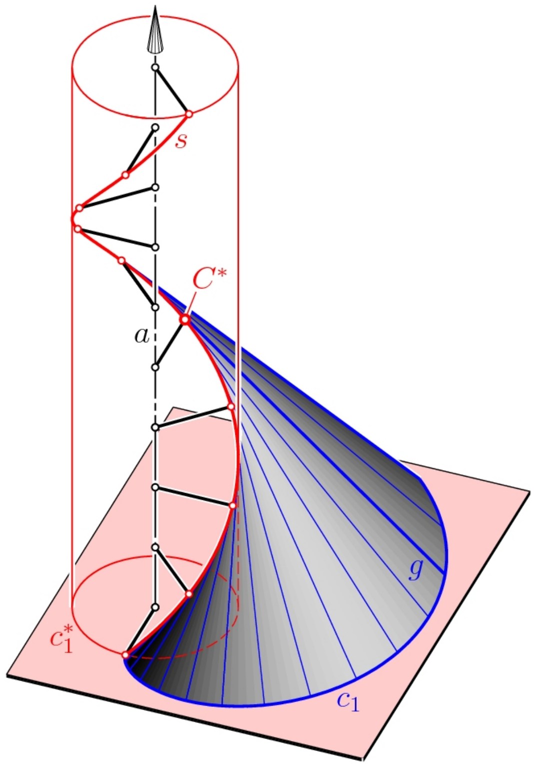

Figure 3: Helical developable.

We assume (see Figure 4) that the axis a of s is vertical and we choose point $C^* \in s$ such that the tangent line g at C^* is parallel to the front view plane π_2. Then the tangent plane τ of

Φ along g is orthogonal π_2, hence the surface normal n at each point C of g is parallel π_2. The front view (upper view in Figure 4) reveals immediately:

Lemma 2: *All normal lines n of the helical developable Φ with radius r and pitch p have the same distance r to the axis a and make the same angle $\alpha = \arctan p/r$ with a. Conversely, the axis a and one single normal line n determine Φ uniquely – up to translations along a.*

The angle $\alpha \neq 0$ is oriented and can be restricted to $-\pi/2 < \alpha < \pi/2$. Distance $r > 0$ and angle α are of course preserved, when g moves along Φ.

Note that all points of the cuspidal edge s are singular points of Φ. The tangent plane τ along g osculates s at its point C^* of contact. The *osculating cone* Ψ of Φ along g is a cone of revolution with axis g^* parallel a and vertex C^* (Figure 4). All regular points C of Φ are parabolic, and per definition the non-vanishing principal curvature at $C \in g$ is reciprocal to the distance $\overline{CM}$, when M is the point of intersection between the corresponding normal n and the axis g^*.

Lemma 3: *Let s be the cuspidal edge of the helical developable Φ with axis a, radius r and pitch $p \neq 0$. Suppose, line n is the normal line at any point $C \in \Phi$.*
When Φ performs a helical motion with pitch q and angular velocity ω along its axis a, while the normal line n is kept fixed, then the point C of intersection[1] between Φ and n moves along n with constant velocity

$$v_C = (q-p)\,\omega\cos\alpha\,. \qquad (2)$$

All postures of Φ are orthogonal to n at C, and the generators g passing through C are mutually parallel. For each of these generators g the osculating cone Ψ has the same axis g^* parallel to a and passing through point $C^* \in n$ closest to a. The non-vanishing principal curvature of Φ at C is reciprocal to the distance $\overline{C^*C}$.

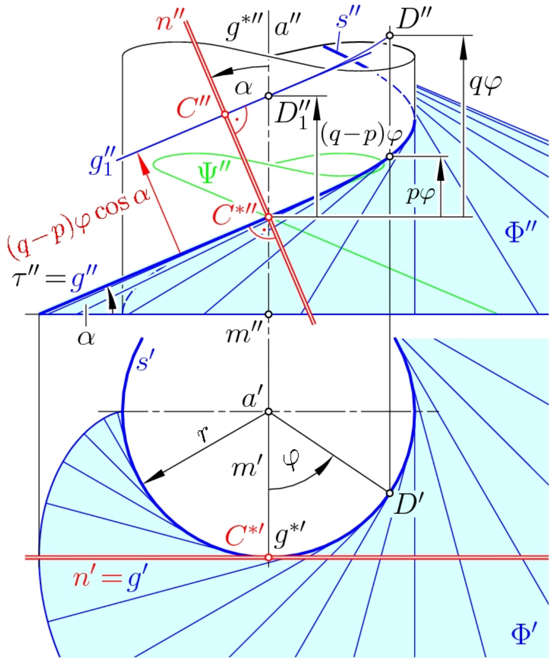

Figure 4: Moving a helical developable.

Proof: Let the initial point C^* be the *striction point* of the generator g, i.e., the point of contact between g and the cuspidal edge s (see Figures 3 and 4). Now we rotate Φ about a through φ und translate it along a by $q\varphi$. Point C^* is moved into D and Φ reaches the posture Φ_1. Then we move D along Φ_1 by rotation through $-\varphi$ and translation $-p\varphi$ and obtain point D_1 which is placed exactly over the initial C^*. The difference in height is $(q-p)\varphi$.[2]

D_1 is the striction point of the generator

[1] To be precise: For each posture of Φ there is an equidistant sequence of points of orthogonal intersection. However, we focus only on a continuous movement of the initial point. And we ignore other points of non-orthogonal intersection.

[2] This proves that due to selfmotions of Φ any posture of the unbounded developable Φ obtained by a helical motion about a can also be reached by a pure translation along a.

$g_1 \subset \Phi_1$ which intersects the normal line n at a point C in the distance $(q-p)\varphi\cos\alpha$ to the initial C^*.
When we set $\varphi = \omega t$ with time t and angular velocity ω we obtain the velocity v_C of the point of intersection as stated in Lemma 2.

At each point $C \in n$ the generator g_1 of Φ_1 passing through is parallel to the initial g. The osculating cone Ψ of g_1 shares with the initial one the axis g^*. This means that during the helical motion of Φ the osculating cone together with the generator g_1 performs a pure translation along g^* or a.

When point $C \in n$ passes C^* which is closest to the axis a and at the same time the striction point of generator g, then the convex side of Φ is replaced by the concave one. □

3. CONSEQUENCES FOR SKEW GEARING

Now we repeat the arguments of Section 1:
Let *two helical developables* Φ_i, $i = 1,2$, be in contact at point C with the common normal line n. Let Φ_i have the axis a_i, radius r_i and pitch p_i.
Suppose that both developables Φ_i simultaneously perform helical motions about a_i with angular velocities ω_i and pitches q_i. Under the condition

$$(q_1 - p_1)\,\omega_1 \cos\alpha_1 = (q_2 - p_2)\,\omega_2 \cos\alpha_2 \quad (3)$$

at each instant the two developables remain in contact since by (3) point C of intersection runs with the same velocity v_C with respect to both developables and also the orthogonality between Φ_i and n is preserved. When substituting $p_i = r_i \tan\alpha_i$ we obtain

$$(q_i - p_i)\cos\alpha_i = q_i \cos\alpha_i - r_i \sin\alpha_i$$

which leads to a condition equivalent to (3).

Theorem 1: *Let two helical developables Φ_i (radius r_i, pitch p_i) be in contact at point C. If then both surfaces Φ_i perform helical motions about their axes a_i with pitch q_i and angular velocities ω_i such that*

$$\frac{\omega_2}{\omega_1} = \frac{q_1 \cos\alpha_1 - r_1 \sin\alpha_1}{q_2 \cos\alpha_2 - r_2 \sin\alpha_2} \quad (4)$$

for $\tan\alpha_i = p_i / r_i$, then the contact at C is preserved. This means, Φ_1 and Φ_2 are two conjugate tooth flanks for a transmission with ratio ω_2/ω_1 between the two helical motions. The meshing point C traces a straight line n fixed in the gear box.

In the case of rotations, i.e., $q_1 = q_2 = 0$, this is exactly the involute gearing invented by J. Phillips [5, 8, 2]. Note that by equ. (4) the gear ratio depends only on the dimensions of the two tooth flanks Φ_i and the pitches q_1, q_2 of the helical motions, but not on their relative position. This means that the gear ratio ω_2/ω_1 is independent from errors of assembly.

In practice, the meshing point C cannot pass the singular points C_1^* or C_2^* of Φ_1 and Φ_2, respectively – as well as at planar involute gearing the cusps of the profiles are excluded. The osculating cones of both tooth flanks at the contact point C will reveal that only positions of C *between* C_1^* *and* C_2^* are possible points of contact for spatial involute gearing. Otherwise the two cones would penetrate each other.

Through each meshing point $C \in n$ there passes a generator g_1 of Φ_1. We call g_1 the *contact generator* of Φ_1. By Lemma 3 all g_1 are mutually parallel. They are orthogonal to n and to the common normal m_1 between g and axis a_1 (see Figure 5). In the same way all contact generators g_2 of Φ_2 are orthogonal to n and m_2 and therefore mutually parallel. Hence, the angle

$$\theta = \measuredangle g_1 g_2 = \measuredangle m_1 m_2$$

remains constant during the mesh (Figure 5).

In order to check the *curvatures* of conjugate tooth flanks at the meshing point C, we replace at both contact generators g_i the flank Φ_i by its osculating cone along g_i. All these cones have a fixed axis g_i^* parallel to a_i and passing through C_i^*. Hence, at each posture of Φ_1 the vertex of the osculating cone is the point of intersection between the fixed g_i^* and the generator g_i which is only translated (Figure 5).

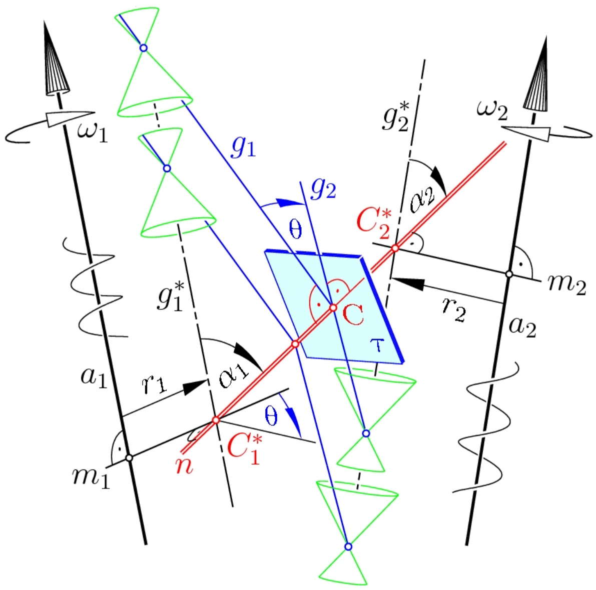

Figure 5: Skew involute gearing.

Theorem 2: *At spatial involute gearing the angle θ between the contact generators g_1 and g_2 remains constant* [8]. *It equals the angle made by the common perpendiculars m_1, m_2 between the meshing line n and the axes a_1, a_2, respectively. During the mesh the contact generator g_i, $i=1,2$, together with the osculating cone of the tooth flank performs a translation along the axis g_i^*, which is parallel to a_i and passes through the point $C_i^* \in n$ closest to a_i.*

In the case $\theta = 0$ the axes a_1, a_2 and the meshing line n are parallel to a fixed plane. Then there is a permanent line contact between the tooth flanks along $g_1 = g_2$ – like ordinary helical gears for parallel axes [4].

It should be noted that (4) is the most general form for the *spatial Law of Gearing*. Line n is a contact normal of tooth flanks for a transmission between helical motions about a_1 and a_2 with pitches q_1, q_2 and angular velocities ω_1, ω_2, resp., if and only if (4) holds. Here α_i is the angle and r_i the distance between n and a_i (see Figure 5). In the rotational case $q_1 = q_2 = 0$ this is equivalent to the equation given in [5], capture of Fig. 2.02, p. 46.

There is also a direct way to prove eq. (4): n is a contact normal if and only if it is included in the linear complex of normals of the relative motion between the two wheels [3, 6]. The screw of this motion in dual vector notation (see, e.g., [10, 1, 7, 9]) is

$$\underline{\mathbf{q}}_{21} = \mathbf{q}_{21} + \varepsilon \hat{\mathbf{q}}_{21} = \underline{\mathbf{q}}_{20} - \underline{\mathbf{q}}_{10},$$

where $\underline{\mathbf{q}}_{i0} = \omega_i (1 + \varepsilon q_i) \underline{\mathbf{p}}_i$ is the screw of the helical motion of Φ_i about the axis a_i with dual unit vector $\underline{\mathbf{p}}_i$. Lines $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon \hat{\mathbf{n}}$ of the normal complex are characterized by

$$\underline{\mathbf{q}}_{21} \cdot \underline{\mathbf{n}} \in \mathbb{R}, \text{ i.e., } \hat{\mathbf{q}}_{21} \cdot \mathbf{n} + \mathbf{q}_{21} \cdot \hat{\mathbf{n}} = 0.$$

Using the formula

$$\underline{\mathbf{p}}_i \cdot \underline{\mathbf{n}} = \cos\alpha_i - \varepsilon\, r_i \sin\alpha_i$$

for the dot product of two dual unit vectors we obtain directly eq. (4).

4. ORTHOGONAL TRAJECTORIES

Let Γ be any ruled helical surface with generators neither orthogonal to the axis a nor intersecting. An orthogonal trajectory o of the rulings is uniquely defined by an initial point C on any generator $n \in \Gamma$. By Lemma 2 there is an unique coaxial helical developable Φ which passes through C and has n (together with all other generators of Γ) included in its normal-congruence.

Φ intersects Γ along a smooth curve through C which at each point is orthogonal to a generator of Γ. Hence this curve is identical with the given orthogonal trajectory o on Γ.

Theorem 3: *On each ruled helical surface Γ with rulings included in the normal-congruence of a coaxial helical developable Φ each orthogonal trajectory of the rulings of Γ is completely located on a translate of Φ (which is also an offset of Φ).*

The slip tracks of the gearing explained in Theorem 1 are such orthogonal trajectories. This results from the fact that relative to the gear box the meshing point C is located on the fixed contact normal n. Relatively to the tooth flank Φ_i, $i = 1,2$, n is sweeping out a helical surface Γ with pitch q_i.

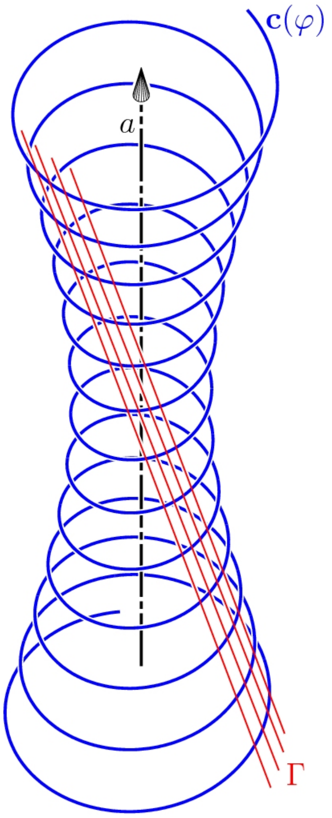

Figure 6: Slip track and orthogonal trajectory.

These curves look similar to the *"bedspring curves"* of the rotational case $q_1 = q_2 = 0$ (Figure 6, notation by J. Phillips [5]).

Theorem 3 can also be verified in an analytic way: Point C traces the slip track on the helical developable Φ when during the helical movement with parameter p through angle φ point C is moving along g by the length $(q-p)\varphi\sin\alpha$ (compare Figure 4). Setting $\sin\alpha = p/\sqrt{p^2+r^2}$ leads to the parametrization

$$\mathbf{c}(\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ p\varphi \end{pmatrix} + \frac{(q-p)p\varphi}{p^2+r^2}\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ p \end{pmatrix},$$

hence

$$\mathbf{c}(\varphi) = \frac{1}{p^2+r^2}\begin{pmatrix} r\cos\varphi(p^2+r^2) - (q-p)pr\varphi\sin\varphi \\ r\sin\varphi(p^2+r^2) + (q-p)pr\varphi\cos\varphi \\ p\varphi(p^2+r^2) + (q-p)p^2\varphi \end{pmatrix}.$$

The same path is obtained when C is performing the helical movement with parameter q through angle φ and traversing n by the length $(q-p)\varphi\cos\alpha$ (Lemma 3, Figure 4). This yields

$$\mathbf{c}(\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ q\varphi \end{pmatrix} + (q-p)\varphi\cos\alpha\begin{pmatrix} -\sin\alpha\sin\varphi \\ \sin\alpha\cos\varphi \\ -\cos\alpha \end{pmatrix}.$$

The substitution $\cos\alpha = r/\sqrt{p^2+r^2}$ gives

$$\mathbf{c}(\varphi) = \frac{1}{p^2+r^2}\begin{pmatrix} r\cos\varphi(p^2+r^2) - (q-p)pr\varphi\sin\varphi \\ r\sin\varphi(p^2+r^2) + (q-p)pr\varphi\cos\varphi \\ q\varphi(p^2+r^2) - (q-p)r^2\varphi \end{pmatrix}.$$

This is in fact the same as before since in both cases the third coordinate equals

$$(pr^2+qp^2)\varphi/(p^2+r^2).$$

Hence the orthogonal trajectories on the ruled helical surface Γ with pitch q, gorge radius r, and angle β $(=\alpha)$ between generators and axis can be parametrized by

$$\mathbf{c}(\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ q\varphi \end{pmatrix} + (q\cos\beta - r\sin\beta)\varphi\begin{pmatrix} -\sin\beta\sin\varphi \\ \sin\beta\cos\varphi \\ -\cos\beta \end{pmatrix}$$

up to helical movements with pitch q along the axis.

5. CONCLUSIONS

This is a new approach to spatial involute gearing based on the 3D-analogue of a property of circular involutes in the plane. Phillips' results on involute gearing are slightly generalized to helical movements of the two gears.

For the first time also the curvature of conjugate helical developables at the point of contact is controlled during the mesh.

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