A REMARKABLE OVERCONSTRAINED SPHERICAL MOTION

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Abstract: The motion under consideration is based on the fact that in the Euclidean space three pairwise orthogonal axes can simultaneously move on a so-called equilateral cone. This defines a non-rational overconstrained spherical motion μ where almost each point path is simultaneously traced by the vertices of an equilateral triangle. One path is a spherical conic. After presenting the projective background, the matrix equation of μ is given. Also some properties of the algebraic completion of μ are discussed.

1. Introduction

The presented spherical motion μ in the Euclidean 3-space $\mathbb{E}^3$ is characterized by the property that the endpoints A, B, C of an orthonormal moving 3-frame trace the same spherical conic c (see Fig. 2). This motion μ is remarkable in many respects:

- μ is overconstrained.
- There is no planar counterpart of μ.
- To the author's knowledge, until recent only trochoid motions have been known as analytic spherical motions with multiply traced point paths. The planar version of this problem is addressed in Müller (1963), p. 96-97. Non-analytic planar motions with a threefold path can e.g. be found in Wunderlich (1970), p. 47-48 (Fig. 32).
- Rational spatial motions have been studied in several papers (see e.g. Jüttler and Wagner (1996)) and even been classified according to the order of their point paths (see Wunderlich (1984), Röschel (1985) and Jüttler (1993)). The considered spherical motion μ is non-rational. The generic point paths are of *spherical order* 24 . This means, that they are projected from the fixed center O of μ by cones of order 24 .

2. The projective background

In the real projective plane $\mathbb{P}^2$ an ordered pair (γ_1, γ_2) of conics is called *apolar*[1], if there is a triangle $P_2Q_2R_2$ self-polar with respect to γ_1 and inscribed in γ_2 (see Fig. 1). A standard result of Projective Geometry says

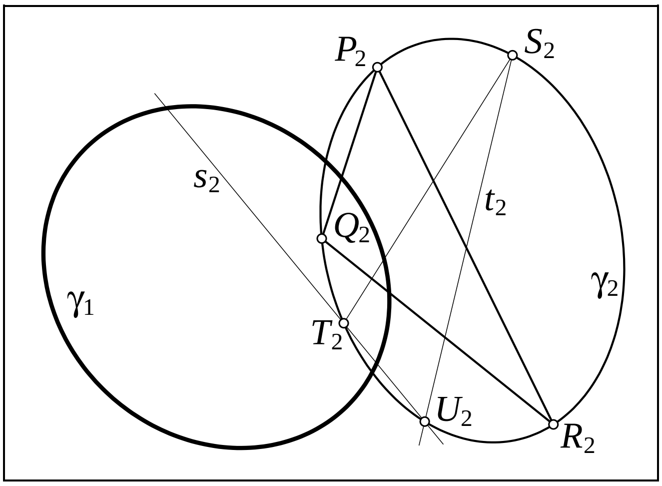

Figure 1. Apolar conics γ_1, γ_2

Lemma 1. *Let (γ_1, γ_2) be a pair of apolar conics. Then there is a one-parameter set of triangles $P_2Q_2R_2$ which are self-polar with respect to γ_1 and inscribed in γ_2.*[2]

Proof: Let $P_2Q_2R_2$ and $S_2T_2U_2$ be two triangles which both are self-polar with respect to γ_1. Then due to von Staudt *the six vertices are located either on two lines or on a conic.* A proof can e.g. be found in Coxeter (1993), p. 87.

Let the apolar pair (γ_1, γ_2) with a defining triangle $P_2Q_2R_2$ be given. Then specify another point $S_2 \in \gamma_2$ such that its polar line s_2 with respect to γ_1 intersects γ_2 at a point $T_2 \neq S_2$ (Fig. 1). Continuity arguments guarantee the existence of S_2 sufficiently near to P_2, Q_2 or R_2. The line s_2 and the polar t_2 of T_2 meet at a point U_2 which completes a second self-polar triangle $S_2T_2U_2$. There must be a conic passing through $P_2, Q_2, R_2, S_2, T_2, U_2$. Since this conic is uniquely defined by the first five points, it coincides with γ_2. $\square$

[1] Baker (1930), p. 33, prefers the unsymmetric notation "γ_1 is *inpolar* to γ_2".

[2] There is also a one-parameter set of triangles $P_1Q_1R_1$ self-polar with respect to γ_2 and circumscribed about γ_1 (see Staude (1915), p. 213). In Baker's notation this means that at the same time "γ_2 is *outpolar* to γ_1". Proofs can be found in Baker (1930), p. 33-34 or Blaschke (1954), p. 84-86. The author would like to thank an anonymous referee for these two references.

Lemma 2. *Let $(x_0 : x_1 : x_2)$ be homogeneous coordinates in $\mathbb{P}^2$. Then the pair of conics (γ_1, γ_2) obeying*

$$\gamma_1\colon \sum_{i,k=0}^{2} c_{ik} x_i x_k = 0\,, \quad \gamma_2\colon \sum_{i,k=0}^{2} d_{ik} x_i x_k = 0$$

with symmetric matrices (c_{ik}) and (d_{ik}) is apolar if and only if in

$$F(\sigma,\tau) := \det(\sigma c_{ik} + \tau d_{ik}) = J_0\sigma^3 + J_1\sigma^2\tau + J_2\sigma\tau^2 + J_3\tau^3$$

the coefficient J_1 is zero.

Proof: (i) The ratio $J_0{:}J_1{:}J_2{:}J_3$ of coefficients in $F(\sigma,\tau)$ does not depend on the choice of the coordinate system.

(ii) For apolar γ_1, γ_2 we use a coordinate system with the fundamental triangle $P_2Q_2R_2$. This implies a matrix (c_{ik}) in diagonal form and vanishing diagonal entries in (d_{ik}), hence

$$(\sigma c_{ik} + \tau d_{ik}) = \begin{pmatrix} \sigma c_{00} & \tau d_{01} & \tau d_{02} \\ \tau d_{01} & \sigma c_{11} & \tau d_{12} \\ \tau d_{02} & \tau d_{12} & \sigma c_{22} \end{pmatrix} \text{ and}$$

$$F(\sigma,\tau) = \det(\sigma c_{ik} + \tau d_{ik}) = c_{00}c_{11}c_{22}\sigma^3 + \tau^2(e\sigma + f\tau)$$

with certain coefficients e, f. Obviously, the coefficient J_1 of $\sigma^2\tau$ is zero.

(iii) In order to prove the converse, we specify a coordinate system which diagonalizes (c_{ik}) and where the fundamental point $P_2 = (1{:}0{:}0)$ is located on γ_2. This implies $d_{00} = 0$.[3] Suppose that in the polynomial

$$F(\sigma,\tau) = \det(\sigma c_{ik} + \tau d_{ik}) = \det\begin{pmatrix} \sigma c_{00} & \tau d_{01} & \tau d_{02} \\ \tau d_{01} & \sigma c_{11} + \tau d_{11} & \tau d_{12} \\ \tau d_{02} & \tau d_{12} & \sigma c_{22} + \tau d_{22} \end{pmatrix}$$

the coefficient of $\sigma^2\tau$ is zero, i.e. $J_1 = c_{00}(c_{11}d_{22} + c_{22}d_{11}) = 0$.

On the line $p_2\colon\ x_0 = 0$ polar to P_2 with respect to γ_1, both conics induce (regular or singular) involutions ι_1, ι_2 of conjugate points, namely

$$\begin{aligned} (0 : x_1 : x_2) \mapsto (0 : x_1' : x_2') \qquad \text{with } c_{11}x_1x_1' + c_{22}x_2x_2' &= 0 \text{ under } \iota_1\,, \\ d_{11}x_1x_1' + d_{12}(x_1x_2' + x_2x_1') + d_{22}x_2x_2' &= 0 \text{ under } \iota_2\,. \end{aligned}$$

For regular (c_{ik}) the condition $J_1 = 0$ is equivalent to the property that the (real or conjugate complex) fixed points Q_2, R_2 of ι_2 are corresponding

[3] Only for $\gamma_1 = \gamma_2$ this choice would be impossible, but then $J_1 \neq 0$ is true.

under ι_1, vice versa.[4] This proves (in the complex extension of $\mathbb{P}^2$) the existence of a triangle $P_2Q_2R_2$ inscribed in γ_2 and self-polar with respect to γ_1. □

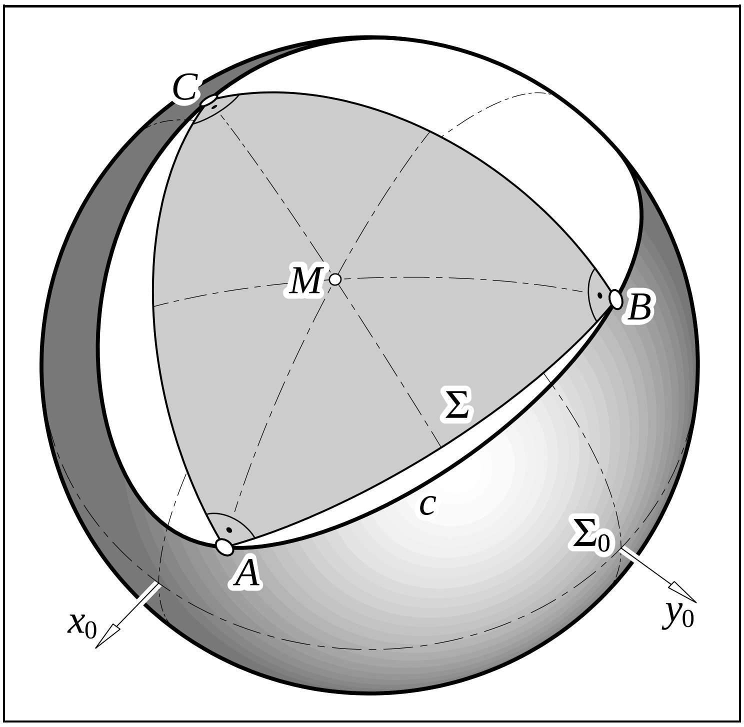

Figure 2. The spherical motion $\mu\colon \Sigma/\Sigma_0$ with the equilateral right triangle ABC inscribed in the equilateral spherical conic c

Let $\mathbb{P}^2$ be the projective extension of the plane $z_0 = -1$ in the Euclidean 3-space $\mathbb{E}^3$, which is equipped with a cartesian coordinate system (x_0, y_0, z_0) with origin O. Lemma 1 remains valid when γ_1 is an empty conic, e.g. with the equation $x_0^2 + y_0^2 + 1 = 0$. In this case two points $P = (x_0, y_0, -1)$ and $P' = (x_0', y_0', -1)$ are conjugate with respect to γ_1 if and only if

$$0 = x_0x_0' + y_0y_0' + 1 = (x_0, y_0, -1)\cdot(x_0', y_0', -1) = \overrightarrow{PO}\cdot\overrightarrow{P'O}.$$

The vanishing dot product shows the equivalence to the orthogonality between the lines connecting the origin O with P and P', respectively. Therefore, when projected from the origin O, the one-parameter set of triangles $P_2Q_2R_2$ according to Lemma 1 yields a one-parameter set of orthogonal 3-bars which all are inscribed in a cone Γ of second order. And this set defines the spherical motion μ to be considered here.

[4]This is exactly the one-dimensional version of apolarity (the two involutions commute, i.e. $\iota_1 \circ \iota_2 = \iota_2 \circ \iota_1$, but $\iota_1 \neq \iota_2$), and this reveals that the n-dimensional version of Lemma 2 can be proved in a similar way by use of induction.

Corollary 1. *When a quadratic cone Γ contains three pairwise orthogonal generators, then this orthogonal 3-bar is even movable on Γ.*

Such a cone is called *equilateral.* According to Lemma 2 its symmetric matrix (c_{ik}) is characterized by a vanishing trace $\mathrm{tr}(c_{ik}) = 0$. When the principal axes of Γ serve as axes of the cartesian coordinate system in $\mathbb{E}^3$, then the equation of Γ can be written as

$$G(x_0, y_0, z_0) := \alpha x_0^2 + \beta y_0^2 - z_0^2 = 0\,, \quad \alpha + \beta = 1, \quad 0 < \alpha \leq \frac{1}{2}. \qquad (1)$$

Only for $\alpha = \beta = \frac{1}{2}$ this is a cone of revolution.

<u>*Remarks:*</u> 1. Let plane ε be a circular section of the cone Γ. Then in each position the axes of the moving frame intersect ε in a triangle $E_1E_2E_3$ inscribed in the fixed circle $k = \Gamma \cap \varepsilon$. All these triangles share the center of the circumcircle and the orthocenter, which is the pedal point of O in ε. Hence, due to the properties of the Euler line, also the center of gravity is common for these triangles in ε. Conversely, these triangles can serve for an elementary approach to Corollary 1.

2. With the following mechanical device the motion μ can be generated: Suppose that the vertices E_1, E_2, E_3 are slot points for the axes of the moving frame. Keep the origin O of this frame fixed while the three slot points move independently from each other on the circle $k \subset \Gamma$.

3. The 3-dimensional versions of Lemma 1 and 2 can be found in Staude (1915), p. 213, the n-dimensional versions in Segre (1928), p. 862, footnote 287.

3. Matrix-representation of the motion μ

From now on (x_0, y_0, z_0) are seen as cartesian coordinates in the fixed space Σ_0 of the motion μ. The curve of intersection between the equilateral cone Γ represented in (1) and the plane $z_0 = 1$ can be parametrized as

$$x_0 = \frac{\cos t}{\sqrt{\alpha}}, \quad y_0 = \frac{\sin t}{\sqrt{\beta}}, \quad z_0 = 1\,, \quad 0 \leq t \leq 2\pi\,.$$

Normalization gives a parameter representation of the curve of intersection between Γ and the unit sphere $\mathbb{S}^2$. In the following c denotes one connected component of this spherical conic. Its parametrization reads

$$\mathbf{c}_1(t) = \frac{1}{\sqrt{r}} \begin{pmatrix} \sqrt{\beta}\cos t \\ \sqrt{\alpha}\sin t \\ \sqrt{\alpha\beta} \end{pmatrix} \text{ with } r := (\alpha - \beta)\sin^2 t + (1+\alpha)\beta, \qquad (2)$$

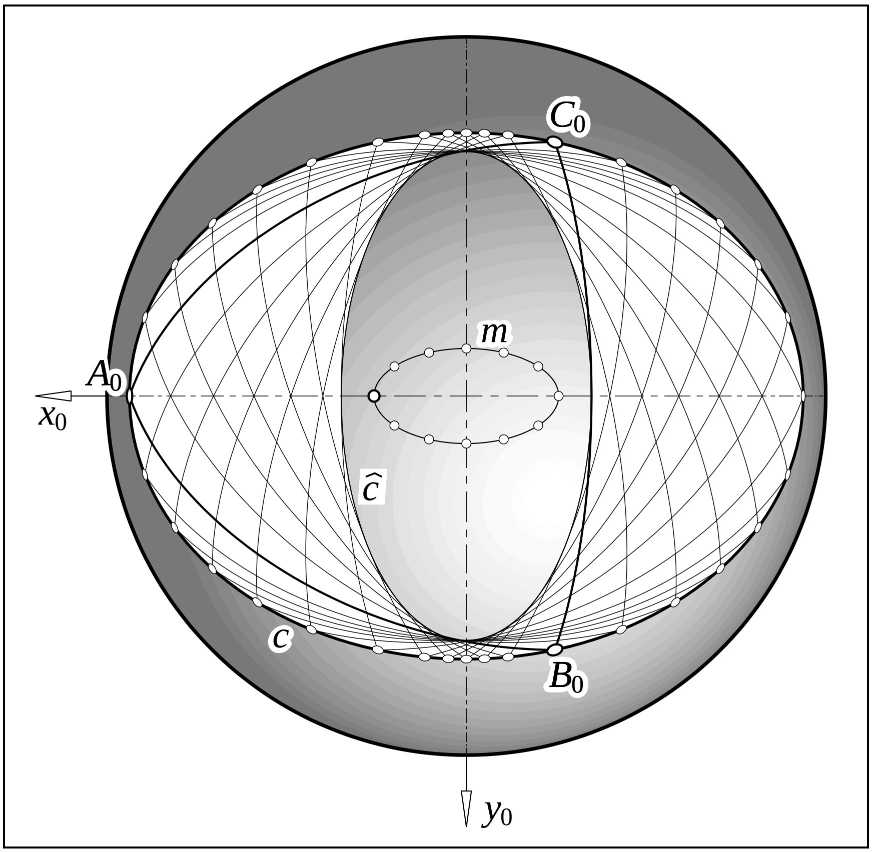

Figure 3. Top view of the motion μ with the initial position $A_0B_0C_0$ of the moving triangle, the path m of the triangle's center and the envelope $\widehat{c}$ of the three sides

as $\alpha + \beta = 1$. For each t the lines of intersection between Γ and the plane

$$x_0\sqrt{\beta}\cos t + y_0\sqrt{\alpha}\sin t + z_0\sqrt{\alpha\beta} = 0 \tag{3}$$

perpendicular to $\mathbf{c}_1(t)$ define the position of the other two axes of the moving frame. We normalize their direction vectors such that the z_0-coordinate is positive. The demand for a right handed frame defines the order of these two unit vectors $\mathbf{c}_2(t), \mathbf{c}_3(t)$ in a unique way.

Let the axes of this moving 3-bar serve as axes of a cartesian coordinate system (x, y, z) in the moving space Σ. Then the vectors $\mathbf{c}_1(t), \mathbf{c}_2(t), \mathbf{c}_3(t)$ are the columns in the orthogonal matrix $\mathbf{C}$ which represents μ. We obtain

Theorem 1. *In matrix-form the motion $\mu\colon \Sigma/\Sigma_0$ can be represented as*

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \Big(\mathbf{c}_1(t)\ \mathbf{c}_2(t)\ \mathbf{c}_3(t)\Big) \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4}$$

Here the first column vector $\mathbf{c}_1(t)$ meets (2). The other two unit vectors $\mathbf{c}_2(t), \mathbf{c}_3(t)$ obey (1) and (3) such that their z_0-coordinates and the triple product $\det(\mathbf{c}_1(t), \mathbf{c}_2(t), \mathbf{c}_3(t))$ are positive.

Eq. (4) enables to visualize the constrained spherical motion μ: In Fig. 2 one position of the moving spherical octant ABC with center M is displayed. Fig. 3 shows several positions of the moving triangle together with the ellipse-shaped path m of M. The selected positions of $ABC \subset \Sigma$ originate from an equal spacing of the path m. In Fig. 3 also the *envelope* $\widehat{c}$ of the moving octant is displayed. $\widehat{c}$ is again a spherical conic; it is located on the cone $\widehat{\Gamma}$ orthogonal to Γ, i.e. $\widehat{\Gamma}$ is tangent to the planes which are orthogonal to the generators of Γ.

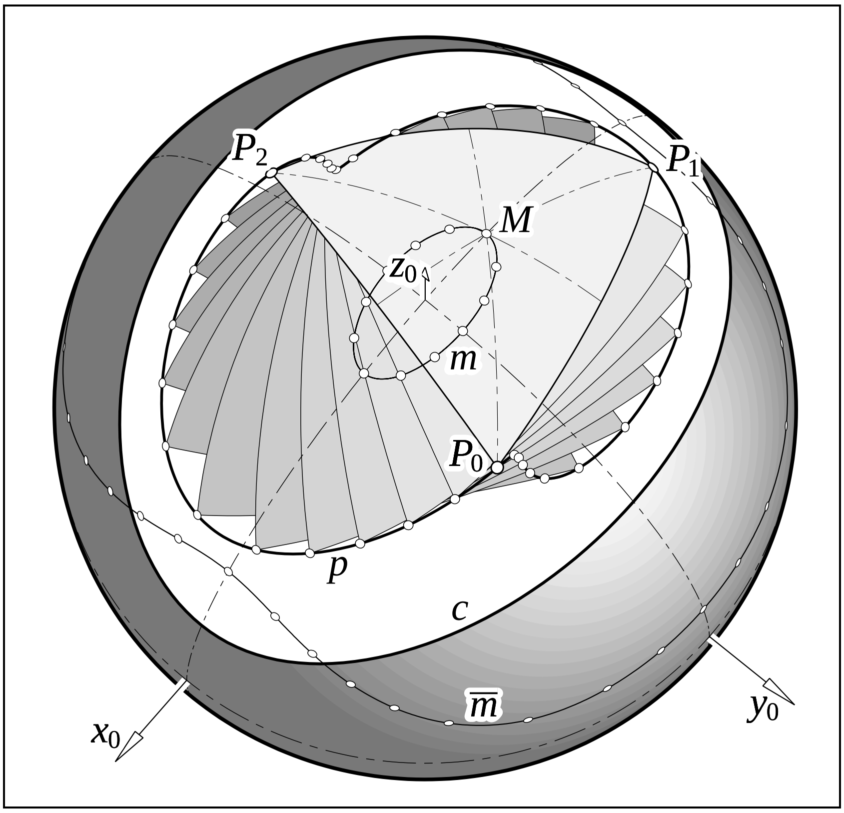

Figure 4. The complete path $\overline{m}$ of M under $\overline{\mu}$ and the path p with the tracing equilateral triangle $P_0P_1P_2$

During one turn of μ the moving triangle ABC returns twice to its initial position $A_0B_0C_0$, however rotated under 120° and 240°, respectively. Let P_0, P_1, P_2 be the corresponding positions of any point P of the moving system for $P \neq M$. Then $P_0P_1P_2$ is again an equilateral triangle with vertices tracing the same spherical path p under μ (see Fig. 4).

4. Algebraic properties of μ

Each position $\Sigma(t)$ of the moving space obtained under μ is uniquely determined either by the corresponding orthogonal matrix $\mathbf{C}$ according to (4) or by the quaternion $\mathbf{q}(t) = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ obeying

$$\mathbf{T} := D\mathbf{C} = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2-b^2-c^2+d^2 \end{pmatrix} \quad (5)$$

with $D := a^2+b^2+c^2+d^2$. For each t the quaternion $\mathbf{q}(t)$ is unique up to a real factor only. The mapping

$$\kappa\colon\ \mathbb{P}^3 \to \mathrm{SO}_3,\ (a:b:c:d) \mapsto \mathbf{C} = \frac{1}{D}\mathbf{T}$$

is the so-called *spherical kinematic mapping.*
Since the unit points $(1,0,0),(0,1,0),(0,0,1)$ of the moving coordinate system trace the same curve c in Σ_0, the column vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ of $\mathbf{T}$ must obey the equation $G(\mathbf{x}) = 0$ of the equilateral cone Γ in (1). This gives

$$\begin{aligned} F_1 &:= G(\mathbf{t}_1) = \alpha(a^2+b^2-c^2-d^2)^2 + 4\beta(ad+bc)^2 - 4(ac-bd)^2 = 0 \\ F_2 &:= G(\mathbf{t}_2) = 4\alpha(ad-bc)^2 + \beta(a^2-b^2+c^2-d^2)^2 - 4(ab+cd)^2 = 0 \\ F_3 &:= G(\mathbf{t}_3) = 4\alpha(ac+bd)^2 + 4\beta(ab-cd)^2 - (a^2-b^2-c^2+d^2)^2 = 0. \end{aligned} \quad (6)$$

However, the three homogeneous polynomials F_1, F_2, F_3 in the indeterminates a, b, c, d are linearly dependent. This results from the representation

$$F_k = G(\mathbf{t}_k) = \alpha t_{1k}^2 + \beta t_{2k}^2 - t_{3k}^2 \text{ with } \mathbf{T} = (t_{ik}),$$

which implies for the row vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of $\mathbf{T}$

$$F_1 + F_2 + F_3 = \alpha\|\mathbf{r}_1\|^2 + \beta\|\mathbf{r}_2\|^2 - \|\mathbf{r}_3\|^2 = D^2(\alpha+\beta-1) = 0\,. \quad (7)$$

In $\mathbb{P}^3$ the set $\mathcal{V}^*$ of zeros $(a\!:\!b\!:\!c\!:\!d)$ of the homogeneous polynomials F_1 and F_2 is an algebraic curve of order 16. But $\mathcal{V}^*$ is reducible for the following reason: The norm of each column vector of $\mathbf{T}$ obeys $\|\mathbf{t}_k\|^2 = D^2$. Hence, $(a:b:c:d)$ is a zero of the polynomials D and $F_1 = G(\mathbf{t}_1)$ if and only if $\mathbf{t}_1$ is isotropic, i.e. $\|\mathbf{t}_1\| = 0$. In this case $\mathbf{t}_2$ is isotropic too and due to $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$ proportional to $\mathbf{t}_1$, provided $\mathbf{t}_1 \neq \mathbf{o}$. Therefore $F_1 = D = 0$ and $\mathbf{t}_1 \neq \mathbf{o}$ imply $F_2 = 0$. A careful analysis proves that $\mathcal{V}^*$ contains two pairs of skew conjugate complex lines on the empty quadric $\Omega\colon\ D = 0$.[5]
The remaining algebraic curve $\mathcal{V}$ of order 12 is mapped under the spherical kinematic mapping κ onto an algebraic one-parameter motion $\overline{\mu}$ which will

[5] We substitute the parameter representation
$a = \frac{1}{2}(\lambda_0\mu_0 + \lambda_1\mu_1)$, $b = -\frac{i}{2}(\lambda_0\mu_0 - \lambda_1\mu_1)$, $c = \frac{1}{2}(\lambda_0\mu_1 - \lambda_1\mu_0)$, $d = -\frac{i}{2}(\lambda_0\mu_1 + \lambda_1\mu_0)$ of Ω in the equations $F_1 = 0$ and $F_2 = 0$ and obtain $\mu_0^2\mu_1^2 P(\lambda_0, \lambda_1) = 0$ and $(\mu_0^2 + \mu_1^2)^2 P(\lambda_0, \lambda_1) = 0$, resp., with $P(\lambda_0, \lambda_1) := (1+\beta)(\lambda_0^4 + \lambda_1^4) - 2(1+2\alpha-\beta)\lambda_0^2\lambda_1^2$.

be called *algebraic completion* of the motion μ. This is a proper extension of μ as the algebraic equations $F_1 = F_2 = 0$ do not rule the orientation of the moving coordinate axes. When therefore $\Sigma(t)$ is a position occupied under μ, then $\overline{\mu}$ contains also all positions which can be achieved from $\Sigma(t)$ under direct displacements which permute the non-oriented coordinate axes. These 24 displacements form a group isomorphic to the rotational symmetries of a cube.

Theorem 2. *The preimage of the motion μ under the spherical kinematic mapping is subset of a non-rational curve $\mathcal{V}$ of order 12 in $\mathbb{P}^3$.*
The path of a generic point under the algebraic completion $\overline{\mu}$ of μ is of spherical order 24 and simultaneously traced by 24 points. The paths of points with $|x| = |y|$ or $|x| = |z|$ or $|y| = |z|$ are symmetric with respect to O; the spherical order reduces to at most 12.
The center M of the moving equilateral right triangle ABC traces a (three-fold covered) path which obeys the equation of fourth order

$$\overline{m}\colon\ (k_1x_0^2 + k_2y_0^2 - k_3z_0^2)(x_0^2 + y_0^2 + z_0^2) - (k_4x_0^2 + k_5y_0^2)^2 = 0\,,$$
$$\begin{array}{lll} k_1 := (5+2\alpha)(1+4\alpha), & \multirow{2}{*}{$k_3 := (\beta-\alpha)^2,$} & k_4 := 3(1+\alpha), \\ k_2 := (5+2\beta)(1+4\beta), & & k_5 := 3(1+\beta). \end{array} \tag{8}$$

Proof: The path of a generic point $\mathbf{x} := (x, y, z)^T$ under $\overline{\mu}$ is located on a cone which is the image of the algebraic curve $\mathcal{V}$ of order 12 under the rational (quadratic) mapping

$$\rho_{\mathbf{x}}\colon\ \mathbb{P}^3 \to \mathbb{P}^2,\ (a:b:c:d) \mapsto (x_0 : y_0 : z_0)\ \text{ for }\ (x_0\ y_0\ z_0)^T = \mathbf{T}\,\mathbf{x}^T$$

according to (5) and (4). Due to standard results of Algebraic Geometry (see e.g. Semple-Kneebone (1959), chapter VIII) the order of this cone equals $2 \cdot 12 - \sigma$ with σ as sum of intersection multiplicities between $\mathcal{V}$ and the set $\mathcal{S}$ of points of indeterminacy under $\rho_{\mathbf{x}}$. This set $\mathcal{S}$ in $\mathbb{P}^3$ obeying $x_0 = y_0 = z_0 = 0$ consists of two skew complex conjugate generators of Ω.[6] For indeterminate (x, y, z) these transcendental lines cannot pass through any point of intersection between $\mathcal{V}$ and Ω. Thus we obtain $\sigma = 0$.

In order to verify the equation of the path $\overline{m}$ of M, we either use the matrix equation (4). Or we express (x_0, y_0, z_0) in terms of $(a, \ldots, d)$ according to (5) and show in accordance with Hilbert's zero point theorem that a power of the resulting polynomial is an element of the ideal which defines $\mathcal{V}$.
In Fig. 3 only one connected component m of the path $\overline{m}$ is displayed which looks like an ellipse. Fig. 4 shows two (real) components of the algebraically completed curve $\overline{m}$. One is traced under μ by the point $(x, y, z) = (1, 1, 1)$, the other simultaneously by $(1, -1, -1), (-1, 1, -1)$ and $(-1, -1, 1)$.

[6] In the notation of footnote 5 these lines obey $2\mu_0\mu_1 x + i(\mu_0^2 + \mu_1^2)y - (\mu_0^2 - \mu_1^2)z = 0$.

Under $\alpha \neq \frac{1}{2}$ the homogeneous equation in (8) defines an irreducible quartic in $\mathbb{P}^2$ without any singularity.[7] Therefore this quartic is non-rational which proves that also $\mathcal{V}$ and the motion $\overline{\mu}$ are non-rational. □

5. Conclusion

The following items are left for future research:

- Each analytic spherical motion can be extended into the dual sphere, which is a model for the set of oriented lines in the $\mathbb{E}^3$ (see e.g. Stachel (1997)). In this sense the motion μ gives rise to a two-parameter spatial motion with the property that the axes of an orthonormal 3-bar trace the same quadratic congruence of lines. There is perhaps a connection with results given in Wunderlich (1980).
- μ can even be generalized to a spherical $\frac{(n-1)(n-2)}{2}$-parameter motion in the Euclidean n-space where the endpoints of an orthonormal n-frame trace the same spherical quadric.

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[7] Singular points could only exist on the lines $x_0 y_0 z_0 = 0$ because of the symmetry.